CLUSTER FORMATION IN A SYSTEM OF GRAVITATING PARTICLES

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Based on a field theory approach to the statistical description of a system of gravitationally interacting particles, the spatially inhomogeneous distribution, a cluster, is described in the Boltzmann limit. The conditions for cluster formation are obtained at a fixed mean density in two cases: for the infinitely large system (in the limit $N \to \infty$, $V \to \infty$ but N/V is fixed) and a large but finite one. It is proved to be that, in the above-mentioned cases, the behavior of the system may be different at the same temperature. The nature of these differences is disclosed using a simple clear consideration.

The formation of a spatially inhomogeneous distribution of interacting particles is a typical problem in condensed matter physics. The conditions for the formation of such structure are determined by the type of interaction [1,2]. For a gas with the gravitational interaction between particles (self-gravitating system), we cannot calculate the virial coefficients for the potential of particle's interaction $1/r^n$ if $n \leq 3$, and such a gas suffers the thermodynamical catastrophe — collapse [16]. As a collapse, we understand the transition of the state with spatially homogeneous distribution to the state with spatially inhomogeneous distribution, namely clusters of finite sizes.

In [1–3], a new approach to the statistical description of interacting particles and phase transitions accompanied by the formation of clusters was proposed. This formation is described by the function of a spatial distribution of particles. This function is the soliton solution of the nonlinear equation which arises in most cases of the statistical description of interacting particles [1]. The original statistical description of a self-gravitating system developed in [1, 3] is based on the application of the apparatus of quantum field theory [4–12], where it was shown that a gravitating gas is equivalent to a single scalar field $\varphi(r)$ with exponential self-action [18], and gives the possibility to find the spatial distribution of particles and to calculate the cluster's size.

In [1, 3], the self-gravitating system was considered in the limit $N \to \infty$, $V \to \infty$, but N/V is fixed. The results the aforesaid paper yield that a collapse happens at any temperature and concentration. However, it is known that real systems have finite volume and finite number of particles. This fact must be accounted in the equation of state, but this brings to a small correction in the equation of state for the systems with short-range interaction between particles [15]. In the case of the systems with long-range interaction (for example, the gravitational one), the situation must be changed cardinally, because the length of such an interaction is equal or bigger than the size of the system. In other words, the finiteness of a system with long-range interaction must lead to nontrivial effects.

In view of the aforesaid, our problem is, by based on the statistical approach [1–3], to describe the formation of a spatially inhomogeneous distribution in a self-gravitating gas, to investigate the differences in its behavior in the case of the infinitely large system ($N, V \to \infty$) and in the case of a large but finite one, and to obtain the conditions for the cluster formation in such a system.

Let's consider the interacting particles' system under conditions when, on the one hand, the wave's thermal length of a particle can be larger than the average distance between them, so that it is necessary to take into account the type of a statistics, but, on the other hand, this length is by far smaller than the average scattering length, which allows one to describe the interaction classically disregarding dynamical quantum correlations. The Hamiltonian of such a system [1–3, 12, 13] is

$$H(n) = \sum_s \varepsilon_s n_s - \frac{1}{2}\sum_{ss'} W_{ss'} n_s n_{s'} + \frac{1}{2}\sum_{ss'} U_{ss'} n_s n_{s'}\,, \quad (1)$$

where ε_s is the additive part of the particle energy in the state s (for example, the kinetic energy or energy in an external field), $W_{ss'}$ and $U_{ss'}$ are the absolute values of the attraction and repulsion energies of particles in the states s and s', respectively. The macroscopic state of the system is determined by occupations numbers n_s. The subscript s corresponds to the variables describing an individual particle state.

In [1, 2] in order to investigate the thermodynamical properties of the interacting particles' system, the Hubbard—Stratonovich [4–7] representation has been used for the partition function

$$Z_n = \frac{1}{2\pi i}\oint d\xi \int D\varphi \int D\psi \exp\left(-S\left(\xi,\varphi,\psi\right)\right)\,, \quad (2)$$

where S is a functional we call an action by analogy with field theory:

$$S\left(\xi,\varphi,\psi\right) = \frac{1}{2\beta}\sum_{s,\acute{s}}\left(W^{-1}_{s,s'}\varphi_s\varphi_{s'} + U^{-1}_{s,s'}\psi_s\psi_{s'}\right) +$$
$$+\delta\sum_s \ln\left(1-\delta\xi\exp\left(-\beta\varepsilon_s+\varphi_s\right)\cos\psi_s\right) +$$
$$+\left(N+1\right)\ln\xi\,, \quad (3)$$

ξ is the activity, $\beta = 1/kT$ is the inverse temperature, s and $s^{'}$ run all the states of the system, ε is the kinetic energy, N is the number of particles, and $\delta = +1$ for Bose particles and -1 for Fermi particles. Two auxiliary fields φ and ψ corresponding to attraction and repulsion are introduced. The partition function (2) is written as a functional integral over these fields. $W^{-1}_{s,s'}, U^{-1}_{s,s'}$ are the inverse operators of the interaction: $\omega^{-1}_{ss'} = \delta_{ss'}\hat{L}_{s'}$, where $\hat{L}_{s'}$ is such an operator, for which the interaction's potential is a Green function.

Integral (2) is calculated by the method of "saddle point" [14, 15] across the point determined by the functional derivatives $\frac{\delta S}{\delta\varphi} = 0$, $\frac{\delta S}{\delta\psi} = 0$ as

$$\frac{1}{\beta}\sum_{s'} W^{-1}_{ss'}\varphi_{s'} - \frac{\xi_s e^{\varphi_s}\cos\psi_s}{1-\delta\xi_s e^{\varphi_s}\cos\psi_s} = 0, \quad (4)$$

$$\frac{1}{\beta}\sum_{s'} U^{-1}_{ss'}\psi_{s'} + \frac{\xi_s e^{\varphi_s}\sin\psi_s}{1-\delta\xi_s e^{\varphi_s}\cos\psi_s} = 0, \quad (5)$$

and the derivative

$$\frac{\partial S}{\partial\xi} = 0 \Rightarrow \sum_s \frac{\xi_s e^{\varphi_s}\cos\psi_s}{1-\delta\xi_s e^{\varphi_s}\cos\psi_s} = N+1\,. \quad (6)$$

These equations provide a solution of the multi-particle problem in the sense that it selects those states of the system, whose contributions in the partition function are dominant.

In the continuum approximation, the subscript s runs through a continuum of values in the system of volume V. When integrating over impulses and coordinates, we bear in mind that the unit of cell's volume in the space of individual states is equal to $\omega = (2\pi\hbar)^3$.

We now consider the system of particles interacting by the gravitational attraction only. For the Newtonian attraction, the inverse operator is known to be $W^{-1}_{rr'} = \frac{-1}{4\pi G m^2}\Delta_r\delta_{rr'}$, where G is the gravitational constant, m is a particle's mass, and Δ_r is the Laplace operator.

Let's consider the system in the usual thermodynamical limit: the number of particles $N \to \infty$ and the volume $V \to \infty$ with N/V fixed. Then, going to Boltzmann limit ($\xi \to 0$) and neglecting the cluster surface contribution (the term $\frac{2}{r}\frac{\partial\varphi}{\partial r}$ can be omitted), we can write action (3) as

$$S = \int_0^V \left\{\frac{1}{2\beta}\frac{(\nabla\varphi)^2}{4\pi G m^2} - \frac{\xi}{\lambda^3}e^{\varphi}\right\}dV + N\ln\xi. \quad (7)$$

It is easy to notice that the theory of a self-interacting gas in thermal equilibrium is equivalent to the field theory of a single scalar field $\varphi(r)$ with exponential self-action. An expression analogous to Eq. (7) was obtained in [18]. However, the term $N\ln\xi$ which fixes the number of particles was absent there.

Let's introduce the dimensionless quantity $r = R/r_m$ instead of R and a new variable $\sigma = \exp\left(\varphi/2\right)$, and let us set $\alpha^2 \equiv r_m^3/\lambda^3$ and $r_m = 2\pi G m^2\beta$. Then action (7) (in spherical coordinates) can be written as

$$S = 4\pi\int_0^\infty \left[\left(\frac{1}{\sigma}\frac{\partial\sigma}{\partial r}\right)^2 - \alpha^2\xi\sigma^2\right] r^2 dr + N\ln\xi\,. \quad (8)$$

The sense of the additional field φ or its equivalent σ is the following: the spatial distribution (a density) can be expressed as

$$\rho\left(r\right) = m\frac{\xi}{\lambda^3}e^{\varphi} \equiv m\frac{\xi}{\lambda^3}\sigma^2\,, \quad (9)$$

with a help of Eq. (6). The equation for "saddle point" is the equation of Lagrange for functional (8). Here, the term $\frac{2}{r}\frac{d\sigma}{dr}$ can be omitted [8] and we have

$$\frac{\partial^2\sigma}{\partial r^2} - \frac{1}{\sigma}\left(\frac{\partial\sigma}{\partial r}\right)^2 + \xi\alpha^2\sigma^3 = 0\,. \qquad (10)$$

This equation has a soliton solution [1,3]

$$\sigma = \frac{\Delta}{\sqrt{\xi}\alpha}\frac{1}{\cosh\Delta r}\,, \qquad (11)$$

where Δ is an integration constant. Any soliton solution corresponds to a spatially inhomogeneous distribution of particles — a finite-size cluster. The corresponding asymptotics are $\sigma^2 = 1$ for $r = d$, where d is the cluster size, and $\sigma \longrightarrow 0$ as $r \longrightarrow \infty$. This solution describes the presence of particles in the inhomogeneous formation of size d and the absence of particles at infinity. This spatial distribution is compressing to the line $\sigma = 1 \Rightarrow \varphi = 0$ as far as $T \to \infty$, $\frac{N}{V} \to 0$. That's why the field $\varphi = 0$ corresponds to a spatially homogeneous distribution in the self-gravitating gas in the aforesaid limit case.

Let's substitute solution (11) into action (8):

$$S = 4\pi\int_0^d \left(\Delta^2 - 2\xi\alpha^2\sigma^2\right) r^2 dr + N\ln\xi\,. \qquad (12)$$

Then we'll integrate by using the decomposition $1/\cosh x \approx 1 - x^2/2$ in a power series of $x \equiv \Delta d \ll 1$:

$$S = -V\frac{\Delta^2}{\alpha^2\lambda^3} + N\ln\xi\,. \qquad (13)$$

Δ^2 is found from the asymptotics, that is, $1 = \frac{\Delta^2}{\xi\alpha^2}\left[1 - \Delta^2 d^2\right] \Longrightarrow \Delta^2 \approx \xi\alpha^2 + \xi^2 d^2\alpha^4$. Thus, we have the result

$$S = -\frac{V}{\lambda^3}\xi + N\ln\xi - \frac{V}{\lambda^3}\xi^2 d^2\alpha^2\,. \qquad (14)$$

Assuming that the average energy of the gravitational interaction of two particles is less than the average kinetic energy $\sim kT$ of a particle, i.e. $r_m^3 N/V \ll 1$ and $\lambda^3 N/V \ll 1$, we can find the activity ξ from the equation $\frac{\partial S}{\partial\xi} = 0$ as

$$\xi \approx \frac{\lambda^3 N}{V} - \frac{2d^2\alpha^2\lambda^6 N^2}{V^2} \equiv \xi_0 + \xi_G\,,\ \ \xi_0 \gg |\xi_G|\,, \qquad (15)$$

where ξ_0 and ξ_G are the activities of the ideal gas and the first correction for gravitation, accordingly. Then, by integrating (2) at the "saddle point" (15), we obtain the partition function as

$$Z_N = Z_N^0 \exp\left[\frac{V}{\lambda^3}\xi_G - N\ln\left(1+\frac{\xi_G}{\xi_0}\right)+\frac{V}{\lambda^3}\xi^2 d^2\alpha^2\right]\,, \qquad (16)$$

where Z_N^0 is the partition function of the ideal gas. Knowing it, we can find the free energy of the system

$$F = F_0 - kT\left[\frac{V}{\lambda^3}\xi_G - N\ln\left(1 + \frac{\xi_G}{\xi_0}\right) + \frac{V}{\lambda^3}\xi^2 d^2\alpha^2\right]\,, \qquad (17)$$

where F_0 is the free energy of the ideal gas. Minimizing (17) by the size of a cluster $d = D/r_m$, when

$$\frac{\partial F}{\partial d} = -kT\frac{d\alpha^2\lambda^3 2N^2}{V} \times \left[1 - 4\frac{d^2\alpha^2\lambda^3 N}{V}\right] = 0\,, \qquad (18)$$

we obtain the optimum radius of the cluster,

$$d_0^2 = \frac{V}{4Nr_m^3} \qquad (19)$$

or, in the dimension values,

$$D_0^2 = \frac{1}{4}\frac{kT}{2\pi G m^2}\frac{V}{N}\,. \qquad (20)$$

This expression means that the equilibrium size of the cluster is defined by balance of two forces. The first force is the gravitational one which aspires to compress the gas. It is represented by the multiplier $m^2 G$ (in sense of the interaction constant). A decrease of the cluster's size with increase in the mean density in the system N/V is related to a closer packing of the particles in the cluster due to the increase of the gravitational energy. The second force is the thermal one, which creates a positive pressure resisting to the gravitational compression. It is presented by the multiplier kT. Such a situation is realized due to the long-range attraction ($\sim 1/R$) of the gravitational interaction.

On the other hand, Eq. 20 can be understood as such characteristic distance in the system, at which the essential deflection from the mean fixed density N/V is observed. Then this expression may be understood as the Jeans length [17].

Since the number of particles in the system $N \to \infty$, but the number of particles in a cluster is finite, this means that our system disintegrates to the infinite number of clusters of the size D_0 each. It is a self-gravitating system too and can undergo a collapse. Then the process repeats again. In others words, the free energy of such a system has no absolute minimum, and each state of the self-gravitating system is analogous to the false vacuum in field theory [8].

Equation (10) has a soliton solution (11) under any thermodynamical conditions $\alpha^2\xi \equiv r_m^3 N/V$ when $N \to \infty$, $V \to \infty$, but N/V is fixed. This means that the gas collapses under any initial conditions, and the collapse does not happen at the infinitely high temperature only. However, it is known that real systems have finite volume and a finite number of particles. In the case of systems with the gravitation interaction between particles, the situation must be changed cardinally, because the range of such an interaction is bigger than the size of the system.

Let the number of particles N be very big, but finite, and let the average density N/V be fixed. Then we can write a condition of normalization for the spatial distribution function (9) as

$$\int_0^V \rho(r)d^3r = \frac{mN}{r_m^3} \text{ or } \int_0^V \sigma^2 d^3r = \frac{V}{r_m^3}. \quad (21)$$

This equation can be regarded as a condition imposed on the solution of Eq. (10). We will find a field σ which minimizes action (8) under condition (21). Then we use the Lagrange method of multipliers. The multiplier χ can be found by solving the system of three equations: two equations are determined by the asymptotics of solution (11), and the third is the condition of normalization (21). However, we can go by a simpler way.

Let's compare two actions *at the saddle point* $\widetilde{\xi}$. One of them is the action of a collapsed gas. It can be obtained by using the condition of normalization (21) for action (12):

$$S_d = 4\pi \int_0^d \left(\Delta^2 - 2\xi\alpha^2\sigma^2\right) r^2 dr + N\ln\xi =$$

$$= \Delta^2 \frac{V}{r_m^3} - 2\xi\alpha^2 \frac{V}{r_m^3} + N\ln\xi. \quad (22)$$

The second one corresponds to a spatially homogeneous distribution with $\varphi \equiv 0$:

$$S_\infty = -\frac{V}{\lambda^3}\xi + N\ln\xi. \quad (23)$$

The saddle points is determined for both actions by the equations (we used Eq. (15) for $\widetilde{\xi}_d$):

$$\frac{\partial S_d}{\partial \xi} = 0 \Rightarrow \widetilde{\xi}_d = \frac{\lambda^3 N}{2V}, \quad (24)$$

$$\frac{\partial S_\infty}{\partial \xi} = 0 \Rightarrow \widetilde{\xi}_\infty = \frac{\lambda^3 N}{V}. \quad (25)$$

It is seen that the thermodynamical conditions determine the saddle point for both actions (22) and (23). In the case of a collapse, the action for the collapsed gas must be less than the action for a spatially homogeneous distribution at the saddle point:

$$S_d(\widetilde{\xi}_d) \le S_\infty(\widetilde{\xi}_\infty)$$
$$\Downarrow$$
$$\Delta^2 \tfrac{V}{r_m^3} - 2\widetilde{\xi}_d\alpha^2 \tfrac{V}{r_m^3} + N\ln\widetilde{\xi}_d \le -\tfrac{V}{\lambda^3}\widetilde{\xi}_\infty + N\ln\widetilde{\xi}_\infty. \quad (26)$$

Here, the equality is reached at the point of the collapse. Inequality (26) can be reduced to the form

$$\frac{r_m^3 N}{V\Delta^2(2+\ln 2)} \ge 1. \quad (27)$$

Let's integrate Eq.(11) over all space using the condition of normalization (21). Using the equality $\int_0^\infty \frac{x^2dx}{\cosh^2 ax} = \frac{\pi^2}{12a^3}$, we find $\Delta = \frac{\pi^3}{3N}$. Then, at temperatures or concentrations, when the inequality

$$\frac{r_m^3 N^3}{V} \ge \left(\frac{\pi^3}{3}\right)^2 (2+\ln 2) \quad (28)$$

is satisfied, the gravitating gas is in the collapsed state. The same parameter figures in works [19, 20] and coincides with the Jeans criterion of the gravitational instability [17]. Moreover, inequality (28) may be understood as a criterion for the realization of solution (11) with parameter (20).

We can see that the finiteness of the volume and the number of particles implies that if the thermodynamical conditions doesn't satisfy inequality (28), then cluster can not be formed. On the contrary, in the limit $N \to \infty$ and $V \to \infty$, the collapse happens always, because condition (28) in this case is satisfied always (if N/V is fixed).

This phenomenon can be further clarified by the following simple consideration. Let the system be a gravitating gas of the mass mN and occupy the volume V. This gas creates the gravitational field $\Delta\phi = 4\pi G\rho$, where $\rho \simeq mN/V$ and $V \sim R_{\max}^3$. If we take such a calibration that the field $\phi = 0$ on the boundary of the system, then the energy of a particle at the center of the system (the depth of the potential well) is

$$U_0 = m\phi \sim -mG\rho R_{\max}^2. \quad (29)$$

On the other hand, the thermal energy of a particle is $\sim kT$. If it is less than its maximal gravitational energy (at the center of the system), then the gas has negative pressure and aspires to compress to a point. On the contrary, if the thermal energy of a particle is more than its maximal gravitational energy (by modulus), then the gas has positive pressure and aspires to expand to infinity. At a compression of the gas, its thermal energy grows, because the full energy of the system is constant.

This leads to the establishment of an equilibrium which determines the spatial distribution function (11).

The point of a collapse (the values of temperature and concentration when a cluster can be formed) is determined from the equality of the above-mentioned energies. That is, we obtain

$$mG\rho R_{\max}^2 \approx kT \Rightarrow \frac{Gm^2N}{kTR_{\max}} \sim 1\,. \tag{30}$$

If we raise this equation to the third power, then it will coincide with Eq. (28) to within a constant. If Eq. (20) is interpreted as the Jeans length, then condition (28) can be obtained as $V \geq D_0^3$.

If $N \to \infty$ and $V \to \infty$, then $U_0 \to \infty$. This means that the thermal energy is less than the gravitational energy at the center of the system always (in this case, any point can be chosen as the center of the system). Hence, such self-gravitating gas collapses at any temperature.

Thus, based on the new method given in [1, 2], we have studied the properties of the model system of a self-gravitating gas with fixed mean density N/V in two cases: the infinitely large system and the large but finite one. The homogeneous spatial distribution in the system of gravitating particles is unstable in the first case. A self-gravitating gas collapses in a multitude of dense formations, namely clusters, at any temperature and a fixed mean density. The size of a cluster is determined by balance of the gravitational and thermal energies. The situation changes in a case of a large but finite size of the system and a large but finite number of particles in it. For a cluster top exist, the temperature and the mean concentration in the system must satisfy inequality (28). Such a behavior of the self-gravitating system is explained by the long-range attractive nature of the gravitational interaction.

We hope that the results of this study will present a certain interest for solving the problems of astrophysics, e.g., the problem of the formation of dense structures (planet-giants, stars, etc.) from a gas-dust matter and their accumulation, in particular.

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КЛАСТЕРОУТВОРЕННЯ У СИСТЕМІ ГРАВІТАЦІЙНО ВЗАЄМОДІЮЧИХ ЧАСТИНОК

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Р е з ю м е

Ґрунтуючись на польовому підході до статистичного опису систем гравітаційно взаємодіючих частинок, описано просторово неоднорідний розподіл — кластер у больцманівській границі. Отримано умови для кластероутворення при фіксованій середній густині у двох випадках: у нескінченно великій системі (у границі $N \to \infty$ та $V \to \infty$, але при фіксованому N/V) та у великій, але скінченній системі. Показано, що поведінка системи у цих випадках різна при одній і тій самій температурі. З'ясовано природу цих відмінностей шляхом простого та прозорого розгляду.