THRESHOLD SPATIALLY PERIODIC REORIENTATION OF THE DIRECTOR IN THE PLANAR CELL OF A NEMATIC LIQUID CRYSTAL

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UDC 532.783; 548-14

The influence of both the director anchoring energy at the surface of a nematic liquid crystal (NLC) planar cell and the ratio r between the Frank elastic constants on the threshold and the period of the spatially periodic (SP) reorientation of the director in an external dc electric field applied in parallel to the cell surface has been considered. The threshold value of the electric field and the spatial period of the director reorientation as functions of the polar and azimuthal anchoring energies and r have been calculated numerically. The range of r-values, where the SP reorientation of the director is possible, has been shown to broaden if the polar anchoring energy decreases and to narrow if the azimuthal anchoring energy decreases.

1. Introduction

The reorientation phenomena of the NLC director under the action of either an electric or magnetic field – in particular, the threshold reorientation — are often used in various electron-optical devices [1–3]. The attention was mainly focused on the director's reorientation which was uniform in the cell plane. Nevertheless, it is known [4] that an SP structure can also emerge in a flexoelectric liquid crystal cell, provided a threshold reorientation of the director from a planar into the homeotropic state. Later, it was shown in work [5] that, if the ratio $r = K_2/K_1$ between the Frank elastic constants is less than $r_c \approx 0.3$, an SP structure can arise in a planarly oriented NLC cell even if there is no flexopolarization. Those works used the simplest model of absolutely rigid director anchoring at the cell surface.

However, in spite of the fact that the threshold reorientation of the director is a bulk effect, its basic characteristics, such as the amplitude of the threshold field and the degree of director reorientation, substantially depend on the interaction between the liquid crystal and the cell surface [6, 7]. One of the most important parameters of this interaction is the director anchoring energy at the surface. A substantial dependence of the character of the SP director reorientation on the amplitude and the kind of its anchoring at the cell surface was pointed out in works [8–10]. In work [11], the dependence of the threshold SP reorientation of the director of a flexoelectric NLC on the anchoring energy in a homeotropically oriented cell was examined. The threshold SP reorientation of the director from a planar state into the homeotropic one in a flexoelectric NLC cell with arbitrary values of the anchoring energy was studied in work! [12].

The authors of works [13–16] have analyzed the influence of the elastic constant K_{24} on the appearance of spontaneous periodic distortions of the director in the cell of a planarly oriented NLC. In work [17], a relation between the elastic constant K_{24} and the parameters of a periodic structure that arises in a planarly oriented nematic cell at the Friedericksz transition in an external magnetic field has been considered.

In this work, a threshold SP reorientation of the director from a planar state into another planar state under the action of an electric field applied in parallel to the cell surface is studied. The opportunity of such a transition at $K_2/K_1 > 2$ has been indicated earlier in work [8].

2. Director Equation

Consider a plane-parallel cell confined by the planes $z = -L/2$ and $z = +L/2$ and containing an NLC possessing the initial planar orientation of its director along the Ox axis. The cell is embedded into an external constant uniform electric field with the strength vector directed along the Oy axis: $\vec{E} = (0, E, 0)$.

The free energy of the NLC cell can be written down in the form

$$F = F_{\rm el} + F_E + F_S \,,$$

where

$$F_{\rm el} = \frac{1}{2}\int\limits_V \Big\{ K_1 \left(\operatorname{div}\vec{n}\,\right)^2 + K_2 \left(\vec{n}\operatorname{rot}\vec{n}\,\right)^2 +$$

$$+ K_3 \left[\vec{n}\times\operatorname{rot}\vec{n}\,\right]^2 \Big\}\, dV -$$

$$-\frac{K_{24}}{2}\int\limits_V \operatorname{div}\big(\vec{n}\operatorname{div}\vec{n} + [\vec{n}\times\operatorname{rot}\vec{n}]\big)\, dV \,,$$

$$F_E = -\frac{\varepsilon_a}{8\pi}\int\limits_V \big(\vec{n}\vec{E}\,\big)^2\, dV, \qquad (1)$$

$$F_S = -\frac{W_\varphi}{2}\int\limits_{S_{1,2}} \cos^2\varphi\, dS - \frac{W_\theta}{2}\int\limits_{S_{1,2}} \cos^2\theta\, dS,$$

$$W_\varphi > 0, W_\theta > 0 \,.$$

Here, $\vec{n}$ is the director, $F_{\rm el}$ the elastic energy of the NLC, F_E the anisotropic contribution of the external electric field to the free energy, F_S the surface free energy of the NLC, $\varepsilon_a = \varepsilon_\parallel - \varepsilon_\perp > 0$ the anisotropy of the static dielectric permittivity, W_θ and W_φ are the polar and the azimuthal energy of director anchoring at the cell surface, respectively, and θ and φ the director deviation angles in the planes xz and xy, respectively. We notice that the surface free energy F_S for the nematic is written in the Rapini approximation [18], taking into account that the variation of the surface energy can differ if the director deviates from its easy axis in either an azimuthal or polar direction [19].

In the geometry specified, the reorientation of the director may result in the emergence of an SP structure along the Oy axis. Therefore, the solution for the director in the NLC bulk is searched in the form

$$\vec{n} = \vec{i}\cos\theta(y,z)\cos\varphi(y,z) +$$

$$+\vec{j}\cos\theta(y,z)\sin\varphi(y,z) + \vec{k}\sin\theta(y,z)\,, \qquad (2)$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the orts of the Cartesian coordinate system.

While determining the reorientation threshold, one may consider only small distortions of the director ($|\varphi|, |\theta| \ll 1$). Then, minimizing the free energy (1) over the angles θ and φ, we obtain the stationary linearized equations

$$r\,\frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} + (1-r)\,\frac{\partial^2\varphi}{\partial y\,\partial z} = 0\,,$$

$$\frac{\partial^2\varphi}{\partial y^2} + r\,\frac{\partial^2\varphi}{\partial z^2} + (1-r)\,\frac{\partial^2\theta}{\partial y\partial z} + \epsilon E^2\varphi = 0 \qquad (3)$$

and the corresponding boundary conditions

$$\left[\frac{W_\theta}{K_1}\,\theta \pm \left(\frac{\partial\theta}{\partial z} + (1-k_{24})\,\frac{\partial\varphi}{\partial y}\right)\right]_{z=\pm L/2} = 0\,,$$

$$\left[\frac{W_\varphi}{K_1}\,\varphi \pm \left(r\,\frac{\partial\varphi}{\partial z} - (r-k_{24})\,\frac{\partial\theta}{\partial y}\right)\right]_{z=\pm L/2} = 0\,, \qquad (4)$$

where the following notations are used: $\epsilon = \dfrac{\varepsilon_a}{4\pi K_1}$, $r = \dfrac{K_2}{K_1}$, and $k_{24} = \dfrac{K_{24}}{K_1}$.

Taking the symmetry of the system of equations (3) into account, its solution is tried in the form

$$\theta(y,z) = \cos(qy)\cdot\theta_1(z)\,, \qquad \varphi(y,z) = \sin(qy)\cdot\varphi_1(z)\,, \qquad (5)$$

where the functions $\theta_1(z)$ and $\varphi_1(z)$ satisfy the equations

$$\begin{pmatrix} \dfrac{d^2}{dz^2} - rq^2 & (1-r)q\dfrac{d}{dz} \\ -(1-r)q\dfrac{d}{dz} & r\dfrac{d^2}{dz^2} - q^2 + \epsilon E^2 \end{pmatrix} \begin{pmatrix} \theta_1(z) \\ \varphi_1(z) \end{pmatrix} = 0\,. \qquad (6)$$

Putting

$$\begin{pmatrix} \theta_1(z) \\ \varphi_1(z) \end{pmatrix} = e^{\lambda z}\begin{pmatrix} \theta_{10} \\ \varphi_{10} \end{pmatrix}, \qquad (7)$$

in Eq. (6), we obtain a homogeneous system of two algebraic equations to determine the unknown coefficients θ_{10} and φ_{10}. The condition that this system of equations possesses a nontrivial solution brings about the equation for λ:

$$r\left(\lambda^2 - q^2\right)^2 + \epsilon E^2\left(\lambda^2 - q^2\right) - \epsilon E^2 q^2\,(r-1) = 0\,,$$

$$r \neq 0\,. \qquad (8)$$

The solutions of Eq. (8) are $\lambda = \pm ip_1$ and $\pm p_2$, where p_1 and p_2 are the real quantities

$$p_1 = \sqrt{\frac{1}{2r}\left[\epsilon E^2 + \sqrt{(\epsilon E^2)^2 + 4r(r-1)q^2\epsilon E^2}\,\right] - q^2}\,,$$

$$p_2 = \sqrt{q^2 - \frac{1}{2r}\left[\epsilon E^2 - \sqrt{(\epsilon E^2)^2 + 4r(r-1)q^2\epsilon E^2}\,\right]}\,. \quad (9)$$

From Eqs. (6), we also find that

$$\frac{\theta_{10}}{\varphi_{10}} = \frac{(r-1)q\lambda}{\lambda^2 - rq^2}\,.$$

The general solution of the system of equations (6) looks like

$$\theta_1(z) = \beta_1(-a_1 \sin p_1 z + a_2 \cos p_1 z) +$$

$$+\beta_2(b_1 \operatorname{sh} p_2 z + b_2 \operatorname{ch} p_2 z),$$

$$\varphi_1(z) = a_1 \cos p_1 z + a_2 \sin p_1 z + b_1 \operatorname{ch} p_2 z + b_2 \operatorname{sh} p_2 z, \quad (10)$$

where

$$\beta_1 = -\frac{(r-1)qp_1}{rq^2 + p_1^2}\,, \qquad \beta_2 = -\frac{(r-1)qp_2}{rq^2 - p_2^2}\,. \quad (11)$$

Here, a_i and b_i $(i = 1, 2)$ are arbitrary constants, the values of which are to be determined from the boundary conditions (4).

3. Influence of the Polar Anchoring Energy

Let us assume that the azimuthal anchoring energy W_φ of the director at the cell surface is infinitely large ($W_\varphi \to \infty$), and the polar one W_θ can acquire an arbitrary value. In this case, the boundary conditions (4) look like

$$\left[\frac{W_\theta}{K_1}\,\theta_1 \pm \frac{d\theta_1}{dz}\right]_{z=\pm L/2} = 0\,,$$

$$\varphi_1|_{z=\pm L/2} = 0\,. \quad (12)$$

Substituting solution (10) into the boundary conditions (12), we obtain a homogeneous system of four algebraic equations to determine the unknown coefficients a_i and b_i $(i = 1, 2)$. The condition for this system to possess a nontrivial solution gives the following equation:

$$\beta_2 \operatorname{ctg}\frac{p_1 L}{2}\left(\frac{W_\theta}{K_1} + p_2 \operatorname{cth}\frac{p_2 L}{2}\right) +$$

$$+\beta_1 \operatorname{cth}\frac{p_2 L}{2}\left(\frac{W_\theta}{K_1} + p_1 \operatorname{ctg}\frac{p_1 L}{2}\right) = 0\,. \quad (13)$$

The solution of this equation, taking into account relations (9) and (11), gives the electric field E as a function of the wave number q. The threshold E_c for the instability to emerge is determined by the minimum in the curve $E(q)$. In order to find the threshold value E_c of the electric field and the corresponding value q_c of the wave number, it is necessary to solve Eq. (13) numerically.

We note that the condition for the threshold SP structure of the director field to appear is equivalent to the inequality

$$\left.\frac{dE}{dq}\right|_{q=0} < 0\,. \quad (14)$$

In this case, differentiating Eq. (13) with respect to q at the point $q = 0$ and carrying out the necessary algebraic transformations, Eq. (14) leads to the inequality

$$(r-1)^2 - \frac{\pi^2}{8}\frac{\varepsilon_\theta + 2}{\varepsilon_\theta}\,r(r-2) < 0\,, \quad (15)$$

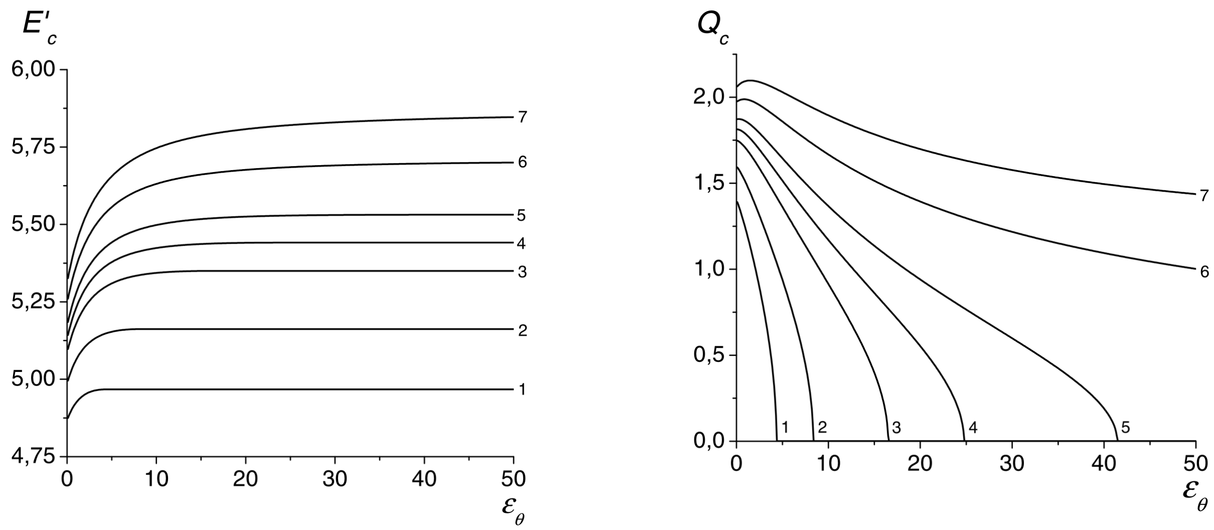
which couples the ratio r between the Frank elastic constants and the value of the polar anchoring energy $\varepsilon_\theta = (W_\theta L)/K_1$ and defines the range of the SP director reorientation.

Fig. 1 exposes the calculated dependences of the dimensionless threshold field $E'_c = \sqrt{\epsilon}E_c L$ and the corresponding value of the dimensionless wave number $Q_c = q_c L$ on the amplitude of the polar anchoring energy ε_θ of the director at the cell surface for various values of the ratio r between the Frank elastic constants. The threshold value E_c of the electric field expectedly increases with the polar anchoring energy ε_θ. In this case, the period $\lambda_c = 2\pi/q_c$ of the arising spatial structure of the director is a nonmonotonous function of the anchoring energy ε_θ at $r \geqslant r_0 = 1 + \frac{\pi}{\sqrt{\pi^2-8}} \approx 3.3$ (Fig. 1,*b*). If $2 < r < r_0$, the period λ_c of the director's spatial structure increases monotonously with ε_θ.

Figure 1,*b* demonstrates that, according to inequality (15), there exists a finite critical value of the polar anchoring energy for every value of r within the interval $2 < r < r_0$

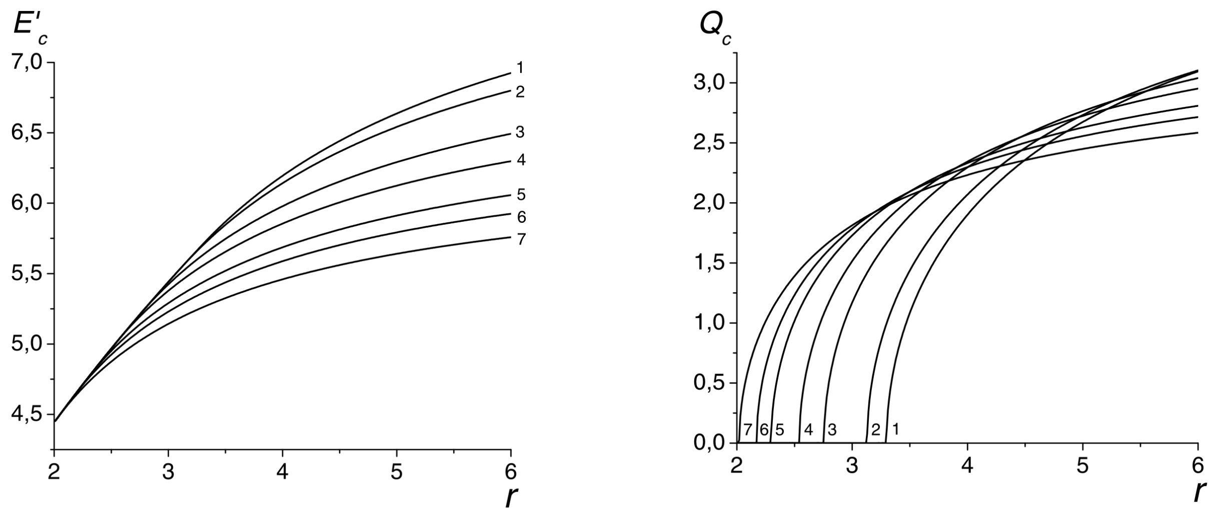
$$\varepsilon_{\theta\mathrm{th}}(r) = \frac{2\pi^2 r(r-2)}{8(r-1)^2 - \pi^2 r(r-2)}$$

at which $Q_c = 0$. If $\varepsilon_\theta < \varepsilon_{\theta\mathrm{th}}$, the Freedericksz transition accompanied by the formation of an SP structure takes place, while, at $\varepsilon_\theta > \varepsilon_{\theta\mathrm{th}}$, only the Freedericksz transition with a uniform (along the Oy axis) distribution of the director is possible. In the case



a **b**

Fig. 1. Dependences of the threshold field E'_c (a) and the threshold wave number Q_c (b) on the amplitude of the polar anchoring energy ε_θ for various values of the ratio $r = 2.5$ (*1*), 2.7 (*2*), 2.9 (*3*), 3.0 (*4*), 3.1 (*5*), 3.3 (*6*), and 3.5 (*7*)



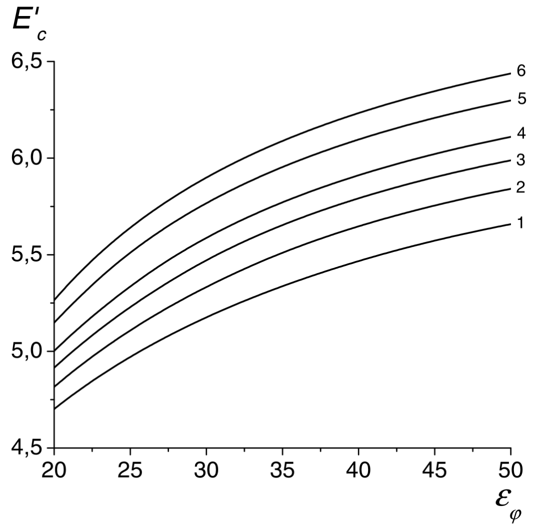
a **b**

Fig. 2. Dependences of the threshold field E'_c (a) and the threshold wave number Q_c (b) on the parameter r for various $\varepsilon_\theta = \infty$ (*1*), 50 (*2*), 10 (*3*), 5 (*4*), 2 (*5*), 1 (*6*), and 0.1 (*7*)

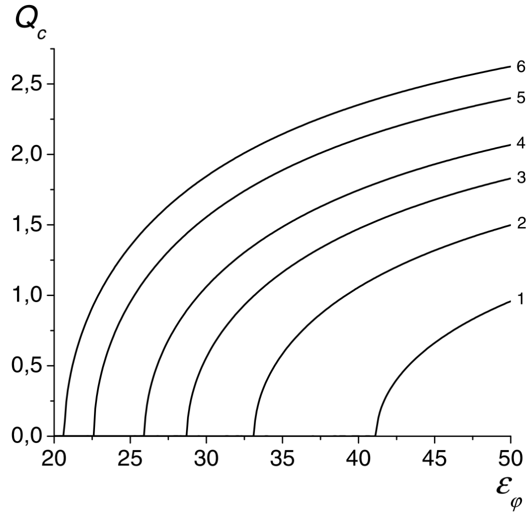
where $r \geqslant r_0$, the critical value of the polar anchoring energy is infinitely large ($\varepsilon_{\theta\mathrm{th}} \to \infty$). For the ratio's values $r \leqslant 2$, only the uniform Fredericksz transition occurs, which is in agreement with the result of work [8].

The dependences of the threshold field E_c and the corresponding wave number q_c on the magnitude of the ratio r for various polar anchoring energies ε_θ are depicted in Fig. 2. As the parameter r grows, the amplitude of the threshold electric field E_c increases, while the corresponding period λ_c of the arising spatial structure of the director falls down monotonously. From Fig. 2,*b*, one can see that for every value of the polar anchoring energy ε_θ, there exists a critical value of the ratio between the Frank elastic constants

$$r_{\mathrm{th}}(\varepsilon_\theta) = 1 + \frac{\pi\sqrt{\varepsilon_\theta + 2}}{\sqrt{(\pi^2 - 8)\varepsilon_\theta + 2\pi^2}}\,.$$



a



b

Fig. 3. Dependences of the threshold field E'_c (*a*) and the threshold wave number Q_c (*b*) on the amplitude of the azimuthal anchoring energy ε_φ for various values of the ratio $r = 4.5$ (*1*), 5.0 (*2*), 5.5 (*3*), 6.0 (*4*), 7.0 (*5*), and 8.0 (*6*)

at which $Q_c = 0$. If $r < r_{\text{th}}$, a uniform Friedericksz transition takes place only, while, if $r > r_{\text{th}}$, the Friedericksz transition with the formation of an SP director structure does. The critical value r_{th} of the ratio between the Frank elastic constants increases monotonously as the polar anchoring energy ε_θ grows, in particular, from $r_{\text{th}} = 2$ at $\varepsilon_\theta = 0$ to $r_{\text{th}} = r_o$ in the case of absolutely rigid anchoring (curve *1*).

From Figs. 1 and 2, one can see that the range of the parameter r, where the SP structure of the director field exists, extends if the value of the polar anchoring energy ε_θ of the director at the cell surface diminishes.

4. Influence of the Azimuthal Anchoring Energy

Now, let us assume that the polar anchoring energy of the director at the cell surface is infinitely large ($W_\theta \to \infty$), and the azimuthal one W_φ can acquire an arbitrary value. In this case, the boundary conditions (4) look like

$$\theta_1|_{z=\pm L/2} = 0\,,$$

$$\left[\frac{W_\varphi}{K_1}\,\varphi_1 \pm r\,\frac{d\varphi_1}{d\,z}\right]_{z=\pm L/2} = 0\,. \tag{16}$$

Note that, as follows from the boundary conditions (12) and (16), the elastic factor K_{24} gives no contribution to the SP reorientation of the director in the framework of the model considered, where either the polar or azimuthal anchoring energy of the director at the surface is assumed infinitely large.

Analogously to what was done in the case of an arbitrary polar anchoring energy, we substitute solution (10) into the boundary conditions – in this case, Eq. (16) – and again obtain the corresponding homogeneous algebraic system of equations to determine the coefficients a_i and b_i ($i = 1, 2$). The requirement of its nontrivial solution gives the equation

$$\beta_2\left(\frac{W_\varphi}{K_1}\,\text{ctg}\,\frac{p_1 L}{2} - r p_1\right) +$$

$$+\beta_1\left(\frac{W_\varphi}{K_1}\,\text{cth}\,\frac{p_2 L}{2} + r p_2\right) = 0 \tag{17}$$

to determine the dispersion dependence $E(q)$. After the numerical minimization of the latter, we obtain the threshold value of the electric field and the corresponding wave number.

The dependences of threshold field E_c and the corresponding wave number q_c on the magnitude of the azimuthal anchoring energy $\varepsilon_\varphi = (W_\varphi L)/K_1$ are exhibited in Fig. 3 for several values of the ratio r between the Frank elastic constants. Analogously to the case of arbitrary values of the polar anchoring energy, the amplitude of the threshold field E_c increases monotonously with the azimuthal anchoring energy ε_φ and the ratio r between the elastic constants. The period of the director orientation $\lambda_c = 2\pi/q_c$ falls down

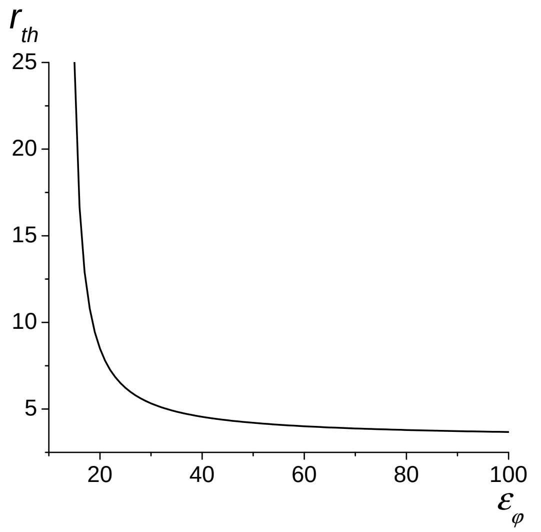


Fig. 4. Dependence of the critical value $r_{\rm th}$ of the parameter r on the amplitude of the azimuthal anchoring energy ε_φ

monotonously with the growth of both the azimuthal anchoring energy ε_φ and the parameter r. In this case, for every given value of the parameter $r > r_0$, there exists such a critical value of the azimuthal anchoring energy $\varepsilon_{\varphi\rm th}(r)$, which corresponds to $Q_c = 0$, that, if $\varepsilon_\varphi < \varepsilon_{\varphi\rm th}$, a homogeneous Friedericksz transition takes place; otherwise, a Friedericksz transition with the formation of the SP structure of the director occurs. The critical value of the azimuthal anchoring energy for the ratio r, provided $r \leqslant r_0$, is infinitely large, i.e. only an ordinary Friedericksz transition is possible in this case.

We did not succeed in obtaining criterion (14) for the emergence of the threshold SP structure of the director in an analytic form for arbitrary values of the azimuthal anchoring energy ε_φ. The results of numerical calculations of the critical value of r as a function of the azimuthal anchoring energy ε_φ are shown in Fig. 4. Here, the range $r > r_{\rm th}$ is the region where the SP structure of the director exists. From Fig. 4, one can see that, provided the azimuthal anchoring energy increases, the critical value of the ratio between the Frank elastic constants decreases and approaches the value of r_o in the limit of absolutely rigid anchoring ($\varepsilon_\varphi \to \infty$). Hence, a reduction of the azimuthal anchoring energy of the director at the cell surface makes the r-range, where the SP structure of the director exists, narrower.

Thus, the values of the polar and azimuthal anchoring energies of the director at the cell surface substantially affect not only the threshold magnitude of the electric field and the period of the arising spatial structure of the director, but also the range of admissible values of the ratio between the Frank elastic constants. In this case, a reduction of the polar anchoring energy expands the interval of admissible values of the parameter r, while a reduction of the azimuthal anchoring energy narrows it.

To summarize, we notice that the values of the ratio K_2/K_1 between the nematic elastic constants, which are quoted in the available scientific literature, are smaller than unity. At the same time, it is known that, while the temperature approaches the nematic–smectic phase transition point, this ratio grows abnormally [1,3]. This is why the phenomenon of the planar periodic reorientation of the director, which has been considered here, is expected to occur just in this temperature range.

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Received 28.04.05.
Translated from Ukrainian by O.I. Voitenko

ПОРОГОВА ПРОСТОРОВО-ПЕРІОДИЧНА ПЕРЕОРІЄНТАЦІЯ ДИРЕКТОРА В ПЛАНАРНІЙ КОМІРЦІ НЕМАТИЧНОГО РІДКОГО КРИСТАЛА

М.Ф. Ледней, І.П. Пінкевич

Р е з ю м е

Розглянуто вплив енергії зчеплення директора з поверхнею планарної комірки нематичного рідкого кристала (НРК) і величини відношення r пружних сталих Франка на поріг і період просторово-періодичної переорієнтації директора у зовнішньому постійному електричному полі, направленому паралельно поверхні комірки. Проведено чисельні розрахунки порогового електричного поля та просторового періоду директора в залежності від значень полярної і азимутальної енергії зчеплення та r. Встановлено, що область значень r, для яких можлива просторово-періодична переорієнтація директора, розширюється при зменшенні полярної енергії зчеплення і звужується при зменшенні азимутальної енергії зчеплення.