VARIANTS OF SUPER-SELF-DUALITY

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New examples of the systems of super-self-duality equations for supersymmetric Yang—Mills theory are proposed. These systems are written both in terms of component fields and in terms of superfields.

Famous self-duality equations that relate magnetic and electric charges provide very important solutions of the Yang–Mills equations such as instantons and monopoles [1–3]. Ward has developed a twistorial method for generating all $N = 0$ self-dual solutions of the Yang—Mills equations [4]. A twistor correspondence for the super-self-duality equations for supersymmetric Yang—Mills theories was shown to be encoded in analytic harmonic superfields satisfying the appropriate generalized Cauchy–Riemann conditions [5].

The $N{=}0$ Yang—Mills self-duality equations attract much attention due to a remarkable conjecture [6, 7] that all integrable systems in lower dimensions are obtainable by the dimensional reduction from 4D self-dual Yang—Mills theory. Unification of self-duality and supersymmetry [8–12] can generate new integrable systems by the dimensional reduction. A natural task is to find all possible systems of super-self-duality equations. In this paper, we obtain some new examples of super-self-dual systems with help of the supersymmetric transformation of $N = 0$ self-duality equations. These systems manifest the complete supersymmetry and are formulated both in terms of component fields and superfields.

In the pure Yang—Mills theory, the self-duality equations read

$$iF_{mn} = \widetilde{F}_{mn} \equiv \frac{1}{2}\epsilon_{mnkl}F^{kl}\,, \qquad (1)$$

where

$$F_{mn} = \partial_m V_n - \partial_n V_m + ig\,[\,V_m\,,\,V_n\,],$$

$$\eta_{mn} = \mathrm{diag}(-1\,,\,1\,,\,1\,,\,1)\,, \quad \epsilon_{0123} = -1\,.$$

In spinor indices $(\alpha, \beta = 1, 2)$

$$F_{\alpha\dot\alpha,\beta\dot\beta} \equiv \sigma^m{}_{\alpha\dot\alpha}\sigma^n{}_{\beta\dot\beta}F_{mn} = \frac{1}{2}\varepsilon_{\alpha\beta}f_{\dot\alpha\dot\beta} + \frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}f_{\alpha\beta},$$

where $f_{\alpha\beta}$, $f_{\dot\alpha\dot\beta}$ are the (1,0) and (0,1) components of the Yang—Mills strength tensor

$$f_{\dot\alpha\dot\beta} \equiv \varepsilon^{\alpha\gamma}F_{\gamma\dot\alpha,\alpha\dot\beta}\,, \qquad f_{\alpha\beta} \equiv \varepsilon^{\dot\alpha\dot\gamma}F_{\alpha\dot\gamma,\beta\dot\alpha}.$$

The σ-matrices are (see notations in [13])

$$\sigma^m = (-1\,,\,\vec\sigma)\,,$$

where 1 is a unit 2×2 matrix, and $\vec\sigma$ are Pauli matrices. The two—spinor indices are raised and lowered by means of the two-dimensional Levi—Civita tensors

$$\varepsilon^{12} = -\varepsilon_{12} = \varepsilon^{\dot1\dot2} = -\varepsilon_{\dot1\dot2} = +1\,.$$

The self-duality equations (1) in spinor indices are written as follows:

$$f_{\alpha\beta} = 0. \qquad (2)$$

Let us consider $N{=}1$ supersymmetric Yang—Mills theory [14, 15] with the Lagrangian

$$L = \mathrm{Tr}\Big\{-\frac{1}{4}F_{mn}F^{mn} - i\bar\lambda\bar\sigma^m\mathcal{D}_m\lambda + \frac{1}{2}D^2\Big\}, \qquad (3)$$

where $\mathcal{D}_m = \partial_m + ig\,[\,V_m\,,\,\cdot\,]$. Together with Bianchi identities, the equations of motion of the theory (3) take the form

$$\varepsilon^{\beta\gamma}\mathcal{D}_{\gamma\dot\beta} f_{\alpha\beta} + 4g\{\lambda_\alpha\,,\,\bar\lambda_{\dot\beta}\} = 0,$$

$$D = 0,$$

$$\mathcal{D}^{\alpha\dot\beta}\lambda_\alpha = 0,$$

$$\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} = 0. \qquad (4)$$

Equations (4) are invariant under the following $N{=}1$ supersymmetric transformations [13, 16]

$$\delta_\xi V_{\alpha\dot\alpha} = -2i(\xi_\alpha\bar\lambda_{\dot\alpha} + \bar\xi_{\dot\alpha}\lambda_\alpha),$$

$$\delta_\xi D = -\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} + \bar\xi_{\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}\lambda_\alpha,$$

$$\delta_\xi\lambda_\alpha = \frac{1}{2}\xi^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D),$$

$$\delta_\xi\bar\lambda_{\dot\alpha} = \frac{1}{2}\bar\xi^{\dot\beta}(f_{\dot\alpha\dot\beta} - 2i\varepsilon_{\dot\alpha\dot\beta}D), \qquad (5)$$

where $\xi_\alpha\,,\,\bar\xi_{\dot\alpha}$ are the parameters of $N = 1$ supersymmetric transformations (constant Weyl spinors).

In $N = 1$ superspace

$$(x^m\,;\,\theta_\alpha\,;\,\bar\theta_{\dot\alpha})$$

a spinor chiral superfield is defined by

$$W_\alpha = -\frac{1}{8g}\overline{\mathcal{D}\mathcal{D}}\big(e^{-2gV}\mathcal{D}_\alpha e^{2gV}\big)\,,$$

where V is a vector superfield in the Wess—Zumino gauge

$$V = -\theta^\alpha\bar\theta^{\dot\alpha}V_{\alpha\dot\alpha} + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta D\,,$$

and the covariant derivatives are

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\,, \qquad \overline{\mathcal{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\partial_{\alpha\dot\alpha}\,.$$

In component fields,

$$W_\alpha = -i\lambda_\alpha - \frac{i}{2}\theta^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) + \theta^\beta\bar\theta^{\dot\beta}\partial_{\beta\dot\beta}\lambda_\alpha +$$

$$+\theta\theta\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} - \frac{1}{4}\theta\theta\bar\theta^{\dot\alpha}\epsilon^{\beta\gamma}\partial_{\gamma\dot\alpha}(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) - \frac{i}{4}\theta\theta\bar\theta\bar\theta\Box\lambda_\alpha,$$

where

$$\Box = \partial_m\partial^m = -\frac{1}{2}\partial_{\alpha\dot\beta}\partial^{\alpha\dot\beta}\,.$$

The equations of motion (4) in the superfield formulation are written as

$$\nabla^\alpha W_\alpha = 0\,, \qquad (6)$$

where the gauge-covariant spinor derivative

$$\nabla_\alpha = \mathcal{D}_\alpha + i\,[\,\mathcal{A}_\alpha\,,\,\cdot\,]\,, \qquad \mathcal{A}_\alpha = -ie^{-2gV}\mathcal{D}_\alpha e^{2gV}\,.$$

It follows from (5) that the bispinor $f_{\alpha\beta}$ is transformed in the following way:

$$\delta_\xi f_{\alpha\beta} = 2i\Big\{(\xi_\alpha\mathcal{D}_{\beta\dot\beta}\bar\lambda^{\dot\beta} + \xi_\beta\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta}) +$$

$$+\bar\xi^{\dot\beta}(\mathcal{D}_{\alpha\dot\beta}\lambda_\beta + \mathcal{D}_{\beta\dot\beta}\lambda_\alpha)\Big\}. \qquad (7)$$

By definition, the system of super-self-duality equations must include the self-duality equations (2) and be invariant under the supersymmetric transformations (5). The invariance under supersymmetric transformations means that the supersymmetric variation of the super-self-duality equations gives the equations which are satisfied on the equations of motion or on the super-self-duality equations. This means that the equations

$$\delta_\xi f_{\alpha\beta} = 0 \qquad (8)$$

must lead to the super-self-duality equations.

Thus, relations (8) and (7) yield two equations

$$\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} = 0\,, \qquad (9)$$

$$\mathcal{D}_{\alpha\dot\beta}\lambda_\beta + \mathcal{D}_{\beta\dot\beta}\lambda_\alpha = 0\,. \qquad (10)$$

The equation of motion (9) can be included into the system of super-self-duality equations together with (2). Equation (10) must be made compatible with the corresponding equation of motion

$$\mathcal{D}^{\alpha\dot\alpha}\lambda_\alpha = 0 \qquad (11)$$

which can be rewritten in the form

$$\mathcal{D}_{\alpha\dot\beta}\lambda_\beta - \mathcal{D}_{\beta\dot\beta}\lambda_\alpha = 0\,. \qquad (12)$$

So, the super-self-duality equations must satisfy both the equations of motion and the equations which are obtained as a result of the supersymmetric transformation of the self-duality equations themselves. In other words, the first step in finding out the super-self-duality equations is to obtain conditions which satisfy the following system:

$$\mathcal{D}_{\alpha\dot\beta}\lambda_\beta + \mathcal{D}_{\beta\dot\beta}\lambda_\alpha = 0$$

$$\mathcal{D}_{\alpha\dot\beta}\lambda_\beta - \mathcal{D}_{\beta\dot\beta}\lambda_\alpha = 0\,. \qquad (13)$$

As the second step, we must investigate the invariance under supersymmetric transformations.

The simplest condition that satisfies both equations (13) is

$$\lambda_\alpha = 0 \tag{14}$$

which was introduced by I. Volovich. Then the variation $\delta_\xi \lambda_\alpha = 0$ gives $f_{\alpha\beta} = 0$, $D = 0$. So, one obtains the following system of super-self-duality equations [11, 12]:

$$f_{\alpha\beta} = 0,$$

$$D = 0,$$

$$\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} = 0,$$

$$\lambda_\alpha = 0, \tag{15}$$

which is closed under the supersymmetric transformations (5) and satisfies the equations of motion (4). In the superfield formalism, system (15) has the form

$$W_\alpha = 0\,. \tag{16}$$

In this paper, we propose another condition which satisfies system (13), namely

$$\mathcal{D}_{\alpha\dot\beta}\lambda_\beta = 0\,. \tag{17}$$

The supersymmetric variation of (17) gives

$$\delta_\xi(\mathcal{D}_{\alpha\dot\alpha}\lambda_\beta) = -\frac{1}{2}\xi^\gamma \mathcal{D}_{\alpha\dot\alpha} f_{\beta\gamma} + i\xi_\beta \mathcal{D}_{\alpha\dot\alpha} D +$$

$$+2g\xi_\alpha\{\bar\lambda_{\dot\alpha}, \lambda_\beta\} + 2g\bar\xi_{\dot\beta}\{\lambda_\alpha, \lambda_\beta\} = 0\,. \tag{18}$$

Taking into account that $f_{\alpha\beta} = 0$, $D = 0$, relation (18) yields

$$\{\lambda_\alpha\,,\, \bar\lambda_{\dot\beta}\} = 0\,, \quad \{\lambda_\alpha\,,\, \lambda_\beta\} = 0\,. \tag{19}$$

It can be proved that Eqs. (19) lead to the system which is closed under supersymmetric transformations [17]. So, we obtain the following system of super-self-duality equations:

$$f_{\alpha\beta} = 0,$$

$$D = 0,$$

$$\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} = 0,$$

$$\mathcal{D}_{\alpha\dot\beta}\lambda_\beta = 0,$$

$$\{\lambda_\alpha, \bar\lambda_{\dot\beta}\} = 0,$$

$$\{\lambda_\alpha\,,\, \lambda_\beta\} = 0. \tag{20}$$

By putting $\lambda_\alpha = 0$ (20) reduces to (15).

In terms of superfields, system (20) looks as

$$\nabla_\alpha W_\beta = 0\,. \tag{21}$$

The fact that the superfield equation (21) leads to super-self-duality was first noticed in [18]. The Abelian analog of (21) in [19] is called "relaxed super-self-duality".

The next way to obtain the systems of super-self-duality equations is the combination of conditions (14) and (17), which results in

$$\lambda^\gamma = 0\,, \quad \mathcal{D}_{\alpha\dot\beta}\lambda_\gamma = 0\,. \tag{22}$$

Here, we have two possibilities depending on $\gamma = 1$ or $\gamma = 2$. The two corresponding self-dual systems are ($\gamma = 1$ or $\gamma = 2$)

$$f_{\alpha\beta} = 0, \quad D = 0, \quad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} = 0,$$

$$\lambda^\gamma = 0, \quad \mathcal{D}_{\alpha\dot\beta}\lambda_\gamma = 0,$$

$$\{\lambda_\gamma, \bar\lambda_{\dot\beta}\} = 0, \quad \{\lambda_\gamma, \lambda_\gamma\} = 0. \tag{23}$$

In part, if $\gamma = 2$, system (23) can be written as

$$f_{\alpha\beta} = 0\,, \quad D = 0\,, \quad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} = 0\,,$$

$$\lambda_1 = 0\,, \quad \mathcal{D}_{\alpha\dot\beta}\lambda_2 = 0\,,$$

$$\{\lambda_2\,,\, \bar\lambda_{\dot\beta}\} = 0\,, \quad \{\lambda_2\,,\, \lambda_2\} = 0\,. \tag{24}$$

The superfield formulation of (23) is given by the superfield system which is a combination of (16) and (21):

$$W^\gamma = 0\,, \quad \nabla_\alpha W_\gamma = 0\,. \tag{25}$$

It should be mentioned that the self-duality equations in the Minkowski space (1) admit only complex solutions. It also refers to the solutions of the supersymmetrized systems (15), (20), (23). In the process of supersymmetrization, the self-dual gauge fields become complexified, because the gauge group is replaced by its complexification [20–22]. The problem of construction of self-dual gauge theories with real gauge fields can be tackled, as it is shown in [23–26].

Let us show that there is one more way to satisfy Eqs. (13), namely by adding the constraint for the spinor field

$$\lambda_1 = k\lambda_2\,,$$

where k is a real number, to Eq. (10). It is easy to verify that the system

$$\mathcal{D}_{\alpha\dot{\beta}}\lambda_\beta + \mathcal{D}_{\beta\dot{\beta}}\lambda_\alpha = 0$$

$$\lambda_1 = k\lambda_2 \tag{26}$$

satisfies system (13), i.e. (26) leads to the equation of motion (12). By the further supersymmetric variation of (26), we obtain the following system of super-self-duality equations:

$$f_{\alpha\beta} = 0$$

$$D = 0$$

$$\mathcal{D}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} = 0$$

$$\mathcal{D}_{\alpha\dot{\beta}}\lambda_\beta + \mathcal{D}_{\beta\dot{\beta}}\lambda_\alpha = 0$$

$$\lambda_1 = k\lambda_2$$

$$\{\lambda_\alpha\, ,\, \bar{\lambda}_{\dot{\beta}}\} = 0$$

$$\{\lambda_\alpha\, ,\, \lambda_\beta\} = 0\,. \tag{27}$$

The superfield formulation of (27) looks as

$$W_1 = kW_2$$

$$\nabla_\alpha W_\beta + \nabla_\beta W_\alpha = 0\,. \tag{28}$$

In fact, system (28) generalizes (25). If $k = 0$, (28) reduces to (25) for $\gamma = 2$. If $k = \infty$, (28) reduces to (25) for $\gamma = 1$.

It is believed that the obtained variants of super-self-duality equations can lead to new integrable systems by the corresponding dimensional reduction.

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ВАРІАНТИ СУПЕРАВТОДУАЛЬНОСТІ

А.М. Павлюк

Р е з ю м е

Запропоновано нові приклади систем рівнянь суперавтодуальності для суперсиметричної теорії Янга—Міллса. Системи записано в термінах компонентних полів і в термінах суперполів.