EQUIVALENCE OF THE REPRESENTATION WITH THE USE OF THE ISOSPIN FORMALISM AND THE REPRESENTATION WITHOUT ISOSPIN FOR A THREE-NUCLEON SYSTEM

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For the systems of three nucleons, (2p,n) and (2n,p), in the doublet state in spin, it is established that, respectively, from six and four spatial components of wave functions in the standard form with the use of the isospin formalism, only two components are independent. The formulas for the construction of the full antisymmetric wave functions in terms of two independent components are obtained. The systems of two equations for independent spatial components in the doublet state of nuclei ^{3}He and ^{3}H and one equation for the quartet state are formulated. It is shown that the physical characteristics, which are calculated in the representation with the use of the isospin formalism and in the representation without isospin, coincide. The perspective of the new approach for the execution of precise studies of few-nucleon systems is discussed.

1. Introduction

The advances recently reached in the precise studies of quantum systems composed of few particles of different nature [1—3] stimulate the development of the methods of high-accuracy calculations of the characteristics of few-nucleon nuclei directly on the basis of potentials of the two-particle interaction between nucleons without using the simplified model ideas of nuclei. The high accuracy of values of the energies of bound states of nucleons in nuclei, which was attained in variational calculations, allows one to simultaneously get the practically exact wave functions which correspond to the Hamiltonian with a given two-particle interaction. Nuclear characteristics calculated on the basis of such wave functions (WFs) can be considered as "standard" ones, with which the corresponding values obtained by various approximate methods should be compared. At the same time, the comparison of the wide collection of characteristics calculated for different nuclei on the basis of the same interaction with experimental data allows one to critically estimate the applied standard model potentials and to search for ways of their improvement.

The precise studies with high controlled accuracy have the own specificity and certain difficulties on the way of their practical realization [2—5]. In the studies of atomic nuclei, the additional difficulties appear due to a complicated spin-isospin structure of nuclear potentials (and hence, of WFs), as well as due to the presence of the Coulomb interaction that should be obligatorily taken into account in high-precision calculations. On such a structure of potentials, the energy states of nuclei are described by many-component WFs (vectors), each component of which being a function of spatial coordinates of all the nucleons of nuclei. For example, the WFs of a nucleus ^{3}He have six components on the use of central exchanged nuclear potentials with regard for the Coulomb interaction between protons, and, hence, the Schrödinger equations involving six internal coordinates are reduced to a system of six coupled differential equations of the second order relative to spatial components of the wave functions (SC WF). As the number of nucleons in nuclear systems increases, the numbers of SC WF and the equations increase rapidly in the general case. For a nucleus ^{4}He, it

is necessary already to solve the system of twelve equations. With regard for the fact that each SC WF in variational calculations (and many other approaches) is a superposition of a great number of basis functions, it becomes obvious that the derivation of reliable high-precision results will meet dufficulties with increase in the number of particles. We note that, as was indicated on the initial stage of the introduction of the isospin formalism for protons and neutrons [6], the generalized Pauli principle for proton-neutron systems is not an additional fundamental position, but only one of the convenient methods for a certain circle of problems. It is worth to note that the use of the isospin variable leads to the additional symmetrization of WF which increases significantly the number of its spatial components and, as a result, complicates essentially the calculations with regard for all the possible components [5].

In the present work, by the example of three-nucleon nuclei ^{3}He and ^{3}H, we develop an approach, in which the determination of all SC WF is reduced to the solution of a system of only two equations. In this case, it is clarified that the additional antisymmetrization is related to the formal recognition of the identity of a proton and a neutron and does not change the number of independent spatial components, i.e., the numbers of independent components in the isospin formalism and in our approach without isospin coincide. We have studied the equivalence of the formulated systems of equations for independent components and the equations in the representation without isospin and have established also the coincidence of the physical averages in the representation without isospin and those in the isospin formalism.

2. Systems of Equations for Three Nucleons in the Isospin Formalism

We formulate the standard system of equations for SC WF of the system of three nucleons, (2p, n), in the doublet spin state (a nucleus ^{3}He in the state with spin $S = 1/2$) in the isospin formalism under the assumption about the central character of the exchange pairwise interaction between nucleons as

$$V(i,j) = V_{11}(r)P_s(\sigma)P_s(\tau) + V_{13}(r)P_s(\sigma)P_t(\tau) +$$

$$+V_{31}(r)P_t(\sigma)P_s(\tau) + V_{33}(r)P_t(\sigma)P_t(\tau) + \frac{e^2}{r}P_iP_j, \quad (1)$$

where $r = |\vec{r}_i - \vec{r}_j|$. The nuclear nucleon-nucleon potential includes the standard operators of projection on the singlet s and triplet t states of two nucleons in the spin σ and isospin τ spaces, and the Coulomb potential does the operators of projection on the proton isospin states.

For the description of the spin-isospin structure of WF, we will use the standard notations (see [7,8]). We denote the spin states of the individual i-th nucleon with $S_z = +1/2$ and $S_z = -1/2$ as α and β, and the proton and neutron isospin states as p and n. The three-nucleon (2p, n) spin functions ζ in the state $S = 1/2$, $S_z = 1/2$ and the isospin functions χ in the state $T = 1/2$, $T_z = 1/2$ are usually represented in the following form (the Young scheme [21]):

$$\begin{cases} \zeta^{'} = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3, \\ \\ \zeta^{''} = \frac{1}{\sqrt{6}}[2\alpha_1\alpha_2\beta_3 - (\alpha_1\beta_2 + \alpha_2\beta_1)\alpha_3], \end{cases}$$

$$\begin{cases} \chi^{'} = \frac{1}{\sqrt{2}}(p_1n_2 - p_2n_1)p_3, \\ \\ \chi^{''} = \frac{1}{\sqrt{6}}[2p_1p_2n_3 - (p_1n_2 + p_2n_1)p_3]. \end{cases} \quad (2)$$

The Coulomb interaction between protons in the system (2p, n) mixes the states with different values of the total isospin, $T = 1/2$ and $T = 3/2$. Therefore, in order to construct the full WF, we need else the symmetric combination of products of one-particle isospin functions (the Young scheme [3])

$$\chi^s = \frac{1}{\sqrt{3}}(p_1p_2n_3 + p_1n_2p_3 + n_1p_2p_3), \quad (3)$$

which corresponds to the total isospin $T = 3/2$ and the projection $T_3 = 1/2$. Then the spin-isospin functions with a certain symmetry relative to permutations have the form

$$\begin{array}{ll} \xi^s = \frac{1}{\sqrt{2}}\left(\zeta^{'}\chi^{'} + \zeta^{''}\chi^{''}\right), & \xi^a = \frac{1}{\sqrt{2}}\left(\zeta^{'}\chi^{''} - \zeta^{''}\chi^{'}\right), \\ \xi^{'} = \frac{1}{\sqrt{2}}\left(\zeta^{'}\chi^{''} + \zeta^{''}\chi^{'}\right), & \xi^{''} = \frac{1}{\sqrt{2}}\left(\zeta^{'}\chi^{'} - \zeta^{''}\chi^{''}\right), \\ \zeta^{'}\chi^s, & \zeta^{''}\chi^s, \end{array} \quad (4)$$

where ξ^s (the Young scheme [3]) and ξ^a (the Young scheme [1^3]) are, respectively, symmetric and antisymmetric relative to permutations of a pair of nucleons. The functions ξ' and ξ'', as well as $\zeta'\chi^s$ and $\zeta''\chi^s$, are transformed on permutations by the two-dimensional representation of the Young scheme [21] just so as ζ' and ζ''.

In the commonly accepted standard notations, the antisymmetric WF of a nucleus ^{3}He in the state with spin $S = 1/2$ looks as

$$\Psi^a_{1/2} = \psi^s\xi^a + (\psi'\xi'' - \psi''\xi') + \psi^a\xi^s + (\varphi'\zeta'' - \varphi''\zeta')\chi^s, \quad (5)$$

where ψ^s and ψ^a are, respectively, the symmetric and antisymmetric spatial components, and ψ', ψ'' and φ', φ'' are SC WF of the Young scheme [21]. After the subsequent projection of the Schrödinger equation with WF (5) onto the spin-isospin basis (4), we get the system of six differential equations for SC WF:

$$\left[\hat{K} - E + \frac{1}{2}\left(U^s_{31} + U^s_{13}\right) + \frac{1}{3}U^s_c\right]\psi^s -$$

$$-\frac{1}{2}\left[\left(U'_{31} - U'_{13}\right)\psi' + \left(U''_{31} - U''_{13}\right)\psi''\right] +$$

$$+\frac{1}{3}\left(U'_c\psi' + U''_c\psi''\right) + \frac{1}{3}\left(U'_c\varphi' + U''_c\varphi''\right) = 0,$$

$$\left[\hat{K} - E + \frac{1}{4}\left(U^s_{33} + U^s_{31} + U^s_{13} + U^s_{11}\right) + \frac{1}{3}U^s_c\right]\psi' -$$

$$-\frac{1}{4}\left[\left(U'_{33} - U'_{31} - U'_{13} + U'_{11}\right)\psi'' +\right.$$

$$\left. + \left(U''_{33} - U''_{31} - U''_{13} + U''_{11}\right)\psi'\right] -$$

$$-\frac{1}{2}\left(U'_{31} - U'_{13} - \frac{2}{3}U'_c\right)\psi^s +$$

$$+\frac{1}{2}\left(U''_{33} - U''_{11} + \frac{2}{3}U''_c\right)\psi^a + \frac{1}{3}\left(U''_c\varphi' + U'_c\varphi''\right) = 0,$$

$$\left[\hat{K} - E + \frac{1}{4}\left(U^s_{33} + U^s_{31} + U^s_{13} + U^s_{11}\right) + \frac{1}{3}U^s_c\right]\psi'' -$$

$$-\frac{1}{4}\left[\left(U'_{33} - U'_{31} - U'_{13} + U'_{11}\right)\psi' -\right.$$

$$\left. - \left(U''_{33} - U''_{31} - U''_{13} + U''_{11}\right)\psi''\right] -$$

$$-\frac{1}{2}\left(U''_{31} - U''_{13} - \frac{2}{3}U''_c\right)\psi^s -$$

$$-\frac{1}{2}\left(U'_{33} - U'_{11} + \frac{2}{3}U'_c\right)\psi^a + \frac{1}{3}\left(U'_c\varphi' - U''_c\varphi''\right) = 0,$$

$$\left[\hat{K} - E + \frac{1}{2}\left(U^s_{33} + U^s_{11}\right) + \frac{1}{3}U^s_c\right]\psi^a +$$

$$+\frac{1}{2}\left[\left(U''_{33} - U''_{11}\right)\psi' - \left(U'_{33} - U'_{11}\right)\psi''\right] -$$

$$-\frac{1}{3}\left(U'_c\psi'' - U''_c\psi'\right) + \frac{1}{3}\left(U'_c\varphi'' - U''_c\varphi'\right) = 0,$$

$$\left[\hat{K} - E + \frac{1}{2}\left(U^s_{13} + U^s_{33}\right) + \frac{1}{3}U^s_c\right]\varphi' +$$

$$+\frac{1}{2}\left[\left(U'_{13} - U'_{33}\right)\varphi'' + \left(U''_{13} - U''_{33}\right)\varphi'\right] +$$

$$+\frac{1}{3}\left(U'_c\psi'' + U''_c\psi' + U'_c\psi^s - U''_c\psi^a\right) = 0,$$

$$\left[\hat{K} - E + \frac{1}{2}\left(U^s_{13} + U^s_{33}\right) + \frac{1}{3}U^s_c\right]\varphi'' +$$

$$+\frac{1}{2}\left[\left(U'_{13} - U'_{33}\right)\varphi' - \left(U''_{13} - U''_{33}\right)\varphi''\right] +$$

$$+\frac{1}{3}\left(U'_c\psi' - U''_c\psi'' + U''_c\psi^s + U'_c\psi^a\right) = 0. \qquad (6)$$

Here, $\hat{K}$ is the operator of kinetic energy of the system of three identical nucleons, and the following notations for two-particle interaction potentials with the given permutation symmetry are introduced:

$$U^s = V(r_{12}) + V(r_{13}) + V(r_{23}),$$

$$U' = \frac{\sqrt{3}}{2}\left[-V(r_{13}) + V(r_{23})\right],$$

$$U'' = -V(r_{12}) + \frac{1}{2}\left[V(r_{13}) + V(r_{23})\right]. \qquad (7)$$

Two last equations in system (6) appear due to the account of the Coulomb interaction and the mixing of states with the isospins $T = 1/2$ and $T = 3/2$. If the Coulomb interaction is switched off, then system (6) of six equations is split into the system of four first equations for SC WF with total isospin $T = 1/2$ and the system of two last equations with isospin $T = 3/2$. The standard system of four equations (see [7]) describes a nucleus ${}^3H$ in the doublet spin state with $S = 1/2$. The system of six equations (6) for SC WF of a nucleus ^{3}He and, respectively, the system of four equations for that of a nucleus ${}^3H$ in the representation of the isospin formalism are the standard basis for both precise studies and the construction of various approximate schemes of calculations of the characteristics of three-nucleon nuclei. We emphasize that the same systems of equations are also the basic ones for the study of various processes of scattering in three-particle systems (2p, n) and (2n, p), respectively, in the isospin formalism.

3. Reduction of the General System of Equations to a System of Two Equations

We will demonstrate now that only two of six SC WF in expression (5) are independent. The connection between components becomes more transparent, if we expand WF in the new full spin-isospin basis

$$|\zeta^{'} n_1\rangle, |\zeta^{'} n_2\rangle, |\zeta^{'} n_3\rangle, |\zeta^{''} n_1\rangle, |\zeta^{''} n_2\rangle, |\zeta^{''} n_3\rangle, \tag{8}$$

where $|n_1\rangle, |n_2\rangle, and |n_3\rangle$ stand for the isospin states in the form of products of two one-particle proton and one neutron isospin functions:

$$|n_1\rangle \equiv |n_1 p_2 p_3\rangle, |n_2\rangle \equiv |p_1 n_2 p_3\rangle, |n_3\rangle \equiv |p_1 p_2 n_3\rangle. \tag{9}$$

We write the full antisymmetric WF of a nucleus ^{3}He in the new basis as

$$\Psi^a_{1/2} = \Phi^{'}_1|\zeta^{'} n_1\rangle + \Phi^{'}_2|\zeta^{'} n_2\rangle + \Phi^{'}_3|\zeta^{'} n_3\rangle +$$

$$+\Phi^{''}_1|\zeta^{''} n_1\rangle + \Phi^{''}_2|\zeta^{''} n_2\rangle + \Phi^{''}_3|\zeta^{''} n_3\rangle. \tag{10}$$

The new SC WF are determined as the projections of WF (10) onto the basis functions (9):

$$\Phi^{'}_1 = \langle\zeta^{'} n_1|\Psi^a_{1/2}\rangle,\ \Phi^{'}_2 = \langle\zeta^{'} n_2|\Psi^a_{1/2}\rangle,\ \Phi^{'}_3 = \langle\zeta^{'} n_3|\Psi^a_{1/2}\rangle,$$

$$\Phi^{''}_1 = \langle\zeta^{''} n_1|\Psi^a_{1/2}\rangle, \quad \Phi^{''}_2 = \langle\zeta^{''} n_2|\Psi^a_{1/2}\rangle,$$

$$\Phi^{''}_3 = \langle\zeta^{''} n_3|\Psi^a_{1/2}\rangle. \tag{11}$$

We will act now on the functions $\Phi^{'}_3$ and $\Phi^{''}_3$ by the operator of permutation of the first and third particles $\hat{P}_{13}$. With regard for (11) and the law of transformation of the spin functions $\zeta^{'}$ and $\zeta^{''}$

$$\left\{\begin{array}{l}\hat{P}_{13}\zeta^{'} = \frac{1}{2}\zeta^{'} - \frac{\sqrt{3}}{2}\zeta^{''},\\ \hat{P}_{13}\zeta^{''} = -\frac{\sqrt{3}}{2}\zeta^{'} - \frac{1}{2}\zeta^{''},\end{array}\right. \quad \left\{\begin{array}{l}\hat{P}_{23}\zeta^{'} = \frac{1}{2}\zeta^{'} + \frac{\sqrt{3}}{2}\zeta^{''},\\ \hat{P}_{23}\zeta^{''} = \frac{\sqrt{3}}{2}\zeta^{'} - \frac{1}{2}\zeta^{''},\end{array}\right. \tag{12}$$

we obtain

$$\hat{P}_{13}\Phi^{'}_3(123) = \Phi^{'}_3(321) = \langle\hat{P}_{13}(\zeta^{'} n_3)|\hat{P}_{13}\Psi_a\rangle =$$

$$= -\langle(\hat{P}_{13}\zeta^{'})n_1)|\Psi_a\rangle = -\frac{1}{2}\langle\zeta^{'} n_1|\Psi_a\rangle + \frac{\sqrt{3}}{2}\langle\zeta^{''} n_1|\Psi_a\rangle =$$

$$= -\frac{1}{2}\Phi^{'}_1(123) + \frac{\sqrt{3}}{2}\Phi^{''}_1(123), \tag{13}$$

$$\hat{P}_{13}\Phi^{''}_3(123) = \Phi^{''}_3(321) = \langle\hat{P}_{13}(\zeta^{''} n_3)|\hat{P}_{13}\Psi_a\rangle =$$

$$= -\langle(\hat{P}_{13}\zeta^{'})n_1|(\Psi_a)\rangle = \frac{\sqrt{3}}{2}\langle\zeta^{'} n_1|\Psi_a\rangle + \frac{1}{2}\langle\zeta^{''} n_1|\Psi_a\rangle =$$

$$= \frac{\sqrt{3}}{2}\Phi^{'}_1(123) + \frac{1}{2}\Phi^{''}_1(123). \tag{14}$$

We rewrite the obtained result in the form of a system of two algebraic equations

$$\begin{array}{l}-\frac{1}{2}\Phi^{'}_1(123) + \frac{\sqrt{3}}{2}\Phi^{''}_1(123) = \Phi^{'}_3(321)\\ -\frac{\sqrt{3}}{2}\Phi^{'}_1(123) + \frac{1}{2}\Phi^{''}_1(123) = \Phi^{''}_3(321).\end{array} \tag{15}$$

This yields the explicit relations between $\Phi^{'}_1$,

$$\begin{array}{l}\Phi^{'}_1(123) = -\frac{1}{2}\Phi^{'}_3(321) + \frac{\sqrt{3}}{2}\Phi^{''}_3(321) =\\ = \hat{P}_{13}(-\frac{1}{2}\Phi^{'}_3(123) + \frac{\sqrt{3}}{2}\Phi^{''}_3(123)),\\ \Phi^{''}_1(123) = \frac{\sqrt{3}}{2}\Phi^{'}_3(321) + \frac{1}{2}\Phi^{''}_3(321) =\\ = \hat{P}_{13}(\frac{\sqrt{3}}{2}\Phi^{'}_3(123) + \frac{1}{2}\Phi^{''}_3(123)).\end{array} \tag{16}$$

Analogously after the application of the operator $\hat{P}_{23}$ instead of $\hat{P}_{13}$, we get the explicit relation between

$$\begin{array}{l}\Phi^{'}_2(123) = -\frac{1}{2}\Phi^{'}_3(132) - \frac{\sqrt{3}}{2}\Phi^{''}_3(132) =\\ = \hat{P}_{23}(-\frac{1}{2}\Phi^{'}_3(123) - \frac{\sqrt{3}}{2}\Phi^{''}_3(123)),\\ \Phi^{''}_2(123) = -\frac{\sqrt{3}}{2}\Phi^{'}_3(132) + \frac{1}{2}\Phi^{''}_3(132) =\\ = \hat{P}_{23}(-\frac{\sqrt{3}}{2}\Phi^{'}_3(123) + \frac{1}{2}\Phi^{''}_3(123)).\end{array} \tag{17}$$

Hence, among six SC WF in (10), only two components, $\Phi^{'}_3$ and $\Phi^{''}_3$, are independent, and relations (15) and (17) give a possibility to reconstruct the completely antisymmetric WF (10) in the isospin formalism.

4. System of Two Equations for the Doublet State in the Isospin Formalism

The obtained relations between SC WF and the established fact that the independent components are only $\Phi^{'}_3$ and $\Phi^{''}_3$ allow us to reduce the system of six equations (6) to a system of only two nonequivalent equations. In the general aspect, the problem of three nucleons is reformulated into the direct determination of the functions $\Phi^{'}_3$ and $\Phi^{''}_3$. Then we will be able to determine any of the components of the wave function (5) or (10) in their terms.

The construction of the system of equations for $\Phi^{'}_3$ and $\Phi^{''}_3$ can be executed as follows. First, we substitute $\Psi^a_{1/2}$ in the form (10) in the Schrödinger equation and project it successively onto the spin-isospin functions $|\zeta^{'} n_3\rangle$ and $|\zeta^{''} n_3\rangle$. Then we replace the components $\Phi^{'}_1$, $\Phi^{''}_1$, $\Phi^{'}_2$, $\Phi^{''}_2$ by $\Phi^{'}_3$ and $\Phi^{''}_3$ in the obtained relations, by

using relations (16) and (17). In this case, the sought system of two equations takes the form

$$\left\{\left(\hat{K}-E+\frac{e^2}{r_{12}}\right)+\left[\frac{1}{4}(U^s_{33}+U^s_{31}+U^s_{13}+U^s_{11})+\right.\right.$$

$$+\frac{1}{4}(U^{''}_{33}+U^{''}_{31}-U^{''}_{13}-U^{''}_{11})+\frac{1}{2}(V_{13}(12)-V_{11}(12))\Big]+$$

$$+\frac{1}{8}[-3V_{33}(13)+3V_{31}(13)+V_{13}(13)-V_{11}(13)]\hat{P}_{13}+$$

$$+\frac{1}{8}[-3V_{33}(23)+3V_{31}(23)+V_{13}(23)-$$

$$\left.-V_{11}(23)]\hat{P}_{23}\right\}\Phi^{'}_3(123)+\left\{\frac{1}{4}[U^{'}_{33}+U^{'}_{31}-U^{'}_{13}-U^{'}_{11}]+\right.$$

$$+\frac{\sqrt{3}}{8}[V_{33}(13)-V_{31}(13)+V_{13}(13)-V_{11}(13)]\hat{P}_{13}+$$

$$+\frac{\sqrt{3}}{8}[-V_{33}(23)+3V_{31}(23)-V_{13}(23)+$$

$$\left.+V_{11}(23)]\hat{P}_{23}\right\}\Phi^{''}_3(123)=0,$$

$$\left\{\frac{1}{4}[U^{'}_{33}+U^{'}_{31}-U^{'}_{13}-U^{'}_{11}]+\frac{\sqrt{3}}{8}[V_{33}(13)-\right.$$

$$-V_{31}(13)+V_{13}(13)-V_{11}(13)]\hat{P}_{13}+$$

$$\left.+\frac{\sqrt{3}}{8}[-V_{33}(23)+3V_{31}(23)-V_{13}(23)+V_{11}(23)]\hat{P}_{23}\right\}\times$$

$$\times\Phi^{'}_3(123)+\left\{\left(\hat{K}-E+\frac{e^2}{r_{12}}\right)+\left[\frac{1}{4}(U^s_{33}+U^s_{31}+\right.\right.$$

$$+U^s_{13}+U^s_{11})+\frac{1}{4}(-U^{''}_{33}-U^{''}_{31}+U^{''}_{13}+U^{''}_{11})+$$

$$+\frac{1}{2}(V_{33}(12)-U_{31}(12))]+\frac{1}{8}[-V_{33}(13)+V_{31}(13)+$$

$$+3V_{31}(13)-3V_{11}(13)]\hat{P}_{13}+\frac{1}{8}[-V_{33}(23)+V_{31}(23)$$

$$\left.+3V_{13}(23)-3V_{11}(23)]\hat{P}_{23}\right\}\Phi^{''}_3(123)=0. \qquad (18)$$

We note that the equations completely equivalent to (18) can be also derived on the projection on $|\zeta^{'}n_1\rangle$, $|\zeta^{''}n_1\rangle$ and $|\zeta^{'}n_2\rangle$, $|\zeta^{''}n_2\rangle$.

The solution of system (18) consists in the determination of eigenfunctions $\Phi^{'}_3$ and $\Phi^{''}_3$ which have the corresponding symmetry (12) relative to the permutations of particles and have the corresponding asymptotes. In view of the derived eigenfunctions $\Phi^{'}_3$ and $\Phi^{''}_3$ and relations (16) and (17), WF (10) is completely determined.

In order that the full antisymmetric function was normed, the components $\Phi^{'}_3$ and $\Phi^{''}_3$ must be normed in a certain way. With regard for the fact that the spin-isospin functions are orthonormed, the condition for the full WF to be normed takes the form

$$\|\Psi^a_{1/2}\|^2=\langle\Psi^a_{1/2}|\Psi^a_{1/2}\rangle=\langle\Phi^{'}_1|\Phi^{'}_1\rangle+\langle\Phi^{'}_2|\Phi^{'}_2\rangle+$$

$$+\langle\Phi^{'}_3|\Phi^{'}_3\rangle+\langle\Phi^{''}_1|\Phi^{''}_1\rangle+\langle\Phi^{''}_2|\Phi^{''}_2\rangle+\langle\Phi^{''}_3|\Phi^{''}_3\rangle=1. \qquad (19)$$

By replacing $\Phi^{'}_1$, $\Phi^{''}_1$, $\Phi^{'}_2$, and $\Phi^{''}_2$ by $\Phi^{'}_3$ and $\Phi^{''}_3$, taking the triviality of the permutations of the numbers of particles into account, and performing simple transformations, we get the normalization condition only in terms of the independent SC WF as

$$\langle\Psi^a_{1/2}|\Psi^a_{1/2}\rangle=3(\langle{\Phi_3}'|\Phi^{'}_3\rangle+\langle{\Phi_3}''|\Phi^{''}_3\rangle)=1. \qquad (20)$$

The full antisymmetric wave function (10) can be written in a more compact form. Indeed, taking the relations $|n_1\rangle=\hat{P}_{13}|n_3\rangle$ and $|n_2\rangle=\hat{P}_{23}|n_3\rangle$ into account and using (16) and (17), we can perform the following transformations:

$$\Psi^a_{1/2}=\zeta^{'}\left\{\hat{P}_{13}\left[\left(-\frac{1}{2}\Phi^{'}_3+\frac{\sqrt{3}}{2}\Phi^{''}_3\right)|n_3\rangle\right]+\right.$$

$$\left.+\hat{P}_{23}\left[\left(-\frac{1}{2}\Phi^{'}_3-\frac{\sqrt{3}}{2}\Phi^{''}_3\right)|n_3\rangle\right]+\Phi^{'}_3|n_3\rangle\right\}+$$

$$+\zeta^{''}\left\{\hat{P}_{13}\left[\left(\frac{\sqrt{3}}{2}\Phi^{'}_3+\frac{1}{2}\Phi^{''}_3\right)|n_3\rangle\right]+\right.$$

$$\left.+\hat{P}_{23}\left[\left(-\frac{\sqrt{3}}{2}\Phi^{'}_3+\frac{1}{2}\Phi^{''}_3\right)|n_3\rangle\right]+\Phi^{''}_3|n_3\rangle\right\}=$$

$$=\left(-\frac{1}{2}\zeta^{'}+\frac{\sqrt{3}}{2}\zeta^{''}\right)\hat{P}_{13}({\Phi_3}'|n_3\rangle)-$$

$$-\left(\frac{1}{2}\zeta^{'}+\frac{\sqrt{3}}{2}\zeta^{''}\right)\hat{P}_{13}({\Phi_3}'|n_3\rangle)+\zeta^{'}\Phi^{'}_3|n_3\rangle+$$

$$+\left(\frac{\sqrt{3}}{2}\zeta^{'}+\frac{1}{2}\zeta^{''}\right)\hat{P}_{13}(\Phi_3{''}|n_3\rangle)+$$

$$\left(-\frac{\sqrt{3}}{2}\zeta^{'}+\frac{1}{2}\zeta^{''}\right)\hat{P}_{23}(\Phi_3{''}|n_3\rangle)+\zeta^{''}\Phi_3^{''}|n_3\rangle, \quad (21)$$

With regard for equality (12), we obtain finally

$$\Psi^a_{1/2}=(1-\hat{P}_{13}-\hat{P}_{23})\left\{[\zeta^{'}\Phi_3{'}+\zeta^{''}\Phi_3^{''}]|n_3\rangle\right\}. \quad (22)$$

For the sake of convenience, we consider that the function $\varphi(1,2;3)\equiv(\zeta^{'}\Phi_3^{'}+\zeta^{'}\Phi_3^{'})|n_3\rangle$ is normed on unity: $\langle\varphi|\varphi\rangle=\langle\Phi_3^{'}|\Phi_3^{'}\rangle+\langle\Phi_3^{''}|\Phi_3^{''}\rangle=1$. . Then we get

$$\Psi^a_{1/2}=\frac{1}{\sqrt{3}}(1-\hat{P}_{13}-\hat{P}_{23})\varphi(1,2;3)|n_3\rangle. \quad (23)$$

We now show directly that the operator $\hat{\Im}=\frac{1}{\sqrt{3}}(1-\hat{P}_{13}-\hat{P}_{23})$ transforms the expression $\varphi(1,2;3)|n_3\rangle$ into a function antisymmetric relative to the simultaneous permutation of the spatial, spin, and isospin variables of any pair of nucleons, if $\varphi(1;2;3)$ is antisymmetric relative to permutations of the spatial and spin variables of the first and second nucleons. Indeed, let

$$\varphi(2,1;3)=-\varphi(1,2;3), \quad (24)$$

Then

$$\Psi^a_{1/2}=\frac{1}{\sqrt{3}}(1-\hat{P}_{13}-\hat{P}_{23})\{\varphi(1,2;3)|n_3\rangle\}=$$

$$\frac{1}{\sqrt{3}}\{\varphi(1,2;3)|n_3\rangle-\varphi(3,2;1)|n_1\rangle-\varphi(1,3;2)|n_2\rangle\}. \quad (25)$$

Acting by the operator of permutations of the first and third nucleons on $\Psi^a_{1/2}$, we get

$$\hat{P}_{13}\Psi^a_{1/2}(1;2;3)=\hat{P}_{13}\frac{1}{\sqrt{3}}\{\varphi(1;2;3)|n_3\rangle-$$

$$-\varphi(3;2;1)|n_1\rangle-\varphi(1;3;2)|n_2\rangle\}=$$

$$=-\frac{1}{\sqrt{3}}\{\varphi(1;2;3)|n_3\rangle-\varphi(3;2;1)|n_1\rangle-$$

$$\varphi(1;3;2)|n_2\rangle\}=-\Psi^a_{1/2}(1;2;3). \quad (26)$$

In the same manner, we can verify the antisymmetry of $\Psi^a_{1/2}$ relative to permutations of any pair of nucleons.

5. Reconstruction of SC WF with Isospin and the Identity of Physical Averages

All the components of the antisymmetric WF $\Psi^a_{1/2}$ written in the traditional form (5) with the use of the isospin formalism can be also presented in terms of $\Phi_3^{'}$ and $\Phi_3^{''}$, because there exists a certain connection between components of functions (5) and (10). We project successively equality (10) onto each basic spin-isospin function in (4) and, as a result, obtain the relation

$$\begin{pmatrix}\psi^s_1\\ \psi'\\ \psi''\\ \psi^a\\ \varphi'\\ \varphi''\end{pmatrix}=\begin{pmatrix}-\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}}\\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}}\\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0\end{pmatrix}\times\begin{pmatrix}\Phi_1^{'}\\ \Phi_2^{'}\\ \Phi_3^{'}\\ \Phi_1^{''}\\ \Phi_2^{''}\\ \Phi_3^{''}\end{pmatrix}. \quad (27)$$

This implies that the transition from one collection of the spatial components of WF to another one is carried out with the help of a certain real orthogonal matrix. The inverse transformation is performed with the help of the transposed matrix.

Using relations (16) and (17), we present the components of functions (5) directly in terms of two independent functions $\Phi_3^{'}$ and $\Phi_3^{''}$ as

$$\begin{pmatrix} \psi_1^s(123) \\ \psi'(123) \\ \psi''(123) \\ \psi^a(123) \\ \varphi'(123) \\ \varphi''(123) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \Phi_3'(123) \\ \Phi_3'(321) \\ \Phi_3'(132) \\ \Phi_3''(123) \\ \Phi_3''(321) \\ \Phi_3''(132) \end{pmatrix}. \quad (28)$$

This is the final result which presents the reconstruction of the completely antisymmetric WF (5) in the isospin formalism in the explicit form only through two independent spatial components in the representation without isospin, namely the function $\Phi_3^{'}$ symmetric relative to the permutation of the coordinates of the first and second protons and, respectively, the antisymmetric function $\Phi_3^{''}$.

The obtained results allow one to execute the precise calculations of the characteristics of a nucleus ^{3}He as follows. First, the solution of a simple system of two equations (18) with the use of some modern schemes of the Galerkin projection variational method and the relevant basis gives two components which minimize the energy. The reduction of the number of equations from six to two for the spatial components of WF allows one to significantly facilitate their solution and to enhance the accuracy of the value of the energy [5]. This ensures, as a result, a high accuracy also for the functions $\Phi_3^{'}$ and $\Phi_3^{''}$. Then we find all the SC WF of a nucleus in the isospin formalism, by using equality (28). With the WF obtained in such a way, we can calculate all structural nuclear characteristics in the routine manner.

The equivalent calculation of the physical characteristics of nuclei can be performed without the construction of the completely antisymmetric functions (5) або (10), but with the use of the WF antisymmetric only in the identical protons and neutrons separately, i.e. in the representation without the use of isospin [5]. For the system of three nucleons (2p, n), it is the function

$$\Psi(1,2;3) = \zeta^{'}\Phi_3^{'} + \zeta^{''}\Phi_3^{''}, \quad (29)$$

which is antisymmerized only relative to the permutations of protons in a nucleus ^{3}He. It is obvious that the calculation of the average values of an arbitrary operator $\hat{A}$ symmetric in particles within both methods will give the same result in all the cases where the condition

$$\langle\hat{\Im}\varphi(1,2;3)|\hat{A}|\hat{\Im}\varphi(1,2;3)\rangle = \langle\Psi(1,2;3)|\hat{A}|\Psi(1,2;3)\rangle. \quad (30)$$

is fulfilled. For example, an operator which does not depend on spin obeys the equality

$$\langle\Psi|\hat{A}|\Psi\rangle = \langle\Phi_3^{'}|\hat{A}|\Phi_3^{'}\rangle + \langle\Phi_3^{''}|\hat{A}|\Phi_3^{''}\rangle. \quad (31)$$

In particular, the Coulomb interaction energy for a nucleus ^{3}He is defined by the formula

$$W_k = \langle\Phi_3^{'}|\frac{e^2}{|\vec{r}_1-\vec{r}_2|}|\Phi_3^{'}\rangle + \langle\Phi_3^{''}|\frac{e^2}{|\vec{r}_1-\vec{r}_2|}|\Phi_3^{''}\rangle. \quad (32)$$

The charge distribution density in a nucleus ^{3}He reads (we recall that numbers 1 and 2 are referred to protons everywhere)

$$\rho(\vec{r}) = e\int\int\left|\Phi_3^{'}(\vec{r},\vec{r}_2,\vec{r}_3)\right|^2 d\vec{r}_2 d\vec{r}_3 +$$
$$+e\int\int\left|\Phi_3^{'}(\vec{r}_1,\vec{r},\vec{r}_3)\right|^2 d\vec{r}_1 d\vec{r}_3 +$$
$$+e\int\int\left|\Phi_3^{''}(\vec{r},\vec{r}_2,\vec{r}_3)\right|^2 d\vec{r}_2 d\vec{r}_3 +$$
$$+e\int\int\left|\Phi_3^{''}(\vec{r}_1,\vec{r},\vec{r}_3)\right|^2 d\vec{r}_1 d\vec{r}_3. \quad (33)$$

6. Representation without Isospin

It is expedient to write the system of two coupled equations (18) for SC WF of a nucleus ^{3}He (and, generally, for any physical process of scattering in the system of three nucleons (2p, n) in the doublet spin state) in the form [5] convenient for the direct execution of calculations

$$\left[\frac{1}{2M_p}(p_1^2+p_2^2) + \frac{1}{2M_n}p_3^2 + \frac{e^2}{r_{12}} + V_{s(pp)}^{+}(r_{12}) - E\right] \times$$

$$\times\Phi_1(123)+\frac{1}{8}\sum_{ij=13,23}\sum_{+,-}[3V^{\pm}_{t(np)}(r_{ij})+$$

$$+V^{\pm}_{s(np)}(r_{ij})][1\pm P(ij)]\Phi_1(123)+$$

$$+\frac{\sqrt{3}}{8}\sum_{ij=13,23}\sum_{+,-}(-1)^{i+j}[V^{\pm}_{s(np)}(r_{ij})-$$

$$-V^{\pm}_{t(np)}(r_{ij})][1\pm P(ij)]\Phi_2(123)=0,$$

$$\left[\frac{1}{2M_p}(p_1^2+p_2^2)+\frac{1}{2M_n}p_3^2+\frac{e^2}{r_{12}}+V^{-}_{t(pp)}(r_{12})-E\right]\times$$

$$\times\Phi_2(123)+\frac{1}{8}\sum_{ij=13,23}\sum_{+,-}[V^{\pm}_{t(np)}(r_{ij})+$$

$$+3V^{\pm}_{s(np)}(r_{ij})][1\pm P(ij)]\Phi_2(123)+$$

$$+\frac{\sqrt{3}}{8}\sum_{ij=13,23}\sum_{+,-}(-1)^{i+j}[V^{\pm}_{s(np)}(r_{ij})-$$

$$-V^{\pm}_{t(np)}(r_{ij})][1\pm P(ij)]\Phi_1(123)=0\,. \qquad (34)$$

Here, the functions $\Phi_1(123)\equiv\Phi_3^{'}(123)$ and $\Phi_2(123)\equiv\Phi_3^{''}(123)$ are, respectively, symmetric and antisymmetric relative to the permutations of protons, and P_{ij} is the operator of permutations of spatial coordinates. To get (34), we generalized Eq. (18) for three nucleons to the case of the possible account of the charge-dependent two-nucleon exchange potentials of interaction in the states which are singlet in spin [with the moment of two nucleons which is even (+) and odd (–) relative to the orbital moment]

$$V_{13}\to V^{+}_{s(nn)},\;\; V^{+}_{s(pp)},\; V^{+}_{s(np)},\quad V_{11}\to V^{-}_{s(np)} \qquad (35)$$

and in the triplet state (with even and odd orbital moments)

$$V_{31}\to V^{+}_{t(np)},\quad V_{33}\to V^{-}_{t(nn)},\;\; V^{-}_{t(pp)},\; V^{-}_{t(np)}. \qquad (36)$$

In (34), we took the difference of the masses of a proton and a neutron into account.

The nuclear three-nucleon system (2n, p) (and a nucleus ^{3}H) in the doublet state with the total spin $S=1/2$ is described also by the system of two equations (34), in which it is necessary to switch off the Coulomb potential and to change the indices, $n\leftrightarrow p$. We will emphasize once more that though the attention was earlier focused on the problem of bound states, the presented systems of equations (18) and (34) can be directly used also for the processes of scattering with regard for the relevant boundary conditions.

The equivalence of the standard formalism with the use of the notion of isospin and the representation without isospin, which was considered above by the example of the three-nucleon problem for the doublet spin state, is preserved without any cautions in the case of the quartet ($S=\frac{3}{2}$) state of three nucleons. For the system of two protons and one neutron (2p, n) in the isospin formalism, the problem is generally reduced to the solution of the system of three equations for SC WF (for the system of (2n, p), two equations should be solved). Let us consider the equivalent statement with the use of the representation without isospin. In order to construct the full wave function of three nucleons, being antisymmetric only relative to the permutation of two protons, it is sufficient to solve only one equation. In the version accounting the difference of the masses of a proton and a neutron, this equation looks as

$$\left[\frac{1}{2M_p}(p_1^2+p_2^2)+\frac{1}{2M_n}p_3^2+\right.$$

$$\left.+\frac{e^2}{r_{12}}+V^{-}_{t(pp)}(r_{12})-E\right]\Phi(123)+$$

$$+\frac{1}{2}\sum_{ij=13,23}\sum_{+,-}V^{\pm}_{t(np)}(r_{ij})\;\;[1\pm P(ij)]\,\Phi(123)=0. \qquad (37)$$

In this equation, the spatial function $\Phi(123)$ is antisymmetric relative to the permutation $(1\leftrightarrow 2)$ of the coordinates of protons. In the case of the three-nucleon system (2n, p) in the quartet state, it is necessary to switch off the Coulomb interaction in Eq. (37) and change the indices, $n\leftrightarrow p$.

7. Conclusions

For three-nucleon systems, we have directly proved that the isospin formalism for protons and neutrons and the representation without the use of isospin, where protons and neutrons are considered to be different particles and interact via pairwise exchange central potentials, lead to the completely equivalent physical results. At the same time, the representation without isospin yields the simpler systems of equations for the spatial components of wave functions, and the number of independent equations is determined by the dimension of the representation of a Young spin scheme.

The minimum number of independent equations, namely two, for spatial components in the doublet state

of three nucleons (both for bound states and an arbitrary process of scattering) is a strict mathematical result (as well as the fact that the quartet state possesses only one independent spatial component). This leads to essential simplifications in all the approaches to the problem of three nucleons (the variational method, method of Faddeev integral equations, method of hyperspherical basis, etc.). We note that the above-developed approach and the construction of the minimum system of independent equations can be also transferred onto three-nucleon systems with regard for the tensor interaction, though this requires constructionally a separate consideration. It is obvious that the scheme proposed in this work can be directly transferred onto nuclear systems with three particles of different nature (nucleons, pions, kaons, etc.) under the assumption of pairwise interactions of the exchange character. Formulated in the representation without isospin, the equations for independent components, whose number in the doublet state is only two, have essential advantages over those in the traditional scheme with the use of the isospin formalism, in the first turn, while performing the precise calculations and studying the fine effects characteristic of few-nucleon systems.

The approach proposed in this work with the use of the representation without isospin can be surely applied to the nuclear systems, where the number of nucleons is more than three. In each specific case, the number of independent SC WF is determined only by the dimension of the corresponding Young scheme for the total spin. This gives the significant advantages to the applications of the representation without isospin as compared with the traditional approach. For comparison, the full antisymmetric wave function of a nucleus ^{4}He with zero spin has twelve spatial components in the isospin formalism, whereas only two components are independent, which is realized in the representation without isospin. In this case, there exists the complete equivalence of the approaches with the use of the isospin formalism and the representation without isospin, though the details of the direct proof of such an equivalence require a separate consideration.

Finally, we emphasize that the results of this work pretend by no means on the revision of qualitative regularities as for the nuclear spectra for light and medium nuclei which are derived on the basis of the approximation of the isotopic invariance of nuclear forces.

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ЕКВІВАЛЕНТНІСТЬ ПРЕДСТАВЛЕННЯ З ВИКОРИСТАННЯМ ФОРМАЛІЗМУ ІЗОСПІНУ І ПРЕДСТАВЛЕННЯ БЕЗ ІЗОСПІНУ ДЛЯ СИСТЕМИ ТРЬОХ НУКЛОНІВ

І.С. Доценко, І.В. Сименог

Р е з ю м е

Для систем трьох нуклонів (2p,n) і (2n,p) у дублетному стані за спіном встановлено, що з шести і відповідно чотирьох просторових компонентів хвильових функцій (ХФ) в стандартній формі з використанням формалізму ізоспіну незалежними є лише дві компоненти. Отримано співвідношення для побудови повних антисиметричних ХФ через дві незалежні компоненти. Сформульовані системи двох рівнянь відносно незалежних просторових компонентів у дублетному стані ядер ^{3}He і ^{3}H, а також одне рівняння для квартетного стану. Показано, що розраховані в представленні з використанням формалізму ізоспіну і в представленні без ізоспіну фізичні характеристики збігаються. Обговорюється перспективність нового підходу для виконання прецизійних досліджень малонуклонних систем.