PECULIAR PROPERTIES OF SU(2) GAUGE FIELD THERMODYNAMICS ON A FINITE LATTICE. CALCULATION OF BETA-FUNCTION

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The new method of nonperturbative calculations of the beta-function in the lattice gauge theory is proposed. The method is based on the finite-size scaling hypothesis.

Ever since the pioneering work by Creutz [1], the approach to the asymptotic scaling, and thus the continuum limit, was one of the central issues in studies of gauge theories on the lattice. Although the first results were promising, the lack of the asymptotic scaling of physical observables has been observed in SU(N) gauge theories. One of the main sources of the nonperturbative results in gauge theories today is the Monte-Carlo (MC) lattice calculations. For the SU(N) pure gauge theories on lattices of size $N_\tau \times N_\sigma^3$, MC results are the dimensionless functions of the bare coupling constant g (another form for the coupling, $\beta \equiv 2N/g^2$, is often used). The transformation of these functions to physical quantities are done by multiplying them by the lattice spacing a in the corresponding powers. The length scale L and the temperature T are given as

$$L = N_\sigma a, \quad T = \left(N_\tau a\right)^{-1}. \tag{1}$$

To define the physical quantities, one needs a connection between the lattice spacing a and the bare coupling constant g. Such a connection is formulated in terms of the beta-function $\beta_f(g)$ through the equation

$$\beta_f(g) = -a\frac{dg}{da}. \tag{2}$$

The perturbation theory gives the asymptotic expansion of the beta-function

$$\beta_f^{\mathrm{AF}} = -b_0 g^3 - b_1 g^5 + O(g^7),$$
$$b_0 = \frac{11N}{48\pi^2}, \quad b_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2, \tag{3}$$

where $N = 2$ in the SU(2) case. The differential equation (2) with $\beta_f^{\mathrm{AF}}(g)$ in (3) leads to

$$a\Lambda_L^{\mathrm{AF}} = \exp\left(-\frac{1}{2b_0 g^2}\right) \cdot \left(b_0 g^2\right)^{-b_1/2b_0^2} \equiv R(g^2), \tag{4}$$

where Λ_L^{AF} is the renormalization group invariant parameter (integration constant of Eq. (2)). Equation (4) is known as the asymptotic freedom (AF) relation.

Using (1) and (4), one can calculate

$$\frac{T_c}{\Lambda_L^{\mathrm{AF}}} = \frac{1}{N_\tau R(g_c^2)}. \tag{5}$$

The values of $T_c/\Lambda_L^{\mathrm{AF}}$ at different N_τ are presented in Table 1. One observes a rather strong dependence of $T_c/\Lambda_L^{\mathrm{AF}}$ on N_τ. This means that the perturbative AF relation (4) does not work even on the largest available lattices. This fact is known as the absence of the asymptotic scaling.

It has been proposed in [3] that a deviation from the asymptotic scaling can be described by a universal non-perturbative (NP) beta-function, i.e. $\beta_f(g)$ is the same one for all lattice observables and does not depend on the lattice size if N_σ and N_τ are not too small.

The following ansatz was suggested [3]:

$$a\Lambda_L^{\rm NP} = \lambda(g^2) R(g^2), \tag{6}$$

where $R(g^2)$ is given by (4), and $\lambda(g^2)$ is thought to describe a deviation from the perturbative behaviour. Equation (4) has been expected at $g \to 0$, so that an additional constraint, $\lambda(0) = 1$, has been assumed. The values of $T_c/\Lambda_L^{\rm NP}$ can be calculated then as

$$T_c/\Lambda_L^{\rm NP} = \frac{1}{N_\tau \lambda(g_c^2) R(g_c^2)}. \tag{7}$$

A simple formula for the function $\lambda(g^2)$ was suggested [3]:

$$\lambda(g^2) = \exp\left(\frac{c_3 g^6}{2b_0^2}\right). \tag{8}$$

The parameter c_3 in (8) and a new one, $T_c^*/\Lambda_L^{\rm NP}$=const, have been considered as free parameters and determined from fitting the MC values of $T_c/\Lambda_L^{\rm NP}$ (7) at different N_τ to the constant value $T_c^*/\Lambda_L^{\rm NP}$. This procedure gives

$$T_c^*/\Lambda_L^{\rm NP} = 21.45(14), \quad c_3 = 5.529(63) \times 10^{-4}. \tag{9}$$

In comparison to $T_c/\Lambda_L^{\rm AF}$, the much weaker N_τ-dependence of $T_c/\Lambda_L^{\rm NP}$, which becomes now close to the constant value of $T_c^*/\Lambda_L^{\rm NP}$ (9), has been obtained.

Despite the phenomenological success of the above procedure [3], the crucial question regarding the existence of the universal NP beta-function which does not depend on the lattice size is not solved and remains just a postulate. A principal difference of our approach is that we do not assume the existence of the universal beta-function and take into account the finite-size effects of the lattice.

As usual, the finite-size scaling (FSS) in the vicinity of a finite-temperature phase transition is discussed for lattice SU(N) gauge models without trying to make contact with the continuum limit, i.e. the scaling properties are studied on lattices $N_\tau \times N_\sigma^3$ with fixed N_τ and varying N_σ, and the model is viewed as a three-dimensional spin system. On the other hand, in the continuum limit, the FSS properties of these non-Abelian models should be discussed, of course, in terms of the physical volume $V = L^3$ and the temperature T. On a $N_\tau \times N_\sigma^3$ lattice, L and T are given in units of the lattice spacing a. Therefore, it is advantageous to introduce the dimensionless combination

$$LT = \frac{N_\sigma}{N_\tau}. \tag{10}$$

The scaling behaviour of the continuum theory emerges from the lattice free energy on arbitrary lattices, i.e. when varying N_τ and N_σ.

Following [2], let us discuss briefly the FSS procedure. The singular part of the free energy density is described by the universal finite-size scaling function

$$f(t, h, N_\sigma, N_\tau) = Q_f\left(g_t\left(\frac{N_\sigma}{N_\tau}\right)^{\frac{1}{\nu}}, g_h\left(\frac{N_\sigma}{N_\tau}\right)^{\frac{\beta+\gamma}{\nu}}\right) \times \left(\frac{N_\sigma}{N_\tau}\right)^{-3}, \tag{11}$$

where β, γ, ν are the critical indices of the theory, the scaling function Q_f depends on the reduced temperature $t = (T - T_c)/T_c$ and the external field strength h through the thermal and magnetic scaling fields

$$g_t = c_t t\,(1 + b_t t)\,, \tag{12}$$

$$g_h = c_h h\,(1 + b_h t) \tag{13}$$

with non-universal coefficients c_t, c_h, b_t, b_h still carrying a possible N_τ-dependence.

The order parameter and the susceptibility are now obtained as derivatives of f

$$\langle L\rangle = -\left.\frac{\partial f}{\partial h}\right|_{h=0} = \left(\frac{N_\sigma}{N_\tau}\right)^{-\beta/\nu} Q_L\left(g_t\left(\frac{N_\sigma}{N_\tau}\right)^{1/\nu}\right), \tag{14}$$

$$\chi = \left.\frac{\partial f^2}{\partial h^2}\right|_{h=0} = \left(\frac{N_\sigma}{N_\tau}\right)^{\gamma/\nu} Q_\chi\left(g_t\left(\frac{N_\sigma}{N_\tau}\right)^{1/\nu}\right). \tag{15}$$

Here, we have used the hyperscaling relation

$$\frac{\gamma}{\nu} + 2\frac{\beta}{\nu} = 3.$$

Taking the fourth derivative of f at $h = 0$, it is easy to see that the quantity

$$g_4 = \left.\frac{\partial^4 f}{\partial h^4}\right|_{h=0} \Bigg/ \chi^2 \left(\frac{N_\sigma}{N_\tau}\right)^3 \tag{16}$$

is directly a scaling function

$$g_4 = Q_{g_4}\left(g_t\left(\frac{N_\sigma}{N_\tau}\right)^{1/\nu}\right). \tag{17}$$

On a finite lattice, g_4 has the form

$$g_4 = \frac{\langle L^4\rangle}{\langle L^2\rangle^2} - 3, \tag{18}$$

i.e. it is the normalized fourth cumulant of the Polyakov loop.

Our approach is based on the two points: i) more conventional statistical mechanical definition of the

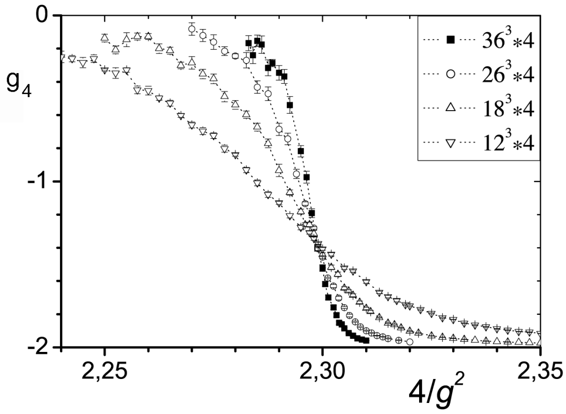

Fig. 1. Fourth Binder cumulant g_4 on the lattices $N_\tau = 4$; $N_\sigma = 12, 18, 26, 36$. MC data are taken from [5]

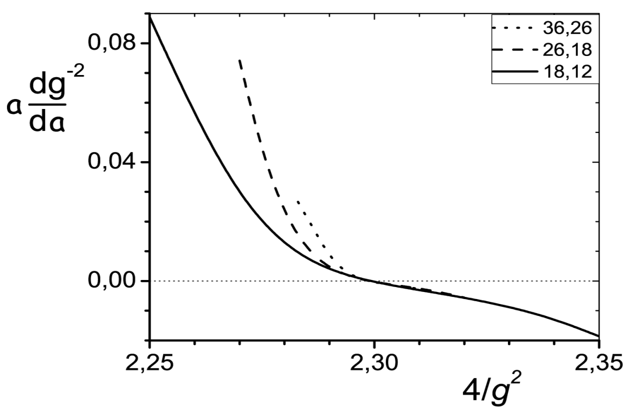

Fig. 2. Beta-function from (24) for the pairs $N_\sigma = 12$, $N'_\sigma = 18$; $N_\sigma = 18$, $N'_\sigma = 26$; $N_\sigma = 26$, $N'_\sigma = 36$

beta-function and ii) FSS and the phenomenological renormalization. Let us make the transformation of the lattice spacing $a \to a' = ba = (1 + \Delta b)a$. Then

$$-a\frac{dg}{da} = -\lim_{b\to 1}\left(a\frac{g(ba) - g(a)}{ba - a}\right) \equiv -\lim_{b\to 1}\frac{dg}{db}. \quad (19)$$

We get the definition of the beta-function for the lattice system as

$$\beta_f(g) = -\lim_{b\to 1}\frac{dg}{db}. \quad (20)$$

Let us first consider the case where N_τ is fixed. Then N_τ can be absorbed in the non-universal constants in g_t and g_h, and we deal with the usual form at the FSS as in the standard spin theory (see, for example [4]). The existence of the scaling function Q allows us to develop a procedure to renormalize the coupling constant g^{-2} by using two different lattice sizes N_σ and N'_σ. Let us fix the spatial size $L = N_\sigma a$ and make a scale transformation

$$a \to a' = ba$$
$$N_\sigma \to N'_\sigma = N_\sigma/b. \quad (21)$$

Then the phenomenological renormalization is defined by the equation

$$Q(g^{-2}, N_\sigma) = Q\Big((g')^{-2}, N_\sigma/b\Big). \quad (22)$$

It expresses that the scaling function Q remains to be unchanged if the lattice size is rescaled by a factor b and the inverse coupling g^{-2} is shifted to $(g')^{-2}$ simultaneously. Taking the derivative of the both sides of (22) with respect to the scale parameter b and using (20), it is easy to obtain the expression

$$a\frac{dg^{-2}}{da} = \frac{\partial \ln Q/\partial \ln N_\sigma}{\partial \ln Q/\partial g^{-2}}. \quad (23)$$

The approximation of the derivative with respect to N_σ by a finite difference yields the formula for the beta-function

$$a\frac{dg^{-2}}{da} = \frac{\ln\dfrac{Q(N'_\sigma)}{Q(N_\sigma)}/\ln(\dfrac{N'_\sigma}{N_\sigma})}{\left[\dfrac{dQ(N_\sigma)}{dg^{-2}}\cdot\dfrac{dQ(N'_\sigma)}{dg^{-2}}/Q(N_\sigma)Q(N'_\sigma)\right]^{1/2}}. \quad (24)$$

Further we consider formula (24) for the fourth cumulant $g_4(g^{-2}, N_\sigma)$, which is the scaling function directly. Figure 1 presents the MC data for g_4 on the lattices $N_\tau = 4$; $N_\sigma = 12, 18, 26, 36$ [5]. One can easily see from (24) that the beta-function has a zero at the fixed point $4/g_c^2 = 2.299$ of the renormalization transformation (22) in full accordance with a second-order nature of the deconfinement phase transition in SU(2) lattice gauge theory.

The results of calculations of the beta-function according to formula (24) are presented in Fig. 2 for three sets $N_\sigma = 12$, $N'_\sigma = 18$; $N_\sigma = 18$, $N'_\sigma = 26$; $N_\sigma = 26$, $N'_\sigma = 36$. Although N_σ and N'_σ in the different pairs are not too close, one can see surprisingly the coincidence of the curves at $4/g^2 \geq 4/g_c^2$. This observation gives the hope that beta-function does not depend on the spatial size of a lattice in the deconfinement phase.

Next we consider fixed $y = N_\sigma/N_\tau$, by varying N_σ, and therefore N_τ accordingly, as is needed to reach the continuum limit. Rescaling N_σ and N_τ by a factor b leads to a phenomenological renormalization by the following identity for a scaling function Q

$$Q\left(g_t\left(g^{-2}, N_\tau\right)\cdot\left(\frac{N_\sigma}{N_\tau}\right)^{1/\nu}\right) =$$

MC data for β_c are taken from [2]. The values of T_c/Λ_L^{AF} are calculated from (4). Our results for T_c/Λ_L are obtained from (30)

N_τ	$\beta_c = 4/g_c^2$	T_c/Λ_L^{AF}	$d\beta_c/dN_\tau$	T_c/Λ_L
2	1.880	29.7	—	—
3	2.177	41.4	0.158	25.22
4	2.299	42.1	0.086	25.46
5	2.373	40.6	0.063	25.38
6	2.427	38.7	0.045	24.13
8	2.512	36.0	0.040	24.24
16	2.739	32.0	0.017	—

$$= Q\left(g_t\left((g')^{-2}, N_\tau/b\right) \cdot \left(\frac{bN_\sigma}{bN_\tau}\right)^{1/\nu}\right), \tag{25}$$

where $g_t(g^{-2}, N_\sigma)$ is determined by (12). If we ignore the possible N_τ-dependence of the coefficients c_t and b_t, then relation (25) yields

$$t\left(g^{-2}, N_\tau\right) = t\left((g')^{-2}, N_\tau/b\right). \tag{26}$$

In the general case, the reduced temperature $t = (T - T_c)/T_c$ is a complicated function of the coupling $\beta = 2N/g^2$ which can be approximated in the vicinity of the critical temperature T_c by [2]

$$t = (\beta - \beta_c)\frac{1}{4Nb_0}\left[1 - \frac{2Nb_1}{b_0}\beta_c^{-1}\right]. \tag{27}$$

This approximation reproduces the correct reduced temperature in the continuum limit, which is easily verified by using (4). Taking the derivatives of the both sides of (26) with respect to the scale parameter b and using (20) and (27), it is easy to obtain the expression for the beta-function

$$\beta_f(g) = -B_0(N_\tau)g^3 - B_1(N_\tau)g^5, \tag{28}$$

where

$$\begin{cases} B_0(N_\tau) = \frac{1}{4N}\left(1 - \frac{2Nb_1}{b_0\beta_c}\right)\frac{d\beta_c}{d\ln N_\tau}, \\ B_1(N_\tau) = B_0(N_\tau)\frac{b_1}{b_0}. \end{cases} \tag{29}$$

Then Eq. (2) leads to

$$a\Lambda_L = \exp\left(-\frac{1}{2B_0g^2}\right)\left(B_0g^2\right)^{-B_1/2B_0^2}. \tag{30}$$

Using (1), one can obtain the critical temperature T_c. The only problem remains to calculate the derivative $d\beta_c/d\ln N_\tau$ in expression (29). The calculation has been made for the SU(2) gauge theory by fitting the MC data for the critical couplings $\beta_c^{MC}(N_\tau)$ with a spline interpolation and a numerical differentiation of this curve. The result of calculations is presented in the table. In comparison to T_c/λ_L^{AF}, the much weaker dependence of the critical temperature T_c/Λ_L on N_τ is observed.

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ОСОБЛИВОСТІ ТЕРМОДИНАМІКИ
SU(2) КАЛІБРУВАЛЬНИХ ПОЛІВ НА ҐРАТЦІ
СКІНЧЕННОГО РОЗМІРУ. РОЗРАХУНОК БЕТА-ФУНКЦІЇ

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Р е з ю м е

Запропоновано новий метод непертурбативного обчислення бета-функції калібрувальної теорії на ґратці. В основі методу покладено гіпотезу скейлінгу скінченного розміру.