DESCRIPTION OF A NEUTRON-DEUTERON SYSTEM ON THE BASIS OF THE BARGMANN REPRESENTATION OF S-MATRIX

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For the effective-range function $k \cot \delta$, we obtain a pole approximation that is optimal for the description of a neutron-deuteron system in the doublet spin state on the basis of the Bargmann representation of the S matrix. By using experimental values of the binding energy of a triton E_T, doublet scattering length a_2 and experimental van Oers–Seagrave phase shifts, we calculated the physical characteristics of a triton in the ground (T) and virtual (v) states: the virtual level energy $B_v = 0.608$ MeV and the dimensionless asymptotic normalizing constants $C_T^2 = 2.866$ and $C_v^2 = 0.0586$. The effective radii of a triton in the ground ($\rho_T = 1.711$ Fm) and virtual ($\rho_v = 74.184$ Fm) states are determined as well.

As an important peculiarity of the nd-system in the doublet state, we mention the presence of a pole of the function $k \operatorname{ctg} \delta_2$ [1, 2]. Since this pole is located at a negative energy near the threshold of elastic nd-scattering, it influences significantly a behavior of the scattering phase in the region of low energies. With regard of this fact, van Oers and Seagrave have first reconstructed the experimental dependence of the quantity $k \operatorname{ctg} \delta_2$ on k^2 [3] for the doublet neutron-deuteron s-scattering, by using the relation

$$k \operatorname{ctg} \delta_2 = -A + Bk^2 - \frac{C}{1 + Dk^2} \tag{1}$$

with the parameters

$$A = 0,3105 \text{ Fm}^{-1}, \quad B = 0,85 \text{ Fm},$$

$$C = 3,138 \text{ Fm}^{-1}, \quad D = 478,5 \text{ Fm}^2. \tag{2}$$

The pole formula for the function

$$k \operatorname{ctg} \delta_2 = -\frac{1}{a_2} \frac{1 + c_2 k^2 + c_4 k^4}{1 + Dk^2} \tag{3}$$

follows directly [4, 5] from the Bargmann representation of the S-matrix [6, 7]

$$S(k) = \frac{(k + i\varkappa_T)}{(k - i\varkappa_T)} \frac{(k + i\varkappa_v)}{(k - i\varkappa_v)} \frac{(k + i\lambda)}{(k - i\lambda)} \frac{(k + i\mu)}{(k - i\mu)} \tag{4}$$

with two physical states. For the sake of convenience, we separate the doublet nd-scattering length a_2 in formula (3) in the explicit form.

As for the nd-system, the quantities $\varkappa_T > 0$ and $\varkappa_v < 0$ in formula (4) are the wave numbers corresponding to the ground and virtual states of a triton, and the parameters $\lambda > 0$ and $\mu > 0$ are superfluous poles of the S-matrix. To within the notation, the pole formula (3) coincides with the empiric van Oers–Seagrave formula (1).

The parameters of the pole formula (3) are simply related [5] to the parameters $\varkappa_T$, $\varkappa_v$, λ, and μ of the Bargmann S-matrix (4):

$$a_2 = \frac{1}{\varkappa_T} + \frac{1}{\varkappa_v} + \frac{1}{\lambda} + \frac{1}{\mu}, \tag{5}$$

$$c_2 = -\frac{1}{\varkappa_T \varkappa_v} - \frac{1}{\varkappa_T \lambda} - \frac{1}{\varkappa_T \mu} - \frac{1}{\varkappa_v \lambda} - \frac{1}{\varkappa_v \mu} - \frac{1}{\lambda \mu}, \tag{6}$$

$$c_4 = \frac{1}{\varkappa_T \varkappa_v \lambda \mu}, \tag{7}$$

$$D = -\frac{1}{a_2} \frac{\varkappa_T + \varkappa_v + \lambda + \mu}{\varkappa_T \varkappa_v \lambda \mu}. \tag{8}$$

It is convenient to represent formula (3) as

$$k\,\mathrm{ctg}\,\delta_2 = -\frac{1}{a_2} + \frac{1}{2} r_0 k^2 + \frac{v_2 k^4}{1 + D k^2}\,, \tag{9}$$

where the effective radius r_0 and the shape parameter v_2 can be given in terms of the parameters of formula (3) by the relations

$$r_0 = \frac{2}{a_2}\left(D - c_2\right)\,, \tag{10}$$

$$v_2 = -\left(\frac{1}{2} D r_0 + \frac{c_4}{a_2}\right)\,. \tag{11}$$

Thus, the pole approximation (9) deduced from the Bargmann representation of the S-matrix is a direct generalization of the effective-range approximation to the case where the system possesses two physical states. For the nd-system, such states are the ground and virtual states of a triton.

By analogy with the effective radius of a deuteron ρ_d in the case of nucleon-nucleon interaction, we introduce the notion of the effective radius of a triton ρ_T for the nd-interaction. We define this radius as the quantity which enters the expansion of the function $k\,\mathrm{ctg}\,\delta_2$ at the point $k^2 = -\varkappa_T^2$:

$$k\,\mathrm{ctg}\,\delta_2 = -\varkappa_T + \frac{1}{2}\rho_T\left(\varkappa_T^2 + k^2\right) + w_2\left(\varkappa_T^2 + k^2\right)^2 + \ldots\,. \tag{12}$$

Formulas (9) and (12) yield the relations which express the wave number $\varkappa_T$ and the effective radius of a triton ρ_T through the parameters of the pole approximation (9):

$$\varkappa_T = \frac{1}{a_2} + \frac{1}{2} r_0 \varkappa_T^2 - \frac{v_2 \varkappa_T^4}{1 - D\varkappa_T^2}\,, \tag{13}$$

$$\rho_T = r_0 + \frac{2 D v_2 \varkappa_T^4 - 4 v_2 \varkappa_T^2}{\left(1 - D\varkappa_T^2\right)^2}\,. \tag{14}$$

Calculating the residue of the S-matrix at the pole $k = i\varkappa_T$ it is easy to prove that the dimensionless asymptotic normalizing constant of the ground state of a triton C_T depends on the effective radius of a triton ρ_T by the formula

$$C_T^2 = \frac{2}{3}\frac{1}{1 - \rho_T \varkappa_T}\,. \tag{15}$$

Relations which are analogous to (12)–(15) are valid also for the quantities $\varkappa_v$, ρ_v, and C_v characterizing the virtual state of a triton.

It is directly seen from formula (7) that, if the system contains the bound and virtual states, the parameter c_4 is a negative quantity. However, in some works [8–10] it was set to zero. In this case, we arrive at the anomalous value for the scattering phase $\delta_2(\infty) = \frac{\pi}{2}$, and, as a result, the Levinson theorem is not satisfied. Nevertheless, at very small energies, the function $k\,\mathrm{ctg}\,\delta_2$ can be approximated by relation (3) with $c_4 = 0$

$$k\,\mathrm{ctg}\,\delta_2 = -\frac{1}{a_2}\,\frac{1 + c_2 k^2}{1 + D k^2}\,. \tag{16}$$

Such an approximation was obtained in [8] on the basis of the dispersion N/D-method. In this case, the parameters c_2 and D are explicitly expressed in terms of the doublet nd-scattering length a_2, the binding energy of a deuteron $\varepsilon_d = \hbar^2\alpha^2/m$, and the asymptotic normalizing constant of a deuteron A_S,

$$c_2 = \frac{5}{4\alpha^2}\left[1 - \frac{2\alpha a_2}{\sqrt{5} z_0}\left(z_0 - \frac{3}{5}\right)\right]\,, \tag{17}$$

$$D = \frac{3}{4\alpha^2 z_0}\,, \tag{18}$$

where

$$z_0 = \frac{\frac{2}{5}\sqrt{3}\,\alpha a_2\left(1 + \frac{2}{3\sqrt{5}}\frac{A_S^2}{\alpha}\right)}{\frac{4}{3\sqrt{3}}\frac{A_S^2}{\alpha} + \frac{2}{\sqrt{3}}\alpha a_2\left(1 - \frac{2}{3\sqrt{5}}\frac{A_S^2}{\alpha}\right)}\,. \tag{19}$$

Using the experimental values of the binding energy of a deuteron, $\varepsilon_d = 2.224575$ MeV [11], asymptotic normalizing constant of a deuteron $A_S = 0.8781\ \mathrm{Fm}^{-1/2}$ [12], and doublet nd-scattering length

$$a_2 = 0.65\ \mathrm{Fm} \quad [13], \tag{20}$$

we get the following values of the parameters c_2 and D

$$c_2 = 43.422\ \mathrm{Fm}^2\,,$$

$$D = 172.678\ \mathrm{Fm}^2\,. \tag{21}$$

Thus, the parameters of the Whiting–Fuda approximation at fixed values of the binding energy of a deuteron ε_d and the asymptotic normalizing constant of a deuteron A_S are completely determined by a single three-particle parameter — the scattering length a_2. The use of formula (16) with parameters (20) and (21) leads to the qualitative description of the nd-scattering phase at energies less than the break-up threshold of a deuteron ($E_L \leqslant 3.34$ MeV) and to the overstated binding energy of a triton $E_T = 5.329$ MeV as compared with its experimental value

$$E_T = 8.481855 \text{ MeV} [14]. \tag{22}$$

By including the experimental value of the binding energy of a triton (22) into the fitting parameters and by somewhat varying the parameter c_2 without changing the parameters a_2, E_T, and D, we can significantly improve the description of the nd-scattering phase. The minimum absolute error of a description of the scattering phase at energies less than the break-up threshold of a deuteron is attained at $c_2 = 38.626$ Fm^2 and is equal to $\sim 1°$. In this case, by the formula

$$c_4 = \left[a_2 \varkappa_T \left(1 - D\varkappa_T^2\right) - 1 + c_2 \varkappa_T^2\right] / \varkappa_T^4 \tag{23}$$

we obtain $c_4 = -75.752$ Fm^4, which leads to the proper description of the nd-system, because $\delta_2(\infty) = 0$.

In the pole approximation (3) of the function $k \operatorname{ctg} \delta_2$ with the parameters

$$a_2 = 0.65 \text{ Fm}, \quad D = 172,678 \text{ Fm}^2,$$

$$c_2 = 38.626 \text{ Fm}^2, \quad c_4 = -75.752 \text{ Fm}^4 (\text{BP}) \tag{24}$$

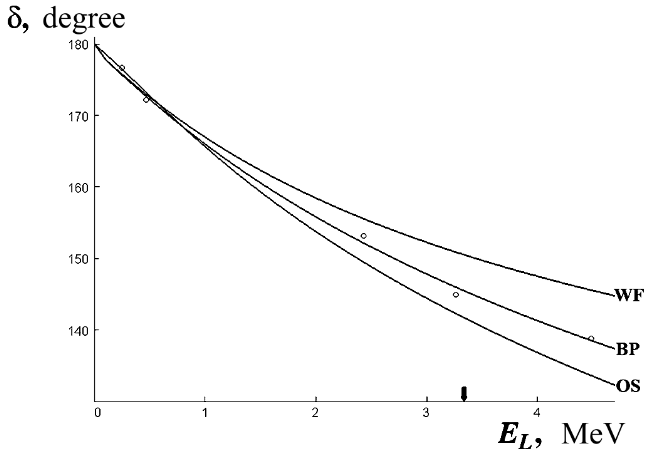
we calculated the doublet nd-scattering phase at laboratory energies less than 4.5 MeV. For the sake of comparison, we calculated also the scattering phase in the van Oers–Seagrave (OS) approximation (1) with parameters (2) and in the Whiting–Fuda (WF) approximation (16)–(19) with parameters (20) and (21).

T a b l e 1. Doublet nd-scattering phase calculated for different versions (WF, OS, and BP) of the pole approximation of the effective-range function

E_L, MeV	Scattering phase, degr.			
	WF	OS	BP	Expt. [2]
0.2517	175.73	176.45	175.63	176.63
0.4699	172.92	173.20	172.63	172.21
2.4333	155.54	149.47	152.14	153.19
3.2640	150.89	142.30	145.99	144.94
4.4891	145.56	133.62	138.52	138.85

T a b l e 2. Low-energy parameters of the doublet nd-scattering corresponding to different versions of the pole approximation of the effective-range function

Version	a_2, Fm	r_0, Fm	v_2, Fm^3	D, Fm^2
WF	0.65	397.71	–34337.84	172.678
OS	0.29	3004.38	–718343.98	478.469
BP	0.65	412.469	–35495.62	172.678



Doublet nd-scattering phase versus the lab.-syst. energy for different versions of the pole approximation of the effective-range function. Points – experimental data from [3]. The arrow indicates the break-up threshold of a deuteron

The results of calculations are given in the Figure and in Table 1. In Table 1, we present also the nd-scattering phase obtained by us by the electronic processing of a scanned figure from work [3] used as experimental data.

As seen from the Figure and Table 1, the pole approximation (3) with parameters (24) allows us to practically exactly describe the doublet nd-scattering phase at energies less than the break-up threshold of a deuteron ($E_L \leqslant 3.34$ MeV). In this case, the absolute error $\sim 1°$, and the relative error is at most 1%. The scattering phases calculated in the pole approximation with the use of the parameters determined in earlier works [3, 4, 8, 9, 15] agree with experimental data significantly worse. In these cases, near the break-up threshold of a deuteron, the absolute error attains $\sim 5°$ and more.

For different approximations of the effective-range function, we calculated the low-energy parameters of the doublet nd-scattering r_0 and v_2 which are given in Table 2. As seen, the effective radius r_0 and the shape parameter v_2 which correspond to the Whiting–Fuda approximation differ insignificantly, as distinct from those in the van Oers-Seagrave approximation, from analogous parameters corresponding to the more correct approximation (BP).

In Table 3, we give values of the main physical characteristics of a triton calculated by us: the position of a virtual level B_v, the dimensionless asymptotic normalizing constants of the ground C_T and virtual C_v states of a triton, and the effective radii of a triton, ρ_T and ρ_v, corresponding to these states.

As seen from Table 3, the results of calculations of the physical values characterizing the virtual state of a triton in the pole approximation (3) with parameters (24) and in the Whiting–Fuda approximation agree well with one another. First of all, this is related to that these quantities are determined at a negative energy in the region which is directly adjacent to the elastic nd-scattering threshold, where the behavior of the function $k\,\mathrm{ctg}\,\delta_2$ is mainly determined by the scattering length a_2 and the pole parameter D which coincide in the cases under consideration.

In conclusion, we will formulate and discuss the main results. The pole approximation (3) obtained on the basis of the Bargmann representation of the S-matrix with the use of the experimental values of the doublet nd-scattering length $a_2 = 0.65$ Fm and the binding energy of a triton $E_T = 8.482$ MeV as input parameters is optimal for the description of the nd-system in the doublet spin state. In this case, approximation (3) with parameters (24) describes almost exactly the doublet phase of nd-scattering at energies less than the break-up threshold of a deuteron ($E_L \leqslant 3.34$ MeV). The relative error does not exceed 1 %.

In this case, for the scattering parameters, the position of a pole of the function $k\,\mathrm{ctg}\,\delta_2$ and the characteristics of a triton in the ground (T) and virtual (v) states, we obtain the values

$$r_0 = 412.469 \text{ Fm},$$

$$v_2 = -35495.62 \text{ Fm}^3, \quad D = 172.678 \text{ Fm}^2\,, \tag{25}$$

$$E_0 = -0.180 \text{ MeV}, \tag{26}$$

$$C_T^2 = 2.866, \quad \rho_T = 1.711 \text{ Fm}, \tag{27}$$

$$B_v = 0.608 \text{ MeV}\ , \quad C_v^2 = 0.0586, \quad \rho_v = 74.184 \text{ Fm}. \tag{28}$$

The positions of a pole $E_0 = -0.180$ MeV and the virtual level $B_v = 0.608$ MeV agree completely with $E_0 = -0.18$ MeV and $B_v = 0.61$ MeV obtained in [16] for the Yukawa potential and differ somewhat from $E_0 = -0.15$ MeV [15, 16] and $B_v = 0.482$ MeV [17]which are used sometimes [16] as experimental data. The asymptotic constant $C_T^2 = 2.866$ obtained by us agrees well with its experimental values $C_T^2 = 2.6 \pm 0.3$ [18], 2.78 ± 1.10 [19], 2.74 ± 0.079 [19, 20] and $C_T^2 = 2.96$, determined in [21] on the basis of the analysis of s-wave pd-scattering within the N/D-method.

T a b l e 3. Physical characteristics of a triton in the ground (T) and virtual (v) states for different versions of the pole approximation of the effective-range function

Version	E_T, MeV	C_T^2	ρ_T, Fm	B_v, MeV	C_v^2	ρ_v, Fm
WF	5.329	1.274	1.509	0.577	0.0548	82.041
OS	8.482	7.489	2.031	0.515	0.0718	64.349
BP	8.482	2.866	1.711	0.608	0.0586	74.184

The effective radii of a triton in the ground ($\rho_T = 1.711$ Fm) and virtual ($\rho_v = 74.184$ Fm) states calculated by us are important characteristics of the nd-system in the doublet spin state which determine the region of the effective neutron-deuteron interaction in a triton. In this case, as contrary to the effective range of the doublet nd-scattering r_0 which has an anomalously great value ($r_0 = 412.469$ Fm), the effective radius of a triton in the ground state possesses a reasonable physical value $\rho_T = 1.711$ Fm and is very close to the effective radius of a deuteron $\rho_d = 1.725$ Fm [12]. The effective radius of a triton in the virtual state $\rho_v = 74.184$ Fm is significantly greater than that in the ground state ρ_T, because the former is determined at the negative energy $E = -0.608$ MeV, being near the pole of the function $k\,\mathrm{ctg}\,\delta_2$, $E_0 = -0.180$ MeV. We note that though the effective radii ρ_T and ρ_v are interesting and important physical quantities characterizing the region of effective interaction in the system of three nucleons, they were not discussed, as far as we know, earlier in the literature.

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ОПИС ДУБЛЕТНОГО СТАНУ nd-СИСТЕМИ НА ОСНОВІ ПРЕДСТАВЛЕННЯ БАРГМАНА ДЛЯ S-МАТРИЦІ

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Р е з ю м е

На основі представлення Баргмана для S-матриці отримано полюсне наближення функції ефективного радіуса $k\text{ctg}\delta$, яке є оптимальним для опису nd-системи у дублетному спіновому стані. З використанням експериментальних значень енергії зв'язку тритона E_T, дублетної довжини nd-розсіяння a_2 і експериментальних фаз ван Оерса–Сегрейва розраховано фізичні характеристики тритона в основному (T) і віртуальному (v) станах: розташування віртуального рівня $B_v = 0,608\,$МеВ та безрозмірні асимптотичні нормувальні константи $C_T^2 = 2,866$, $C_v^2 = 0,0586$. Також розраховані ефективні радіуси тритона в основному ($\rho_T = 1,711$ Фм) і віртуальному ($\rho_v = 74,184$ Фм) станах.