ENERGY THRESHOLDS OF STABILITY OF THREE-PARTICLE SYSTEMS

I.V. SIMENOG[1,2], YU.M. BIDASYUK[1,2], B.E. GRINYUK[1], M.V. KUZMENKO[1]

[1]M.M. Bogolyubov Institute for Theoretical Physics, Nat. Acad. Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03143; e-mail: ivsimenog@bitp.kiev.ua),

[2]Taras Shevchenko Kyiv National University
(6, Academician Glushkov Prosp., Kyiv 03127)

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We have studied the general properties of the energy thresholds of stability for a three-particle system with short-range interaction. A wide region of the interaction constants and various ratios of the masses of particles are considered. The specific effects characteristic of the near-threshold stationary energy levels of three particles are revealed. The asymptotic estimates are obtained at same limiting cases, and the high-precision variational calculations of the thresholds for various values of the interaction constants and the masses of particles are carried out.

1. Introduction

The studies of the quantum systems of three particles of different nature, which are performed within various theoretical approaches, remain to be actual for a long period (see survey [1]). This is related to both the presence of nontrivial effects, which appear in this simplest many-particle system, such as the famous Efimov effect [2], and the importance of the theoretical consideration of real three-particle and three-cluster systems of different nature, e.g., nuclei with three nucleons or with three clusters, hypothetical systems of the type of trineutrons, molecular trimers, etc. Important and insufficiently studied are the fine near-threshold effects in the systems of three particles. The present work is devoted to the analysis of the general properties of these effects and the thresholds of stability for three-particle systems. The study of properties of the three-particle thresholds is executed qualitatively on the basis of the analysis of the asymptotic estimates, and also using the high-precision variational calculations with the use of optimized Gaussian bases. This approach allows us to investigate, with a high controlled accuracy, even such fine effects as the Efimov effect, as well as the structural functions of these near-threshold levels [3] that are characterized by a very small energy.

2. Statement of the Problem

In the present work, we consider a system of three particles, among which two particles are identical (we set their masses $m_1 = m_2 = 1$ without loss of generality), and the third one can differ from them by the mass and the pairwise interaction potential. We write the Hamiltonian of such a system in the case of pairwise interactions in the form (in the system of units with $\hbar = 1$)

$$\hat{H} = -\frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2 - \frac{1}{2m}\Delta_3 + V(r_{12}) + U(r_{13}) + U(r_{23}). \quad (1)$$

Let the intensities of two-particle interaction potentials be defined by the dimensionless interaction constants g (for the pair of identical particles (*12*)) and λ (for the pairs of particles (*13*) and (*23*)) on the given form of the interaction, where $V(r) = gv(r)$ and $U(r) = \lambda u(r)$. We consider the potential functions $v(r)$ and $u(r)$ mainly with positive values. Negative values of g and λ correspond to the attraction, and positive ones to the repulsion. We will study the properties of the energy thresholds of stability for three-particle systems, i.e. the regions of such values of the constants g and λ, at which the n-th energy level of the three-particle system appears below the two-particle thresholds or below zero if the two-particle subsystems are not bound,

$$E_n(\text{three}) - E_0(\text{two}) \le 0, \quad E_0(\text{two}) \le 0,$$

$$E_n(\text{three}) < 0, \text{the coupling of two particles is absent.} \quad (2)$$

We will study the energy thresholds of stability for three-particle systems with the zero total angular momentum, $L = 0$, for various ratios of the masses of particles, the form of pairwise interaction potentials, and the symmetry of the wave function relative to the permutations of identical particles (for the symmetric $\varphi^s(1,2;3) = \varphi^s(2,1;3)$ and antisymmetric $\varphi^a(1,2;3) = -\varphi^a(2,1;3)$ states).

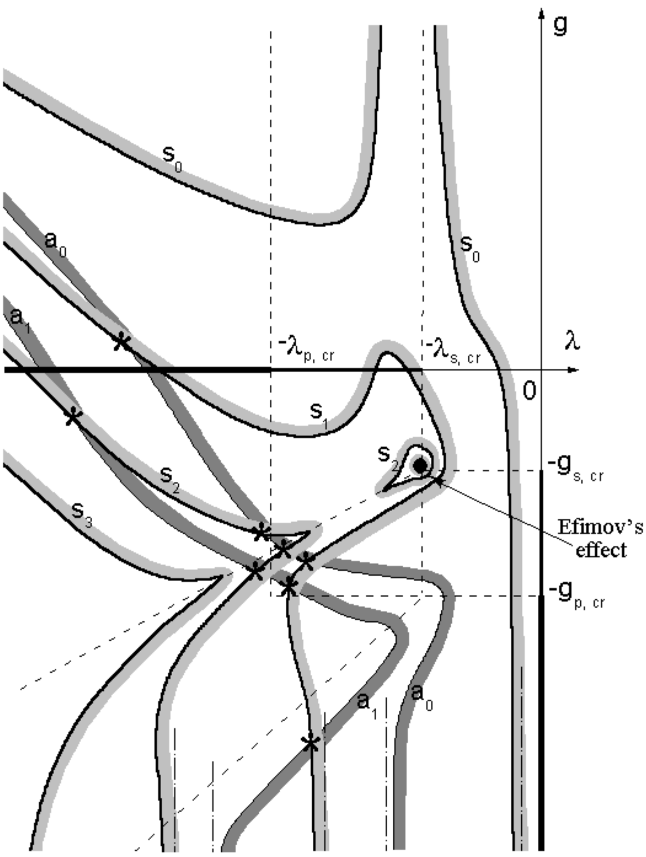

Fig. 1. Schematic image of the energy thresholds of stability of a three-particle system for short-range interaction potentials: s_n — symmetric, a_n — antisymmetric states relative to the permutations of particles (12). The asterisks indicate the intersections of the lines of the thresholds of states with different symmetries, the vertical dash-dotted lines in the lower part of the figure are the asymptotes for the lines of thresholds, and the dashed inclined lines correspond to the equality of the binding energies $\epsilon_{12} = \epsilon_{13}$ of different pairs of particles

On the plane (g, λ), we will present the results of studies in the form of the diagrams of the thresholds of energy levels, namely the thresholds of stability of the energy levels of three particles with the zero angular momentum (Fig. 1). The systematization of the significant amount of calculations and the qualitative analysis allow us to represent the thresholds of stability in the universal form of the diagrams of thresholds. In Fig. 1, we give, for the sake of specificity, the example of the diagrams of stability corresponding mainly to potentials $v(r)$ and $u(r)$ which have the same Gaussian form and correspond to almost equal masses. We emphasize that the characteristic properties of a diagram are preserved for nonsingular short-range potentials of more general forms. In Fig. 1, we show the main peculiarities of such diagrams schematically without holding the scale, but with the preservation of all main regularities. Each line of the threshold of the n-th level (for the states s_n symmetric relative to the permutation of the identical first and second particles, and for the antisymmetric states a_n) separates the region of the existence of a bound state (the region of stability) with the corresponding n-th level of three particles from the region, where this level does not exist (this side of the curve is shaded). Because the bound states of three particles can appear only under the attraction between different particles (the potential $U(r)$ with the negative constant λ) while the interaction between identical particles (the potential $V(r)$) can be both attractive ($g < 0$) and repulsive ($g > 0$), the three-particle thresholds are positioned to the left from the axis of ordinates. On the axes, we marked the points where two-particle ground bound states appear: the s-state with the orbital moment $l = 0$ ($-\lambda_{s,\mathrm{cr}}$, $-g_{s,\mathrm{cr}}$) and the p-state with the orbital moment $l = 1$ ($-\lambda_{p,\mathrm{cr}}$, $-g_{p,\mathrm{cr}}$). We also show the corresponding two-particle thresholds on the axes by lines with different thicknesses. The inclined dashed lines separate the regions, where the lowest threshold from two two-particle ones is the threshold for two identical particles (due to the potential $gv(r)$, below the dashed line) or for two different particles (due to the potential $\lambda u(r)$, above the dashed line). The three-particle levels exist below the lowest two-particle threshold. The dash-dotted vertical lines in the lower part of Fig. 1 are the asymptotes of three-particle thresholds as $g \to -\infty$. In this case, the lower part of the figure is correlated with the rest parts so as it follows from direct calculations. In the diagram of the thresholds of stability, the bound states of the system of three particles exist, by starting from the lines of thresholds, as a rule, towards the increase in the intensities of the attraction (i.e. to the left and down on the diagram). On the energy diagrams of stability, we can distinguish eight different regions by characteristic peculiarities of the ground and excited three-particle levels:

region I — the asymptotic region, where $-\lambda \approx g \to \infty$,

region II — the asymptotes of thresholds as $g \to -\infty$, $\lambda = \mathrm{Const}$,

region III — the region of the infinite series of Efimov levels near the values of the two-particle interaction constants critical as for the appearance of the bound s — states $g \approx -g_{s,\mathrm{cr}}$, $\lambda \approx -\lambda_{s,\mathrm{cr}}$; on the diagram of thresholds, the nonmonotonous and closed curves correspond to the Efimov states (they are marked by the words "Efimov's effect"),

region IV — the axis λ in the case of the absence of the interaction between identical particles (12), when $g = 0$ and the conditions of the Thomas theorem are satisfied,
region V — the region where the effect of a "tube" is manifested for $\lambda \approx -\lambda_{s,\mathrm{cr}}$ and arbitrary positive g,
region VI — we indicate the characteristic behavior of thresholds for the symmetric three-particle states in the region $g \approx \lambda \ll -g_{s,\mathrm{cr}}$ with the appearance of an acute "wedge" on the line of equality of the threshold binding energies of different pairs of particles, $\varepsilon_{12} = \varepsilon_{13}$,
region VII — the region of a "rearrangement" of the thresholds of energy levels,
region VIII — the region of the nonmonotonicity of the curves for the thresholds of antisymmetric levels.

The characteristic peculiarities of the behavior of the energy thresholds of stability of the three-particle system in different regions on the plane (λ, g), the general established phenomenon of the nonmonotonicity of thresholds, and the effect of "traps" are studied analytically and with the use of numerical calculations within the Galerkin variation method with a Gaussian basis and various high-precision optimization schemes.

3. Asymptotics of Thresholds and the Effect of a "Tube"

Consider region I for the ground and excited symmetric states on the diagram of thresholds, where $\lambda \to -\infty$ and simultaneously $g \to +\infty$ (the left upper part of the diagram of thresholds in Fig. 1). We rewrite Hamiltonian (1) in the center-of-mass system as

$$H = -\frac{1}{2m}\left(1 + \frac{m}{2}\right)\Delta_\rho - \Delta_r + gv(r) + \lambda u\left(\left|\boldsymbol{\rho} + \frac{\mathbf{r}}{2}\right|\right) + \lambda u\left(\left|\boldsymbol{\rho} - \frac{\mathbf{r}}{2}\right|\right), \tag{3}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\boldsymbol{\rho} = \mathbf{r}_3 - (\mathbf{r}_1 + \mathbf{r}_2)/2$ — Jacobi relative coordinates. Let a short-range repulsive potential $gv(r)$ have a maximum at zero and monotonically decrease, and let the potential $\lambda u(r)$ have a minimum value U_0 ($U_0 < 0$ — the attraction, and this occurs not obligatorily at $r = 0$). In the limit of strong coupling, the main contribution to the ground-state energy is determined by the minimum of the full three-particle potential well, which is positioned at $\rho_{\min} = 0$ and at some $r_{\min}$, minimizing the effective total potential energy $V(r) + 2U(r/2)$. For nonsingular short-range potentials V and U, the threshold line is determined by the formula

$$\min_r (gv(r) + 2\lambda u(r/2)) = \min_x \lambda u(x), \tag{4}$$

which determines the value of $r_{\min}$, where the minimum of the effective attractive potential energy of three particles $gv(r) + 2\lambda u(r/2)$ is attained, and establishes the connection between the constants g and λ in the considered region of the diagram of thresholds. In the simple case of the interaction potentials V and U in the Gaussian form with unit radii, which is thoroughly studied numerically, the condition for the threshold has the form

$$ge^{-r_{\min}^2} - 2|\lambda| e^{-r_{\min}^2/4} = -|\lambda|. \tag{5}$$

From the condition of the minimum of the left-hand side of (5), we get

$$r_{\min} = \sqrt{\frac{4}{3}\ln\left(\frac{2g}{|\lambda|}\right)}, \tag{6}$$

and, with regard for (5),

$$r_{\min}^2 = 4\ln(3/2) \approx 1.62. \tag{7}$$

Then, in the limit $|\lambda| \to \infty$, the asymptotic formula with regard for the next correction for the lines of the thresholds of the ground and excited levels has finally the following form:

$$g = \frac{27}{16}\left(|\lambda| + C_{p,q}\sqrt{2|\lambda|} + O(1)\right). \tag{8}$$

Here,

$$C_{p,q} = \frac{9}{2}\sqrt{1 + \frac{1}{m}} - (2p+1)\frac{3r_{\min}}{2} - (2q+3)\sqrt{\left(1 + \frac{2}{m}\right)\left(\frac{3}{2} - \frac{r_{\min}^2}{4}\right)}, \quad p, q = 0, 1, 2, \ldots \tag{9}$$

We note that the quantum numbers $p = q = 0$ in (8) and (9) correspond to the ground state. The series of states in p is due to one-dimensional small oscillations along the coordinate $x \equiv r - r_{\min}$, whereas the series of states in q corresponds to three-dimensional oscillations along the coordinate ρ near the minimum of the potential well in the three-particle system. Indeed, if we expand the potential energy (in the case of the Gaussian potentials V and U) in Hamiltonian (3) averaged over angles near the minimum in ρ^2 and in the squared deviation $x^2 \equiv (r - r_{\min})^2$, we get, instead of (3), the approximate oscillatory Hamiltonian as a function of both coordinates

$$\tilde{H}_{\rho,x} = -\frac{1}{2m}\left(1 + \frac{m}{2}\right)\Delta_\rho + \frac{2}{3}\left(2 - \frac{4}{3}\ln\left(\frac{3}{2}\right)\right)|\lambda|\rho^2 -$$

$$-\frac{1}{x^2}\frac{\partial}{\partial x}x^2\frac{\partial}{\partial x}+2\ln\left(\frac{3}{2}\right)|\lambda|\,x^2. \tag{10}$$

The energy states of the oscillatory Hamiltonian (10) generate the lines of thresholds (8) and (9). The motion of the system of three particles in symmetric states in approximation (10) is oscillations relative to the following configuration. At the center of the system, a particle with mass m is located. On the same diameter with it, particles 1 and 2 ($\rho = 0$) are positioned, but the wave function is spherically symmetric in angles (the sphere diameter $r_{\min} \approx 2\sqrt{\ln(3/2)} + \mathrm{O}\left(1/\sqrt{g}\right)$). The distance between particles 1 and 2 does not depend in the main approximation on the mass of the third particle. That is, such a configuration of the system (the third particle is at the center between the two identical particles) occurs even in the case where the mass m of the third particle is small.

The more detailed consideration of the threshold in region I shows that asymptote (8) for the ground state is defined as

$$g \to \frac{27}{16}\left(|\lambda| + C_{0,0}\sqrt{2\,|\lambda|}+...\right), \tag{11}$$

and the coefficient

$$C_{0,0} = -\frac{3}{\sqrt{2}}\left\{\sqrt{\frac{2+m}{m}\left(3-2\ln\left(\frac{3}{2}\right)\right)}+\right.$$

$$\left.+\sqrt{2\ln\left(\frac{3}{2}\right)}-3\sqrt{\frac{1+m}{2m}}\right\} \tag{12}$$

is positive at very small masses. Moreover, $C_{0,0} \to 3\left(3/2-\sqrt{3-2\ln(3/2)}\right)/\sqrt{m} \approx 0.06/\sqrt{m}$ as $m \to 0$, is negative at greater masses and reaches a minimum value at $m \approx 0.057$ ($C_{0,0} \approx -1.39$). At $m \to \infty$, it approaches the constant, $C_{0,0} \approx -0.55$. Thus, the linear asymptote of the threshold for the ground state in region I of the threshold diagram in Fig. 1 is reached from the top for very small masses ($m \leq\sim 0.001$) and is reached from the bottom for the greater mass of the third particle.

We also note that, in region I of the diagram of thresholds, the result which is asymptotic in the coupling constant, $|\lambda| \to \infty$, is easily generalized to other potentials. In particular, let the interaction potential U between different particles be chosen in the Gaussian form with another interaction radius $R \neq 1$. Then we get the same configuration of three particles (the third particle is located between two identical ones). Moreover, instead of (7), we obtain (for $R \geq 1/\sqrt{2}$, when $r_{\min}^2 \geq 0$)

$$r_{\min}^2 = 4R^2\ln\frac{4R^2-1}{2R^2}. \tag{13}$$

In the main approximation, we get

$$g = B(R)\,|\lambda| + O\left(\sqrt{|\lambda|}\right), \tag{14}$$

where

$$B(R) = \frac{\left(4R^2-1\right)^{4R^2-1}}{\left(2R^2\right)^{4R^2}}. \tag{15}$$

The least value of the coefficient, $B(R) = 1$, is reached in (14) at $R = \sqrt{2}/2$, when $r_{\min} = 0$ and all three particles approach one another at small distances. As $R \to \infty$, the coefficient $B(R)$ grows indefinitely. For $R < \sqrt{2}/2$, all three particles are, all the more, at small distances, where the small oscillations of the particles relative to the equilibrium position occur. The asymptotics of a threshold remains invariable: $g = |\lambda| + \mathrm{O}\left(\sqrt{|\lambda|}\right)$. Hence, in the region of the diagram of thresholds where $g \to \infty$ and $\lambda \to -\infty$, the three-particle system can possess different configurations even in the case of the simplest interactions. Qualitatively, analogous results are obtained for a wider class of the pairs of potentials V and U. For example, for potentials in the form of exponentials, $V(r) = ge^{-r}$ and $U(r) = -|\lambda|\,e^{-r/R}$, we have, for $R \geq 1$,

$$r_{\min} = 2R\ln\frac{2R-1}{R},\ B(R) = \frac{1}{R}\left(2-\frac{1}{R}\right)^{2R-1}. \tag{16}$$

In this case, the least value of the coefficient $B(R) = 1$ is also attained at $R = 1$, when $r_{\min} = 0$, and both $r_{\min}$ and $B(R)$ grow with increase in R. Moreover, for an arbitrary pair of the monotonous repulsive potential $V(r)$ between the identical particles and the attractive potential $U(r)$ between different particles in region I, a linear dependence between the intensities of the potentials takes place, which separates the region of stability of three particles from the region, where the coupling is absent. If the repulsive potential V is nonmonotonous and has a sufficient decrease at zero, and if the attractive potential U has a minimum at finite distances $r_1 > 0$, a two-cluster configuration of the system of three particles is also possible: in this case, the identical particles are positioned near each other, and the third one is located at a distance r_1 from them.

All the main qualitative and analytic results concerning the asymptotics of the thresholds of stability in region I are confirmed by the high-precision systematic calculations with two-particle potentials of

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the Gaussian form for a great variety of masses within the variational approach with the use of Gaussian bases. The separate calculations were executed for other potentials, and they also support the general schematic Fig. 1.

We now consider the asymptotic limit of the strong coupling between the identical first and second particles (region II, where $g \to -\infty$ in the lower part of the diagram of thresholds (Fig. 1)). In this limit, the size of subsystem (12) tends to zero, and the variables in (1) are separated in the cluster approach. Then the three-particle wave functions for Hamiltonian (3) can be chosen as

$$\Psi_n(\boldsymbol{r},\boldsymbol{\rho}) \approx \varphi_0^{(\mathrm{osc})}(r) f_n(\rho) \ , \quad (17)$$

where $\varphi_0^{(\mathrm{osc})}(r)$ is the ground state oscillatory wave function of the relative coordinate $\boldsymbol{r}$ of pair (12), and the wave function of the third particle, being in the effective field of two other particles, depends on the coordinate $\boldsymbol{\rho}$ and is the eigenfunction of the effective Hamiltonian reckoned from the two-particle threshold,

$$h = -\frac{1}{2\mu}\Delta_\rho + U_{\mathrm{eff}}(\rho) \ , , \quad (18)$$

with the reduced mass $\mu = 2m(m+2)^{-1}$ and the effective averaged potential

$$U_{\mathrm{eff}}(\rho) = -\frac{2\,|\lambda|}{\pi^{3/2}} \int d\boldsymbol{r} e^{-r^2} u\left(\left|\boldsymbol{\rho} - \boldsymbol{r}/\left(2\,|g|^{1/4}\right)\right|\right) . \quad (19)$$

Consider the motion of the third particle relative to the pair of particles (12) in potential (19) which looks, in particular for the Gaussian form, as

$$U_{\mathrm{eff}}(\rho) = -\frac{2\,|\lambda|}{\left(1 + 1/\left(4\sqrt{|g|}\right)\right)^{3/2}} e^{-\rho^2/\ 1+1/\ 4\sqrt{|g|}} \ .(20)$$

Then, for the critical constants $\lambda_{\mathrm{cr}}^{(n)}$, at which the n-th three-particle near-threshold level in the three-particle system appears, we get the asymptotics as $g \to -\infty$:

$$\lambda_{cr}^{(n)} \approx -\frac{m+2}{8m} g_{s,\mathrm{cr}}^{(n)} \left(1 + \frac{1}{8\sqrt{|g|}}\right) \ . \quad (21)$$

Here, $g_{s,\mathrm{cr}}^{(n)}$ are the critical constants of the appearance of the n-th level in the system of two particles with the zero orbital moment $l = 0$ (for the Gaussian potential with unit radius in the case of the unit masses of particles, the critical constants of the s-states are as follows: $g_{s,\mathrm{cr}}^{(0)} \equiv g_{s,\mathrm{cr}} = 2.684005$; $g_{s,\mathrm{cr}}^{(1)} = 17.79570$; $g_{s,\mathrm{cr}}^{(2)} = 45.57348$; $g_{s,\mathrm{cr}}^{(3)} = 85.96340$, etc.). The expressions analogous to (21) can be also obtained in the case where the subsystem of two particles in the state with the wave function (17) has a nonzero orbital moment. It follows from (21) that the threshold of each three-particle level (in the states symmetric in the permutations of the pair of particles (12)) is positioned always to the left from the corresponding vertical asymptote for $\lambda_{\mathrm{cr}}^{(n)}$ (they are marked by dash-dotted vertical lines in the lower part of Fig. 1). We can show analogously that, for the states antisymmetric relative to the permutations of the pair of particles (12), the thresholds approach, on the contrary, the corresponding vertical asymptotes from the right side. The high-precision calculations for different masses and potentials in the form of a superposition of Gaussian functions confirm completely the above-presented conclusions about the asymptotics of thresholds for different levels as $g \to -\infty$.

Consider region III in the vicinity of the critical constants of the appearance of the two-particle ground bound s-states: $g \to -g_{s,\mathrm{cr}}$ and $\lambda \to -\lambda_{s,\mathrm{cr}}$ in Fig. 1. This is the region where the Efimov effect [2] is manifested, and the infinite series of Efimov weakly bound levels for the system of three (generally saying, different) particles is observed. In the diagram of thresholds, it is seen as the infinite collection of closed self-similar (with the universal scale for highly excited states) convex curves are accumulated at the critical point $(-\lambda_{s,\mathrm{cr}}, -g_{s,\mathrm{cr}})$ for three pairs of particles interacting in a resonance way. The lines of thresholds for the series of Efimov energy levels are considerably stretched along the dashed inclined line (which starts from the region of the Efimov effect), where the two-particle thresholds of different pairs of particles in the s-state coincide, and are somewhat elongated vertically upward along the g axis, when only two pairs of particles interact in a resonance way. The fundamental Efimov effect is described in detail by V. Efimov himself and many other researchers. Here, we should like to emphasize only separate points. Firstly, the infinite series of symmetric three-particle Efimov levels with the zero angular momentum is realized only in the limit $g \to -g_{s,\mathrm{cr}}$, $\lambda \to -\lambda_{s,\mathrm{cr}}$, i.e. in the case where all three pairs of particles in singlet states are at a resonance. Outside this region, the number of levels is always bounded, though these levels have the certain specific properties of properly Efimov levels especially in the case of two resonating pairs of particles (13) and (23) ($\lambda \to -\lambda_{s,\mathrm{cr}}$) even under a sufficiently large decrease in the attraction between the particles of the third pair, (12). Secondly, by the example of the

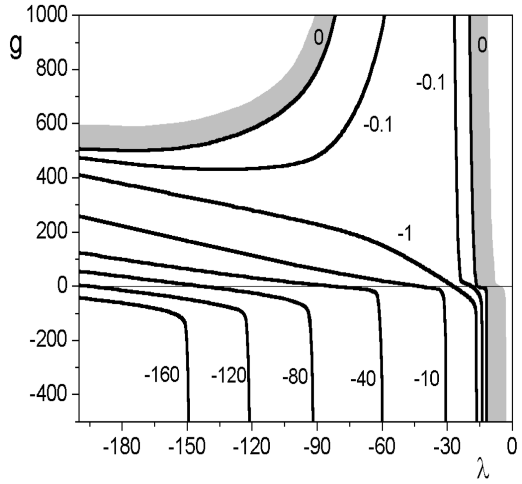

Fig. 2. Isolines of the ground state energy of the three-particle system for $m = 0.06$ with Gaussian potentials. At the calculated isolines, we indicate the difference of the energies of three and two particles

series of Efimov levels, we have the ideal demonstration of the energy "traps" (see Section 5 for more details), when the strengthening of the attraction on the diagram of thresholds leads to a nonmonotonous variation of the number of energy levels: the three-particle levels at first separate from the two-particle threshold and then disappear on it. The Efimov levels exist only in a certain interval of the coupling constants of all three pairs of particles interacting in a resonance way. Thirdly, for three-particle states antisymmetric in the pair of the identical particles (12), the singlet point $(-\lambda_{s,\mathrm{cr}}, -g_{s,\mathrm{cr}})$ is nonsingular (see Fig. 1, region VIII). At the same time, the limiting region near the point $(-\lambda_{s,\mathrm{cr}}, -g_{p,\mathrm{cr}})$, where one pair (12) of the identical particles resonates in the state with the angular momentum $l = 1$, is, as if, "attractive" for the corresponding states. In this region of Fig. 1, there exists a certain anomaly: the curves for the thresholds of antisymmetric states are significantly stretched towards small coupling constants along the line where the two-particle thresholds of two different pairs of particles in the singlet and triplet states coincide. Finally, we note that we do not discuss the three-particle antisymmetric states with nonzero angular momentum, where there exists the possibility for a collapse to arise [4], which requires a separate consideration.

Consider the case where the interaction of one pair of particles, (12) is absent (when $g = 0$). This corresponds to region IV on the λ axis in Fig. 1, where the conditions of the Thomas theorem [5] are satisfied. Then, there always exists the bound ground state of three particles in a wider region of coupling constants λ of two pairs of particles as compared with an isolated pair of two particles. This means that the three-particle threshold of the ground state for $g = 0$ is always positioned to the right from the two-particle critical point in the constant λ. First of all, we note that the important common point for the Thomas and Efimov effects is the presence of pairs of particles interacting in a resonance way. Moreover, the regions of these effects are positioned near each other in the diagram of thresholds (Fig. 1). At the same time, the Thomas effect is referred only to the ground state of three particles, whereas the Efimov effect concerns the infinite series of near-threshold weakly bound states. Secondly, the universality of the Thomas theorem consists in that it holds for Hamiltonian (3) with $g = 0$ for an arbitrary attractive potential $\lambda u(r)$ between two pairs of particles and an arbitrary finite mass m of the third particle. Moreover, the less the mass m, the wider is the region of the manifestation of the Thomas effect relative to the interaction constant. We note that, for small masses, the number of three-particle levels becomes great, when the two-particle subsystems are not bound yet. Thirdly, there occurs the effect of a "trap" for the first excited state at $g = 0$ for particles with close masses (under certain conditions, this concerns higher excited states too), as distinct from the ground state. In this case, the existence of an excited level depends nonmonotonically on strengthening the attraction by variation of $|\lambda|$. In addition, the intersection points of the lines of the thresholds of excited levels with the abscissa axis (Fig. 1) from the side of great values of $|\lambda|$ are located to the right from the corresponding two-particle critical values of the coupling constants, where the binding energies of two particles are equal to zero. We emphasize once more that the schematic Figure 1 represents all main regularities so as they follow from the asymptotic estimates and the high-precision calculations.

A special attention should be paid to the effect of a "tube" for symmetric states at the diagram of thresholds (region V in the upper part of Fig. 1), where the bounded region near $\lambda \sim -\lambda_{s,\mathrm{cr}}$ contains at least one bound state of three particles even under the unlimited increase of the repulsion between the pair of identical particles (g is arbitrary). The effect of a "tube" is seen more clearly in Fig. 2, where, on the real scale, we show the results of high-precision calculations for the threshold of the ground state and the isolines of the energy of three particles, which is reckoned from the two-particle threshold, for the Gaussian potentials with unit interaction radii for the mass $m = 0.06$. The form of the energy surface in the region $g \gg 1$ and near $\lambda \sim -40$, indeed, reminds of a "tube" positioned vertically. It is seen that the isoenergetic lines ΔE = const for the ground state of three particles with small values of $\Delta E \approx -0.1$ are pulled into the "tube" in the upper part of the figure for $\lambda \approx -40$ and great repulsive constants g. The isolines do not form a "tube" already at $\Delta E \approx -1$ and greater values, but reveal the clear asymptotic behavior characteristic of the isolines with great values of ΔE, showing a sharp change of the modes near the abscissa axis (on the given scale). It is important that, on the increase of the attraction between particles (13) and (23) (the motion along the horizontal axis in Fig. 2) under a significant repulsion between the pair of particles (12), the nonmonotonicity in the number of bound states is always observed (we call this as the effect of a "trap" related to the presence of the "tube").

The phenomenon of a "tube" can be explained on a qualitative level. We will show firstly that the number of levels in a "tube" is always bounded, but it grows infinitely as $m \to 0$. We use the fact [3] that the pair correlation functions $G_{(n)}(r) \equiv \langle\varphi_n|\,\delta(\mathbf{r}-\mathbf{r}_{12})\,|\varphi_n\rangle$ in Efimov states $|\varphi_n\rangle$ for $r = 0$ for the adjacent levels satisfy the relation

$$G_{(n+1)}(0)\,/G_{(n)}(0) \to \frac{1}{\sqrt{\Lambda_0}}\,, \tag{22}$$

where Λ_0 is the ratio of the energies of adjacent Efimov levels in the limit $n \to \infty$ ($\Lambda_0 = e^{2\pi/s_o} = 515.035\ldots$ in the case of three identical particles, and the Danilov—Minlos—Faddeev—Efimov constant $s_0 = 1.00623$ determines the index of singularity of the limiting equation of Skornyakov—Ter-Martirosyan in the case of the zero interaction radius). Relation (22) takes place due to the fact that, as $n \to \infty$, the spatial region, where the pair correlation function behaves itself as $G(r) \sim r^{-2}$ [3], is spreading. It is true from distances of the order of the radius of forces to distances of the order of the size of the system $\sim \left(\sqrt{\Lambda_0}\right)^n R_0$ (here, R_0 is the characteristic size of the system in the ground state, n is the level number, and $\sqrt{\Lambda_0} \approx 22.69$ is the ratio of sizes of the system for adjacent Efimov levels). Therefore, the pair correlation function for sufficiently great n has the following normalizing factor of $\sim r^{-2}$:

$$C_n \to \left(\int\limits_{r_0}^{\left(\sqrt{\Lambda_0}\right)^n R_0} r^{-2} d\mathbf{r}\right)^{-1} \to\sim \left(\sqrt{\Lambda_0}\right)^{-n} R_0^{-1}\,. \tag{23}$$

At small distances, the pair correlation function is of the same order as that at $r \sim r_0$, therefore we obtain relation (22).

In order to approach the region of the "tube" from the region, where the Efimov effect manifests itself, we consider some additional short-range repulsion $W(r_{12})$ between the pair of particles (12) which is taken as a perturbation of the zero Hamiltonian, for which the Efimov effect is observed. The greater the number of a level, the greater is the size of the three-particle system, which reminds of a spherical concentric "halo" [3] with the ratio of the radii of the adjacent spheres $\sim \sqrt{\Lambda_0}$. The greater the size of a system of particles with short-range interaction, the better are satisfied the conditions for the "gas" approach [6] taking into account a contribution of the short-range repulsion $W(r_{12})$ to the energy. In the principal approximation in the "gas" parameter, a shift of the n-th level is determined by the two-particle T-matrix:

$$\Delta E_n = \langle\varphi_n|\,T_{12}\,|\varphi_n\rangle \to \frac{4\pi}{m} a G_{12,n}(0)\,. \tag{24}$$

Here, T_{12} is the two-particle T-matrix defined for the potential $W(r_{12})$, a is the scattering length by the potential $W(r_{12})$ (for the repulsive potentials, $a > 0$). These quantities are finite for short-range repulsive potentials of the given form in the limit of the infinitely great repulsion, $g \to \infty$. In this case, the effective radius and other low-energy parameters which are present in the next terms of the expansion of the energy in the "gas" parameter are also finite. These conclusions allow us to consider the most wide region in the constant g in the diagram of thresholds from the region of the Efimov effect to the region of the "tube" with $g \to \infty$, if the summary potential $-g_{s,\mathrm{cr}}u(r_{12}) + W(r_{12})$ is replaced by $gu(r_{12})$. Hence, if the intensity of the repulsive potential increases, $g \to \infty$, the energy shift ΔE_n obtained due to the repulsion has a finite limit. Moreover, the greater the number n of an excited level, the more exact is the relation $\Delta E_n \to\sim aG_{(n)}(0) \sim a\left(\sqrt{\Lambda_0}\right)^{-n}$. Let ε_0 be the quantity of the order of the binding energy of the ground state of three particles for the critical two-particle constant. With regard for the energy $\sim -\varepsilon_0\Lambda_0^{-n}$ of the n-th level of the "zero" Hamiltonian near the critical constants (near the Efimov region), the total energy of the n-th level in the region of the "tube" looks as

$$E_n = E_n^{(0)} + \Delta E_n \approx -\varepsilon_0\Lambda_0^{-n} + B_0 a \Lambda_0^{-n/2}\,, \tag{25}$$

and $B_0 > 0$. Since $\Lambda_0 \gg 1$ (for $m \approx 1$), the energy E_n becomes positive with increase in n for an arbitrary small repulsion which is characterized by the scattering length a. This clarifies the above-mentioned conclusion about the finiteness of the number of levels under a

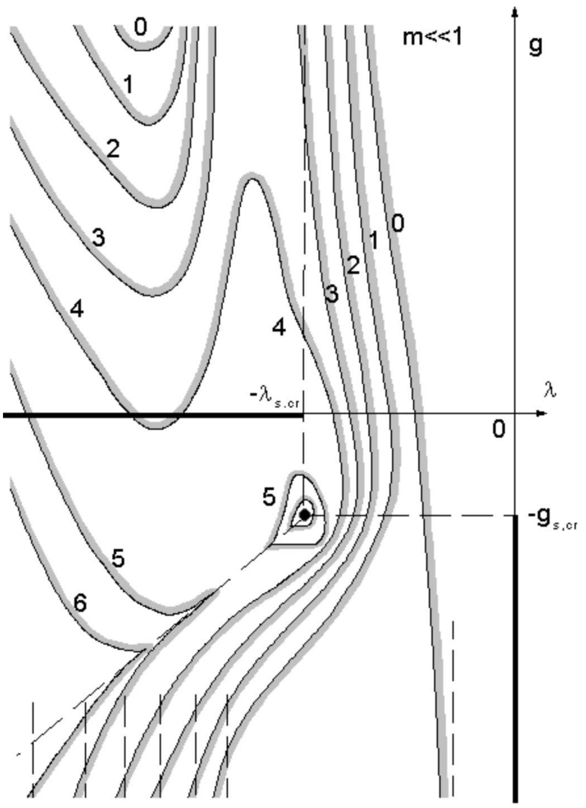

Fig. 3. Diagram of thresholds (schematically) in the case of a small mass of the third particle. The numbers numerate the states symmetric in the particles (12) permutations, the other designations are the same as in Fig. 1

deviation from the accumulation point of Efimov levels by the constant g. That is, for the infinite number of levels to exist, it is necessary that the interaction of all three pairs of particles be resonant, $g \to -g_{s,\mathrm{cr}}$ and $\lambda \to -\lambda_{s,\mathrm{cr}}$ simultaneously. All the more, the number of levels becomes finite, when the repulsion $gu(r_{12})$ reaches a significant intensity (in this case, the T-matrix is bounded, and the length a tends to a finite limit of the order of the radius of forces). At the same time, the arbitrariness of the repulsion intensity for the pair (12), the finiteness of the t-matrix, and the conditions for the resonance and hence the appearance of the effective long-range interaction for two pairs of particles do not contradict the possibility for the bound states of three particles to exist. The presence of at least one bound state of three particles in the region under study is confirmed by the numerical calculations for different forms of potentials and for different masses. We note that the "tube" (see Figs. 1 and 2) is asymmetrically positioned relative to the vertical asymptote $\lambda = -\lambda_{s,\mathrm{cr}}$ (the resonance region for two pairs of particles). To a great degree, it is positioned to the left from the asymptote, where the attractive constant λ is greater by modulus. But, with the further increase in the attractive constant λ by modulus, we come away from the resonance region. And at sufficiently large repulsive constants g, the three-particle level can disappear, arising again only in the limit of a strong coupling (see Figs. 1 and 2). We emphasize once more that all main qualitative conclusions of this section are confirmed by the numerical calculations.

The interesting universal phenomenon of a "wedge" (region VI in Fig. 1 and, respectively, in Figs. 3 and 4) is observed on the line where the binding energies of two different pairs of particles are equal, $\varepsilon_{12} = \varepsilon_{13}$ (on the dashed inclined lines). For the sake of specificity, we consider Fig. 3. As above, we have correctly preserved all the regularities, which are obtained in the calculations, though the scale of the figure is rather arbitrary. The effect of a "wedge" consists in the following. If we move along the dashed line (from the left to the right) from the region of a strong coupling, $|g| \approx |\lambda| \gg 1$, to the side where the attraction decreases, then the threshold line of the n-th symmetric state of three particles (in Fig. 3, they are the lines of the 6-th and 5-th excited states) is finished by a sharp cusp in the form of a "wedge" oriented from the left to the right. The lines of thresholds approach the dashed inclined line from the top and from the bottom with different slopes, and the derivative of the threshold line undergoes a break here. The "wedge" arises not due to the nonanalyticity, but as a consequence of the competition of thresholds. On the further motion to the right (within Fig. 3), firstly the 6-th excited state and then the 5-th one stop to exist. But with the decrease of the attraction along the dashed inclined line, where the energies of the two-particle thresholds are equal, the 5-th excited state appears again, while we approach the Efimov region. Its threshold line has the form of a "wedge" oriented from the right to the left. For higher excited states, the effect of a "wedge" holds analogously. Inside the closed region of the thresholds of Efimov states, the bulbous curves of thresholds have a cusp in the form of a "wedge" on the dashed inclined line. We note that, for antisymmetric states, no similar regularities are observed.

4. Dependence of the Three-Particle Thresholds on the Mass

Firstly, we consider the case of great values of the mass m of the third particle. If the mass is infinite (the three-

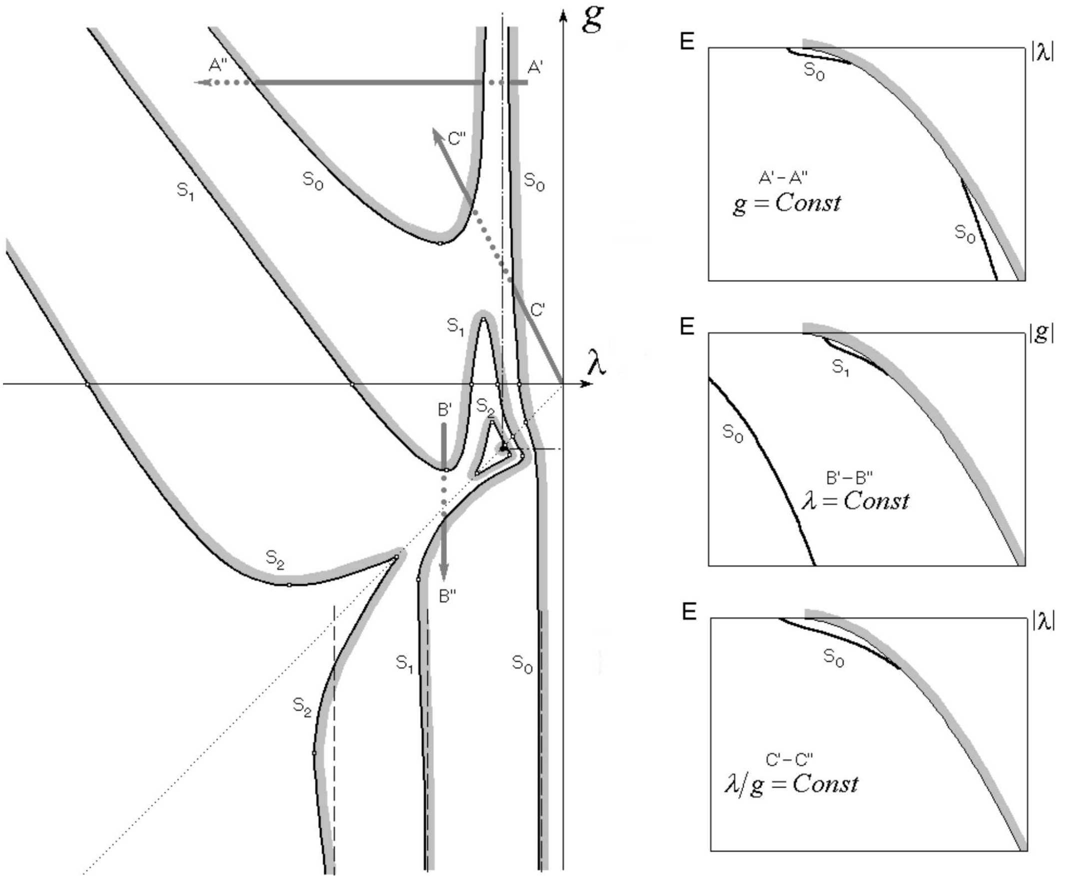

Fig. 4. To the left, the calculated diagram of the thresholds of stability (given on an arbitrary scale) is shown, the arrows at three different places indicate the direction of an increase in the attraction, the other notations are analogous to those in Fig. 1. To the right, the dependence of the energies of three particles on the strengthening of the attraction is shown: $A^{'} - A^{''}$ corresponds to the dependence of the ground state energy on the constant λ (at $g = \text{const}$) with the available "trap" in the resonance region, $B^{'} - B^{''}$ corresponds to the dependence of the energies of the ground and first excited levels on the constant $-g$ (at $\lambda = \text{const}$) with the available "trap" for the excited state, $C^{'} - C^{''}$ demonstrates the presence of a "trap" for the ground state (at $\lambda/g = \text{const}$)

particle model passes into a model with two particles in the short-range field of a fixed center), then it is obvious from (1) that, in the absence of the potential $V(r_{12})$, the problem of three particles is equivalent to that of two independent particles in the field of the attractive center $U(r_i)$. The ground state of such a system arises under the same conditions as those for each of the particles in the field of a center:

$$\lambda_{0,\text{cr}} = -\frac{1}{2} g^{(0)}_{s,\text{cr}} \,. \qquad (26)$$

In the general case of arbitrary masses, $\lambda_{0,\text{cr}} = -(1 + 1/m) g^{(0)}_{s,\text{cr}}/2)$, and, for the Gaussian potential with unit radius, we get $\lambda_{0,\text{cr}} = -1.342$. We emphasize that, for the infinite mass of the third particle in the case where $g = 0$, the Thomas theorem becomes trivial, and the three-particle state exists, by beginning exactly from the two-particle critical constant. For the excited states in the case of the infinite mass $m = \infty$, the couplings for three and two particles also appear at the same points on the λ axis. In addition, the energies monotonically increase with $|\lambda|$. Generally, for $m = \infty$, the thresholds of stability are transformed only slightly on the whole plane (λ, g) as compared with the case where $m \sim 1$, though, in the region of the "tube", there remains only the ground state which differs cardinally from the excited states for this reason. As $m = \infty$, the whole "tube" is positioned to the left from the asymptote $\lambda = -\lambda_{0,\text{cr}}$. Moreover, the right edge of the threshold of a three-particle state crosses the abscissa axis at the resonance point of two pairs of particles $\lambda = -\lambda_{s,\rm cr}$ and rises sharply (almost vertically) upward. In the resonance region of all three pairs of particles ($\lambda \to -g_{s,\rm cr}/2$, $g \to -g_{s,\rm cr}$), the conditions for the appearance of the infinite Efimov spectrum, which possesses the universal properties by beginning already from the second excited level, are realized in the standard way. In other regions in the diagram of the thresholds of stability in the case $m = \infty$, the general peculiarities seen in Fig. 1 are preserved as well.

We now consider the other limiting case of very small mass $m \to 0$ (a two-center model of three particles). In the region where $g > -g_{s,\rm cr}$, heavy particles (12) are not coupled in the absence of the third light particle, and $\lambda \geq -(1+1/m)g_{s,\rm cr}/2$, as well as in the case where there exists the coupling in the pair of heavy and light particles (pair (13) or (23)), we can adiabatically separate the variables. If we make averaging over the fast movement of the light particle (in the state φ), then we get the equation for the wave function of the coordinate of the relative motion of particles (12)

$$\{-\Delta_x + mg_{12}\exp(-mx^2) + W(x)\}\Phi(x) = \varepsilon_{(12)}\Phi(x) \quad (27)$$

(in the case of the Gaussian potentials with unit radii) with the additional potential

$$W(x) \equiv \int d\boldsymbol{\rho}\, |\varphi|^2 \left\{ \frac{(1+m)}{2} g \left(\exp\left(-\left(\boldsymbol{\rho} - \frac{\sqrt{m}}{2}\mathbf{x}\right)^2 \right) + \right.\right.$$
$$\left.\left. + \exp\left(-\left(\boldsymbol{\rho} + \frac{\sqrt{m}}{2}\mathbf{x}\right)^2 \right) \right) - \left(1 + \frac{m}{2}\right) g \exp\left(-\rho^2\right) \right\} \to$$
$$\to -m\,|g|\,C_0 + m\,|g|\,C_1 x^2 + \ldots\,, \quad (28)$$

where $\mathbf{x} \equiv (\mathbf{r}_2 - \mathbf{r}_1)/\sqrt{m}$, and all quantities of the dimension of energy are multiplied by m. Then, as $m \to 0$ and for small deviations from the position of equilibrium, we have the oscillatory potential ($C_1 > 0$). Since the oscillatory frequency $\omega_0 \sim \sqrt{m}$, the spectrum becomes denser for smaller masses. Hence, for constants $g_{s,\rm cr}/2 < |\lambda| < g_{s,\rm cr}/2m$ and small masses ($m \to 0$), we have the growing number of bound levels of the equidistant spectrum. Respectively, the thresholds for small masses will represent the equidistant spectrum in this region.

In the diagrams of the thresholds of stability (see Fig. 3), the indicated regularities are manifested, as $m \to 0$, in the enlargement of "islands" ("traps") of the infinite series of Efimov levels, the separation (due to the merging with the earlier isolated thresholds in the places of a "wedge") of the increasing number of the new lines of thresholds from the islands, the "pulling" of them into the region of the "tube", and the gradual filling of the whole region $|\lambda| > g_{s,\rm cr}/2$, $g > -g^{(0)}_{s,\rm cr}$. It is easy to trace the change of the general pattern in Fig. 3 with decrease in the mass of the third particle or, conversely, an increase in this mass. We can qualitatively explain the growth of the number of thresholds in the "tube" by the following reasoning. Because Λ_0 depends on the mass of particles [3] and tends to 1 with decrease in the mass of the third particle, energy (25) depends more and more weaker on the level number n. Therefore, the number of levels, for which energy (25) remains negative, increases. That is, the less the mass, the greater is the number of thresholds which correspond to excited levels and can be "pulled" into the "tube". This is completely confirmed by the high-precision calculations for small masses. In particular, the first excited level begins to move into the "tube" on the transition from great masses to $m \sim 1$ (Fig. 1). In the limit $m \to 0$, the number of levels in the "tube" tends to infinity, but, for any finite mass, the number of levels is finite (Fig. 3). In the general aspect, the less the mass of the third particle, the more the symmetric states become similar to the first excited one (for $m = 1$). This implies that, in a certain sense, the effects of Efimov and Thomas and the effect of a "tube" have many common features. On the contrary, we may expect that, in the case of the model of two particles in the field of a fixed center and under a variation of the forms of interaction potentials, all three effects can be significantly impoverished, so that they will have few common features.

5. Effect of "Traps", and Effects of "Rearrangement" of Energy Levels

The nonmonotonous change of the number of levels of the three-particle system with increase in the interaction constants in the region of attraction, where the levels appear and then disappear on a two-particle threshold with increase in the attraction (the effect of a "trap") is characteristic (see the diagrams of thresholds in Figs. 1 and 3) of a wide region of the interaction constants which can be positioned even sufficiently far from the Efimov region of the resonance interaction. If the general constant of attraction in the three-particle system grows so that g and λ are linearly related to each other, we can easily reveal a nonmonotonous character of the behavior (appearance and disappearance) of levels. That is, the effect of a "trap" has a sufficiently universal character. In Fig. 4, the lines with arrows show some directions of the coordinated increase in the intensities of the interaction potentials, where the three-particle levels appear and then disappear with increase in the interaction constants (the continuous and dotted lines correspond, respectively, to the absence and presence of the given bound state). In this figure, we present (schematically) the results of high-precision calculations with the Gaussian potentials of unit radii and $m = 1$ (but, for the convenience of a perception, we choose the scale to be arbitrary). The figure demonstrates the presence of a "trap" even in a very narrow interval of the constant λ near a critical value of the resonance interaction constant $\lambda \approx -g_{s,\mathrm{cr}}$, though this region is small. For the sake of convenience, we schematically present (to the right in Fig. 4) the dependence of the binding energies on the coupling constants with strengthening the attraction along the mentioned directions. In this case, we see the appearance of "traps" for the energy levels in both the ground and excited states near two-particle thresholds. In the general aspect, the effect of a "trap" for three-particle levels, which demonstrates the essentially different mode of the dependence of the ground-state energy of two particles and that of the three-particle levels on the interaction constants, is a consequence of the two-particle structure of the full interaction potential in the three-particle system.

Quite nontrivial is the possibility to vary the modes of behavior of the thresholds (and of the energy levels) with the help of the interaction potentials which involve at least two modes of attraction with essentially different radii. In this case, we can nontrivially realize a certain generalization of the well-known Zel'dovich two-particle effect of a "rearrangement" of the energy spectra in the system of three particles. For the pairwise interaction potentials with the components of different radii, for example, of the type

$$U(r) = V(r) = -g\left(\exp\left(-r^2\right) + b\exp\left(-\left(r/r_0\right)^2\right)\right) \quad (29)$$

(where $r_0 \gg 1$ and $b > 0$), one observes a "rearrangement" (Fig. 5) of the energy spectrum in the three-particle system (a three-particle analog of the Zel'dovich effect). Due to a great value of the ratio of the interaction radii of the two-component attractive potential ($r_0 \gg 1$), the spectrum of three particles is close to the superposition of the spectra of two problems involving three particles with the separately taken components of the interaction potential, and the quasidegeneration of the levels occurs. But, at the points of the expected intersection of levels of these two spectra, we observe the effects related to the "rearrangement" of levels. For

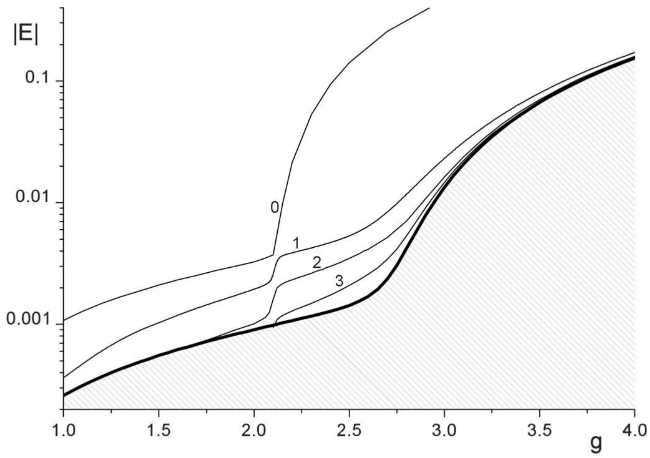

Fig. 5. Dependence of the binding energies of three particles on the coupling constant for the two-component potential with different radii (29); 0, 1, 2, 3 are the numbers of the ground and excited levels. The shaded region corresponds to the continuous spectrum for three particles

example, the first excited level replaces the ground-state level with increase in the total attraction (with increase in g). In its turn, the ground-state level sharply changes the mode of the dependence on the interaction constant. In Fig. 5, on the real scale, we present the results of calculations of a three-particle completely symmetric energy spectrum (the binding energies) as a function of the interaction constant g for a specific example of potential (29) ($m = 1$, $b = 0.001$, $r_0 = 100$). At the same interaction constant $g \approx 2.1$, the "rearrangement" of the spectrum of four levels occurs successively: n-th level replaces $(n+1)$-th level. The ground state level is the only one which, after the "rearrangement", corresponds to the binding energy with a quite different mode of the dependence on g. The effect of a "rearrangement" of the three-particle spectrum is much more pronounced than that for the two-particle threshold and occurs for a less attraction. By changing the class of the corresponding interaction potentials and their parameters, we can vary the number of levels in the region of a "rearrangement" and can also create a "trap" for other levels (see results in [7] for nonlocal separable potentials). The close conditions for a realization of the effects of "rearrangement" and "traps" for the energy states are known in solid state physics for a long time in the case where, for example, the conditions for a "quasidegeneration" of the phonon and optical branches of oscillations are realized, or the crystals with admixtures are considered.

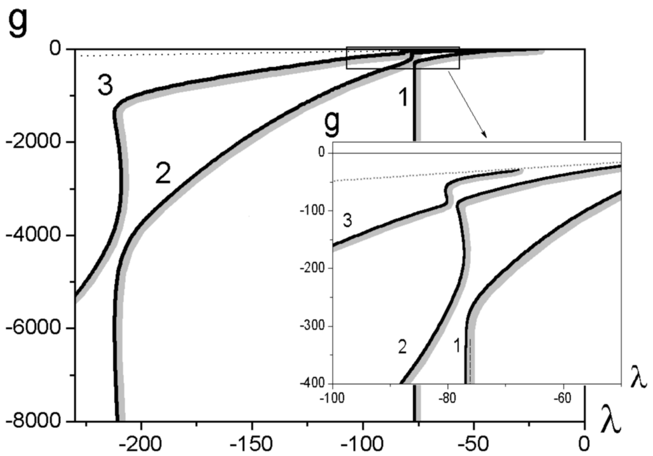

Fig. 6. Calculated diagram with the "rearrangement" of the lines of the thresholds of stability (on the real scale) for the second and third excited levels at $m = 0.06$ and for the Gaussian potential. The insert shows the "rearrangement" effect of thresholds on a greater scale. Dots show the line of the two-particle threshold

Even for simple potentials in the Gaussian form, the performed high-precision studies of the energy thresholds for three particles reveal one more effect of a sharp change of the modes in the diagrams of the thresholds of stability which can be also named the effect of a "rearrangement". This effect has no analog on the two-particle level. There exist the regions of parameters (in the λ axis, these are the regions close to the vertical asymptotes in Fig. 3), where the "rearrangement" of the adjacent energy thresholds occurs in the digram of thresholds. That is, on the monotonous change of the interaction intensities, the threshold of the $(n+1)$-th level sharply approaches that of the n-th one and occupies its place. This effect is especially clearly manifested in the case of small masses of the third particle (see Fig. 6, where the lower part of the diagram of the thresholds of three first excited levels calculated for the mass $m = 0.06$ is shown). We note that, in Fig. 3, this effect of the rearrangement of thresholds is not marked in order to avoid the complication of a perception of the general scheme of three-particle thresholds. The presence of the effect of a "rearrangement" is probably related to the following. In addition to the considerable short-range attractive interaction between particles (12) present in Hamiltonian (1), there appears an additional effective oscillatory interaction having a great radius at small masses of the third particle. This additional interaction between particles (12) is generated by the motion of the third light particle. Then, as $m \to 0$, the radius of the oscillatory well grows. As known [7], in the presence of two potentials with essentially different radii, there occurs the "rearrangement" of the energy spectrum on increasing the intensities of the potentials. In our case where the intensity of the short-range attraction becomes close to the critical constant of the appearance of the n-th level in this well, the lowest level of the other well falls into the well with less radius. On its previous place, the next level falls, etc. In this case, the spectrum for the well with a greater radius is practically renewed and is preserved, until the increase of the intensity of the short-range attraction induces the appearance of a new, $(n+1)$-th level, and the "rearrangement" repeats. As seen from Fig. 6, this region of the rearrangement of thresholds contains also "traps", because the lines of thresholds depending on g are not monotonous.

Finally, we note that, for the three-particle problem, the nonmonotonicity of the thresholds of stability is a general regularity, which is rather a rule than the exception. One can assume that the nonmonotonicity of thresholds in a three-particle system is a consequence of the difference of the modes of dependence of the ground-state energy of two particles and three-particle energies for the chosen, even simple, interaction potentials. Indeed, for simple attractive potentials in a two-particle system, one mode is always realized; we will say that this is one configuration. At the same time, for the system of three particles with pairwise interaction potentials and greater number of coordinates, various spatial configurations of three particles can be realized. This can induce a change of the modes of the dependence of three-particle energies and thresholds and, therefore, the nonmonotonous behavior of the number of levels of three particles in various regions of parameters of the interaction potentials. We may expect a more complicated behavior of the thresholds of stability in the problems with four and greater number of particles with pairwise interaction potentials.

6. Conclusions

In conclusion, we emphasize that we have systematically studied the energy thresholds of stability for the general quantum problem of three particles interacting by short-range potentials. Analytically and by the high-precision calculations of the energy spectra of three particles, we have established the general nonmonotonic dependence of the thresholds of stability on the strengthening (or the weakening) of the attraction between particles. As a result, we have revealed a number of new nontrivial universal effects in the behavior of the three-particle thresholds of stability. In the first turn, we mark the effect of a "trap", when the number of three-particle bound states nonmonotonically depends on the strengthening of the interaction between particles. Secondly, we have discovered the effect of a "rearrangement" of the energy levels and the thresholds of stability, which has no analog in the two-particle systems. At the same time, we have constructed a three-particle generalization of the Zel'dovich effect of a "rearrangement" of the energy levels for the attractive interaction potentials with the modes of attraction which essentially differ by their interaction radii. Thirdly, we have revealed both the effect of a "tube" for the energy states in the region of a significant repulsion between a pair of identical particles and the effect of a "wedge" on the boundary, where the binding energies of two different pairs of particles coincide. We have made a sufficiently full and clear analysis of the dependences of the thresholds of stability on the masses of particles and the form of a short-range interaction, and have also considered different symmetries of states.

The revealed regularities are important for the comprehension of the general properties of three-particle systems of different nature. In addition, the possibilities to describe the characteristic peculiarities of thresholds in three-particle systems with the Coulomb interaction, as well as those of the states with nonzero angular momenta, are open. These problems will be analyzed in further publications. Challenging is also the necessity to study the thresholds of stability for one- and two-dimensional systems, where one can expect significant differences in properties from those in three-dimensional problems.

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ЕНЕРГЕТИЧНІ ПОРОГИ СТАБІЛЬНОСТІ ТРИЧАСТИНКОВИХ СИСТЕМ

І.В. Сименог, Ю.М. Бідасюк, Б.Є. Гринюк, М.В. Кузьменко

Р е з ю м е

Вивчено загальні властивості енергетичних порогів стабільності системи трьох частинок із короткодійною взаємодією. Розглянуто широку область констант взаємодії та різні співвідношення мас частинок. Виявлено специфічні ефекти, характерні для біляпорогових стаціонарних енергетичних рівнів трьох частинок. Для порогів отримано асимптотичні оцінки в певних граничних випадках. Виконано прецизійні варіаційні розрахунки порогів при різних значеннях констант взаємодії та мас частинок.