ON THE INFLUENCE OF THE RESISTIVE IONOSPHERE ON THE BALLOONING STABILITY OF A MAGNETOSPHERIC PLASMA

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We investigate the stability of pressure-driven MHD perturbations of the magnetospheric plasma in the framework of a dipolar model of the geomagnetic field with regard for the boundary conditions of the ionosphere. We consider the latter as a thin layer with finite conductivity. We especially emphasize the investigation of the influence of the ionospheric conductivity on the plasma stability. We demonstrate that the stability criterion does not depend on the ionospheric conductivity and is determined by flute modes. An analytical stability criterion is derived.

1. Introduction

Pressure-driven MHD perturbations are currently believed to determine the plasma stability both in the inner magnetosphere and the magnetotail. Such perturbations are called ballooning modes. They include a special subclass of flute modes (see [1—3] and references therein for more information). In this work, we determine the stability limit of these modes taking into account the finite ionospheric conductivity.

In many earlier works [2, 4, 5], it was pointed out that the oscillation spectrum and the stability of ballooning perturbations depend on the ionospheric conductivity. A significant progress in understanding this dependence was achieved in [6]. The results of this work prompted us to derive an analytical stability limit taking it into account.

We use an axially symmetric dipolar model of the geomagnetic field which was introduced in our previous works [7, 8]. Following [9], we assume that the magnetospheric plasma equilibrium is provided by a toroidal current. We describe ballooning perturbations in terms of the equations for small perturbations that were previously obtained by us in [8] in the eikonal approximation [10]. The boundary conditions describing the finite ionospheric conductivity were obtained in [11, 12] following the approach developed in [6, 13].

2. Initial Equations

In our preceding works [7, 8], we showed that the most unstable MHD perturbations in the plasma geometry with a dipolar magnetic field and a pressure gradient are described by two components ξ and τ of the displacement vector $\vec{\xi}$

$$\vec{\xi} = \xi\frac{\nabla\Psi}{|\nabla\Psi|^2} + \eta\frac{\vec{B}\times\nabla\Psi}{\left|\vec{B}\right|^2} + \tau\frac{\vec{B}}{\left|\vec{B}\right|^2},$$

where Ψ is a poloidal magnetic flux, and $\vec{B}$ is a magnetic field. These components satisfy the equations for small perturbations

$$\Omega^2\frac{c^6}{a}\xi + \left(\frac{\xi'}{a}\right)' + \frac{4c^4}{a^2}\left(T_0 + \frac{\alpha\beta}{\gamma}\xi\right) = 0, \quad (1)$$

$$\Omega^2\tau + \frac{T_0'}{c^3} = 0, \quad (2)$$

where

$$T_0 = \frac{\beta}{c^3}\left[\left(\frac{c^6}{a}\tau\right)' - \frac{4c^4}{a^2}\xi\right],$$

$$x = \sin\theta, \quad a = 1 + 3x^2, \quad c = 1 - x^2,$$

$$\alpha = -\frac{L}{p}\frac{dp}{dL}, \quad \beta = \frac{\gamma p}{B_0^2}, \quad \gamma = \frac{5}{3}.$$

Here, L is McIlwain parameter, B_0 is the norm of a magnetic field on the geomagnetic equator, θ is a poloidal angle counted from the geomagnetic equator northwards (geomagnetic latitude), $\Omega = \frac{\omega}{\omega_A}$, ω is a complex frequency of perturbations, $\omega_A = \frac{B_0}{L\sqrt{\rho}}$ is the Alfvén frequency, a prime denotes the derivative with respect to x, and all other notations are conventional.

Equations (1) and (2) must be supplied with ionospheric boundary conditions. Since the magnetosphere is significantly larger than the ionosphere, the latter can be considered in the first approximation as a rigid thin spherical plasma layer with finite conductivity. On this layer, the usual conditions of the zero normal component of the velocity and the closure of currents [10,13] should hold. For the considered perturbations, these two conditions lead to the following boundary conditions [11]:

$$\Omega\xi + \frac{i\delta}{a}\left[2x\xi' - c^5\left(T_0 + \frac{\alpha\beta}{\gamma}\xi\right)\right]\bigg|_{x=\pm x_0} = 0, \tag{3}$$

$$\xi + 2\tau x c|_{x=\pm x_0} = 0. \tag{4}$$

Here, $\delta = \frac{1}{\Sigma_P \omega_A}$ is the squared skin layer thickness, $x_0 = \sin\theta_0$, θ_0 is the geomagnetic latitude of the "magnetic surface — ionosphere" intersection, and Σ_P is the integral Pedersen conductivity.

The linear homogeneous boundary-value problem (1)—(4) is an eigenvalue problem. The frequencies Ω are the eigenvalues (eigenfrequencies), and the corresponding amplitudes ξ and τ are eigenfunctions. Thus, the formulated boundary-value problem determines the spectrum of MHD perturbations in the plasma equilibrium in the dipolar geometry of the geomagnetic field with a resistive ionosphere.

When the ionosphere is perfectly conducting ($\Sigma_P \to \infty$, $\delta \to 0$), Eqs. (1), and (2) remain true and the boundary conditions take the form

$$\xi|_{x=\pm x_0} = 0, \tag{5}$$

$$\tau|_{x=\pm x_0} = 0. \tag{6}$$

When the ionosphere is insulating ($\Sigma_P \to 0, \delta \to \infty$), the boundary condition (3) takes the form

$$2x\xi' - c^5\left(T_0 + \frac{\alpha\beta}{\gamma}\xi\right)\bigg|_{x=\pm x_0} = 0, \tag{7}$$

$$\xi + 2\tau x c|_{x=\pm x_0} = 0. \tag{8}$$

Note that Eqs. (1) and (2) with the boundary conditions (3) and (4) are equivalent to the spectral problem for a non-self-adjoint operator, while the same equations with the boundary conditions (5), (6) or (7), (8) are equivalent to the spectral problem for a self-adjoint operator.

3. Energy Analysis

If the eigenfrequencies Ω of Eqs. (1) and (2) have positive imaginary parts, an instability can occur. Solving these equations with variable coefficients can be a complicated problem. A certain information on the stability and eigenfrequencies of perturbations can be acquired with the help of the energy principle [14]. For this purpose, let us multiply (1) and (2), respectively, by a complex conjugate amplitude ξ^* and $\frac{c^6}{a}\tau^*$, add them, and integrate with respect to x. These transformations give us

$$K\Omega^2 + iS\Omega - W = 0, \tag{9}$$

where

$$K = \int_{-x_0}^{+x_0} \frac{c^6}{a}\left(|\xi|^2 + c^3|\tau|^2\right)dx, \tag{10}$$

$$S = \frac{|\xi|^2}{2\delta x}\bigg|_{-x_0}^{+x_0}, \tag{11}$$

$$W = \int_{-x_0}^{+x_0}\left[\frac{|\xi'|^2}{a} + \beta\left|\left(\frac{c^6}{a}\tau\right)' - \frac{4c^4}{a^2}\xi\right|^2 - \right.$$

$$\left. -\frac{\alpha\beta}{\gamma}\frac{4c^4}{a^2}|\xi|^2\right]dx - \left(\frac{\alpha\beta}{\gamma}\frac{c^5}{xa}|\xi|^2\right)\bigg|_{-x_0}^{+x_0} = 0. \tag{12}$$

To derive Eqs. (9)—(12), we used the boundary conditions (3), (4).

Note that all coefficients in (9) are real quantities and allow a simple physical interpretation. K is the kinetic energy of the perturbation, S is the energy lost in the ionospheric layer due to its finite conductivity, and W is the potential energy of the perturbation.

A graphical solution of Eq. (9) is presented in Fig. 1. One can see that $W > 0$ implies $\mathrm{Im}\,\Omega < 0$, and the perturbations are stable. The case where $W > \frac{S^2}{4K}$ corresponds to decaying oscillations. When $W < 0$,

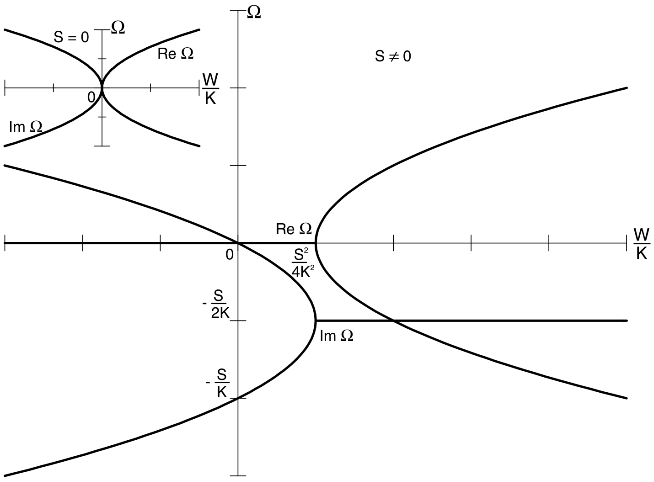

Fig. 1. Dimensionless eigenfrequency and the growth rate of perturbations versus the ratio of their potential and kinetic energies in the presence (main plot) and in the absence (small plot) of energy losses in the ionospheric layer

$\operatorname{Im}\Omega > 0$, and the perturbations become unstable. Thus, the equation

$$W\left(\xi,\tau\right)=0 \tag{13}$$

defines the stability limit of perturbations *regardless of the ionospheric conductivity*.

Usually, the considered perturbations (see, e.g., [1, 2, 8]) become unstable due to the combined influence of the negative field line curvature and the outward pressure gradient [the last integrand in (12)]. However, a new source of instability is present in the case of a resistive boundary associated with the pressure gradient in the ionospheric layer [the last term in (12)].

4. Boundary-Value Problem for the Stability Limit

One can see from Fig. 1 that the growing perturbations have zero frequencies. Thus, the potential energy functional W is self-adjoint in the instability region and can be rewritten as

$$W=\int\limits_{-x_0}^{+x_0}\left[\frac{|\xi'|^2}{a}+\beta\left|\left(\frac{c^6}{a}\tau\right)'-\frac{4c^4}{a^2}\xi\right|^2-\right.$$

$$\left.-\frac{\alpha\beta}{\gamma}\frac{4c^4}{a^2}\left|\xi\right|^2\right]dx-\left.\left(\frac{\alpha\beta}{\gamma}\frac{c^5}{xa}\left|\xi\right|^2\right)\right|_{-x_0}^{+x_0}. \tag{14}$$

Now the amplitudes ξ and τ are real quantities.

Using Eqs. (13) ad (14), we will find the stability limit of pressure-driven perturbations through minimizing the potential energy functional W. To minimize it, we follow the method in [14] which is based on omitting the terms which are always positive through the corresponding choice of amplitudes ξ and τ. Equating the minimized functional to zero, we simultaneously obtain both the expression for the stability limit and the class of amplitudes ξ and τ which ensures the most unstable situation.

Considering the amplitudes ξ and τ to be independent, we minimize the input of the stabilizing terms. First, we minimize the functional over τ. Because τ is present only in the second term of (14) which is proportional to T_0^2, the functional W can be minimized for fixed values of ξ for all values of τ. Varying W with respect to τ, we obtain

$$\left[\frac{1}{c^3}\left(\left(\frac{c^6}{a}\tau\right)'-\frac{4c^4}{a^2}\xi\right)\right]'=0,$$

whence

$$\frac{1}{c^3}\left(\left(\frac{c^6}{a}\tau\right)'-\frac{4c^4}{a^2}\xi\right)=C. \tag{15}$$

The constant C can be derived, by integrating (15) with respect to x along the field line:

$$C=-\frac{\frac{c_0^5}{2a_0x_0}\left(\xi|_{+x_0}+\xi|_{-x_0}\right)+\int\limits_{-x_0}^{+x_0}\frac{4c^4}{a^2}\xi dx}{\int\limits_{-x_0}^{+x_0}c^3dx}. \tag{16}$$

Substituting (16) into (14), we obtain the following expression for W:

$$W=\int\limits_{-x_0}^{+x_0}\left[\frac{(\xi')^2}{a}-\frac{\alpha\beta}{\gamma}\frac{4c^4}{a^2}\xi^2\right]dx+$$

$$+\frac{\beta}{\int\limits_{-x_0}^{x_0}c^3dx}\left(\frac{c_0^5}{2a_0x_0}\left(\xi|_{+x_0}+\xi|_{-x_0}\right)+\right.$$

$$\left.+\int\limits_{-x_0}^{x_0}\frac{4c^4}{a^2}\xi dx\right)^2-\frac{\alpha\beta}{\gamma}\frac{c_0^5}{2a_0x_0}\left(\xi^2\big|_{+x_0}+\xi^2\big|_{-x_0}\right). \tag{17}$$

The stability limit of the considered MHD perturbations is defined now by the equation

$$W\left(\xi\right)=0. \tag{18}$$

The further minimization of (17) depends on ξ which should satisfy the boundary condition

$$2x\xi' - \frac{\alpha\beta}{\gamma}c^5\xi + \beta c^5 \frac{\frac{c_0^5}{2a_0x_0}\left(\xi|_{+x_0} + \xi|_{-x_0}\right)}{\int\limits_{-x_0}^{x_0} c^3 dx} +$$

$$\left. + \beta c^5 \frac{\int\limits_{-x_0}^{x_0} \frac{4c^4}{a^2}\xi dx}{\int\limits_{-x_0}^{x_0} c^3 dx}\right|_{x=\pm x_0} = 0, \qquad (19)$$

as follows from (3) and (16).

Now let us derive a differential equation for the amplitude ξ, by solving problem (18) for the stationary value of a functional (equal to zero) with the boundary condition (19). The problem formulated by us coincides with the main problem of the variation calculus which comes to finding such amplitude ξ that the functional W remains equal to zero with slight variations of ξ. To find the sought amplitude ξ, we calculate the first variation derivative of W and equate it to zero. This gives us a differential equation for the amplitude ξ at the stability limit as

$$\left(\frac{\xi'}{a}\right)' + \frac{\alpha\beta}{\gamma}\frac{4c^4}{a^2}\xi - \beta\frac{4c^4}{a^2}\frac{\int\limits_{-x_0}^{x_0} \frac{4c^4}{a^2}\xi dx}{\int\limits_{-x_0}^{x_0} c^3 dx} -$$

$$-\beta\frac{4c^4}{a^2}\frac{\frac{c_0^5}{2x_0a_0}\left(\xi|_{+x_0} + \xi|_{-x_0}\right)}{\int\limits_{-x_0}^{x_0} c^3 dx} = 0. \qquad (20)$$

This together with the boundary condition (19) is equivalent to the above-mentioned variation problem. Thus, from the mathematical point of view, the problem of determination of the stability limit reduces to the boundary-value problem (19), (20). The latter has a solution only if a certain relation between α and β parameters exists. This relation determines the stability limit.

In conclusion of this section, we mention that, as one can easily persuade oneself with the help of Eqs. (1), (2), (7), and (8), Eqs. (19) and (20) hold for the insulating ionosphere. Thus, the stability limits for resistive and insulating boundaries coincide. This result conforms to Theorem 2 in [6] which states that "the ballooning instability occurs for resistive bounding ends if, and only if, it occurs when the ends are insulators".

5. Stability Limit

Prior to solving the boundary-value problem (19), (20), we will analyze functional (17). It contains two stabilizing terms. A term proportional to $(\xi')^2$ describes the stabilization by resistive boundaries, i.e. the ionosphere. The last term in W describes the stabilization by compressibility. These two terms cannot vanish simultaneously. Thus, we consider analytically only such situations when only one of them vanishes. Other situations will be considered numerically.

5.1. Flute modes

Flute modes, which are also called interchange modes, have a constant amplitude ξ = const. This means that the corresponding field line shifts as a whole without changing its shape. In this case, the plasma stability is provided only by its compressibility. These perturbations are well known in magnetospheric studies, and the stability criterion for them

$$\frac{\alpha}{\gamma} = 4 \qquad (21)$$

was obtained in [15]. Equations (19), (20) have a solution $\xi = \text{const} \neq 0$, when the following condition holds:

$$\frac{\alpha}{\gamma} = \frac{\int\limits_{-x_0}^{x_0} \frac{4c^4}{a^2} dx + \frac{c_0^5}{x_0a_0}}{\int\limits_{-x_0}^{x_0} c^3 dx}. \qquad (22)$$

The stability limit (22) strongly differs from (21) and describes less stable perturbations.

5.2. Incompressible perturbations

Let us now consider incompressible perturbations which correspond to the condition

$$\text{div}\vec{\xi} \sim T_0 \sim C = 0.$$

In this case, the functional W loses a term proportional to T_0^2. This means that

$$\frac{c_0^5}{2x_0a_0}\left(\xi|_{+x_0} + \xi|_{-x_0}\right) + \frac{\int\limits_{-x_0}^{x_0} \frac{4c^4}{a^2}\xi dx}{\int\limits_{-x_0}^{x_0} c^3 dx} = 0. \qquad (23)$$

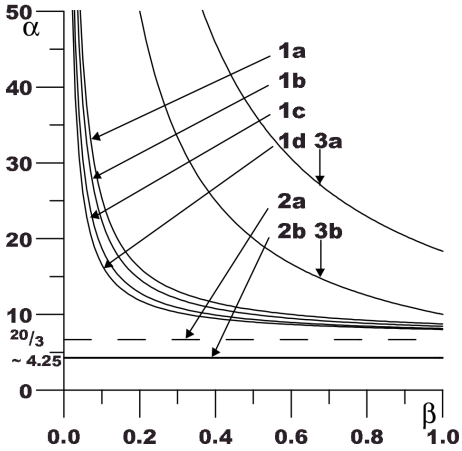

Fig. 2. Stability limits for various classes of eigenperturbations. *1* — ballooning modes with perfectly conductive ionosphere (a — L=2, b — L=3, c — L=4, d — L=7); *2* — flute modes (a — Gold criterion (21), b — Eq. (22)); *3* — ballooning modes with the resistive or insulating ionosphere (a — compressible, b — incompressible)

For such perturbations, Eq. (20) takes the form

$$\left(\frac{\xi'}{a}\right)' + \frac{\alpha\beta}{\gamma}\frac{4c^4}{a^2}\xi = 0, \tag{24}$$

and the boundary condition (19) becomes

$$2x\xi' - \frac{\alpha\beta}{\gamma}c^5\xi\Big|_{x=\pm x_0} = 0. \tag{25}$$

It may seem that Eq. (24) contains the excess number of boundary conditions, but Eq. (23) obviously follows from Eqs. (24) and (25), and the problem is self-adjoint and thus solvable numerically.

5.3. Numerical analysis

One could expect that, besides two above-considered classes of perturbations, other MHD perturbations may occur. Their stability limit and the stability limit of the incompressible modes can be calculated only numerically. The numerical results are given in Fig. 2 as the $\alpha(\beta)$ plot. It is seen that most unstable are the flute perturbations with the stability limit (22). They exist with the insulating and resistive ionospheres. They are absent with a perfectly conductive boundary. In this case, as one can persuade oneself with the help of Eqs. (1), (2), (5), and (6), only the ballooning perturbations exist (see details in [7]) with the stability limits lying higher than limit (22). Above them lies the stability limit of the incompressible modes described by Eqs. (24) and (25) with a resistive boundary. And even higher lies the stability limit of the compressible ballooning modes ($\mathrm{div}\vec{\xi} \neq 0$).

Thus, the numerical analysis states that the magnetospheric MHD stability is determined by the flute modes with a resistive boundary and is described by Eq. (22). It lies lower than one given by the Gold criterion (21).

6. Conclusion

In this work, we have investigated the magnetospheric plasma stability with respect to pressure-driven MHD perturbations. This investigation was aimed at determining the stability limit of these perturbations and its dependence on the ionospheric conductivity.

We have found out that the stability criterion is determined by the flute modes and does not depend on the ionospheric conductivity, with the exception for the special case of a perfectly conductive boundary, when the stability is determined by the ballooning modes. We also obtained the analytical stability criterion (22) for the flute modes.

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ПРО ВПЛИВ РЕЗИСТИВНОЇ ІОНОСФЕРИ НА СТІЙКІСТЬ БАЛОННИХ ЗБУРЕНЬ МАГНІТОСФЕРНОЇ ПЛАЗМИ

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Р е з ю м е

Досліджено проблему стійкості магнітосферної плазми відносно МГД-збурень, що генеруються градієнтом тиску, у рамках дипольної моделі геомагнітного поля з урахуванням граничних умов на іоносфері. Остання розглядалась як тонкий шар зі скінченною провідністю. Основну увагу приділено вивченню впливу провідності іоносфери на границю стійкості збурень. Показано, що границя стійкості не залежить від величини провідності іоносфери та визначається жолобковими модами. Отримано аналітичний вираз для границі стійкості.