SELF-ORGANIZATION OF AN UNSTABLE SYSTEM BY THE HOPF BIFURCATION SCENARIO

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Self-organization in a synergetic system that undergoes perturbations of the fold catastrophe type has been considered. It has been shown that, provided a special choice of the system parameters, which leads to the Hopf bifurcation and the emergence of stable limit cycles, stable states may appear in an unstable system. The corresponding bifurcation and phase diagrams have been obtained. The conditions of the formation of stable and unstable focuses have been revealed.

1. Introduction

It is well known that the behavior of large ensembles of objects (complex systems) possessing the reserve of internal energy reveals common regularities, despite whether these objects are related to animate or dead nature [1–3]. For example, monitoring a moving sand in desert dunes or the emergence of snow avalanches results in some feeling of a reasonable background behind those processes, such as the Solaris ocean in Stanisław Lem's "Solaris". The same feeling also arises, while supervising the collections of representatives of animate nature deprived of intellect—ant hills, a swarm of bees, a flock of birds, and so on. On the other hand, the behavior of people's crowds, although every person in them is gifted with intelligence, frequently brings one to a conclusion about their unreasonableness. (The herds of sport fans are the most illustrative example of such a behavior.) A special place among the indicated phenomena is occupied by the behavior of economic systems, where purposeful actions of a huge number of those who participate in the process can bring about spontaneous financial crashes, catastrophic distribution of means, etc. Similar problems are also caused by the researches of the transitions from nonliving material to living substance, social phenomena, human mentality, and many others.

The problems pointed out, which are reduced to the self-organization in complex systems, are a subject of the interdisciplinary scientific activity, which has acquired the name *synergetics* (see books [1–3] and references therein). The phase transitions, such as the boiling of a liquid, are the elementary physical example of self-organization. Their description is based on the thermodynamic approach, within the framework of which a small subsystem is selected from the big thermostat and the state of this subsystem is described by the *order parameter*. The thermostat governs a state of the subsystem through changes of its mechanistic and thermal parameters which are reduced to the *field conjugated to the order parameter* and the *control parameter* (for the boiling of a liquid, their role is played by the pressure and the volume or the temperature and the entropy of the subsystem, respectively). The basic feature of the thermodynamic approach is that the thermostat can influence the considered subsystem, but the latter (owing to its smallness) does not affect the state of the former. Under such conditions, the influence of the thermostat results in a transition of the system into a local minimum of the effective potential which corresponds to an ordered state [4]. Such a situation is realized in the case where the time of the order parameter relaxation substantially exceeds the characteristic temporal scales of changes of the conjugated field and the control parameter [5].

A quite different picture is observed during the process of self-organization, when it is impossible to choose a small subsystem within the thermostat. In this connection, all degrees of freedom—the order parameter, the conjugated field, and the control parameter—acquire equal rights. The most popular example of such a type is the spontaneous emission of a laser, where the specified quantities are reduced to the strength of the induced field, the electric polarization of the medium, and the inverse population of electron levels (see work [1]). Since the self-organizing system is compatible by size with the thermostat, it becomes open, and its description requires the self-consistent description of the evolution of the order parameter, the conjugated field, and the control parameter. The most complicated scenario, which cannot be interpreted in terms of the effective potential, is realized, when the corresponding time scales are comparable, when the enhancement of the external factor action forces the system to transit into the mode of deterministic chaos which manifests itself by the appearance of a strange attractor [6–10].

The most known example of self-organization is the Belousov—Zhabotinsky chemical reaction, in the course of which a wave that changes one color of the liquid into another one periodically runs for a rather prolonged period. Since the reaction runs at high temperatures, such a behavior means that the molecules of the liquid, moving chaotically, are included periodically into the self-consistent process that spreads quickly over the bulk, changing the collective behavior of the system. Similar changes of the collective behavior are observed when tornados emerge, in oceanic streams and cyclones, in the behavior of organic cells at their morphogenesis, and so on.

Notwithstanding such a wide spread of the self-organization phenomenon, its microscopic picture is lacking now, because it requires the comprehension of the mechanisms of the process, which are of different nature in every specific case. For example, concerning the Belousov—Zhabotinsky reaction, the details of intermediate reactions, their constants, the conditions for the proper choice of reagents, and so on are not known enough. On the other hand, the opportunity for the oscillation mode to occur follows already from the consideration of simple models. In this connection, there is a question: Why do these simple models allow such complex collective phenomenon as the self-organization to be explained? The answer to this question is that the set of possible degrees of freedom of the complex system can be classified into the infinite number of microscopic degrees of freedom and a small number of macroscopic ones (for the designation of the latter, the term *hydrodynamic modes* is adopted). According to the principle of synergetic hierarchy [1], the hydrodynamic modes suppress the microscopic degrees of freedom in the course of the evolution, thus completely governing the self-organization scenario. As a result, the collective behavior of the system is controlled by several parameters which represent the amplitudes of hydrodynamic modes. On the other hand, in the course of self-organization in a nonequilibrium system, the dissipation, which is caused by the processes of diffusion, viscosity, and heat conductivity, which provide a transition into the stationary state, have to play a principal role.

This work is devoted to the research of the autowave mode of self-organization in a synergetic system, such as the Belousov—Zhabotinsky reaction. For definiteness, we study the phenomenon of periodically modulated laser emission [11], when the system passes to its stationary state which is characterized by the formation of a stable autowave process and corresponds to a limit cycle [12]. A typical example of such a process is the formation of space-time structures in the system that is described by the Brusselator model [13].

Since the mode concerned emerges due to the external influence on the self-organizing system, the following question arises: What kind has such an influence to be of to ensure the transition of the system into the limit cycle mode which corresponds to the Hopf bifurcation? The answer to this question is given by the catastrophe theory which makes it possible to describe various situations qualitatively, by modeling the potential of external factor by fold, cusp, and swallow-tail catastrophes [14].

Our researches are based on the Lorenz system [15], which allows the effect of self-organization to be introduced in the simplest manner [16]. Section 2 is devoted to the formulation of the problem, which allows us to pass from the standard form of the Lorenz equations to the system of two differential equations describing the self-organization process with the formation of a limit cycle. In Section 3, we present basic formulae which allow us to describe the self-organization as a result of the Hopf bifurcation [12]. In Section 4, we quantitatively illustrate our consideration by applying it to the elementary case of the fold catastrophe. It turns out that, in such a case contrary to the standard situation, the limit cycle emerges from an unstable state.

2. Model

In Lorenz's original work [15], the velocity of a convective stream X, the temperature difference between opposite streams Y, and the deviation Z of the temperature gradient from a constant value played the roles of principal variables, while describing the self-organizing system physically. Contrary to this work, it is convenient now to use the order parameter, the field conjugated to it, and the control parameter [3]. According to Haken [1], while describing the spontaneous laser emission phenomenologically, the order parameter is reduced to the strength of the radiation electric field ($E \propto X$), the polarization of the active medium P plays the role of the conjugated field ($P \propto Y$), and the control parameter $S \equiv S_e - Z$ corresponds to the inverse population of electron levels (S_e being the level of external pumping). As a result, the Lorenz system is composed of the equations

$$\begin{cases} \tau_E \dot{E} = -E + g_E P, \\ \tau_P \dot{P} = -P + g_P ES, \\ \tau_S \dot{S} = (S_e - S) - g_S EP, \end{cases} \tag{1}$$

where the time scales of the evolution are given by the quantities τ_E, τ_P, and τ_S. The corresponding intensities of feedbacks g_E, g_P, and g_S are exclusively positive. It is typical that the field strength depends linearly on the medium polarization, while the evolutions of the polarization and the level populations are governed by nonlinear components, with the increase of polarization being the reason of self-organization, if the field strength and the inverse population interact with each other. Such a positive feedback is compensated by a negative one in the third equation, which is provided by the interaction between the field strength and the polarization, and results in the reduction of the inverse population according to Le Chatelier's principle.

While investigating system (1), it is convenient to pass to dimensionless quantities and measure the time t, the field strength E, the polarization P, and the inverse population S in units of

$$\tau_E, \quad E_s = (g_P g_S)^{-1/2},$$

$$P_s = (g_E^2 g_P g_S)^{-1/2}, \quad S_s = (g_E g_P)^{-1}, \tag{2}$$

respectively. Then, the Lorenz system acquires the simple form

$$\begin{cases} \dot{E} = -E + P, \\ \sigma \dot{P} = -P + ES, \\ \varepsilon \dot{S} = (S_e - S) - EP \end{cases} \tag{3}$$

which is characterized by the dimensionless parameter of the external factor S_e and the ratios between the characteristic times, $\sigma \equiv \tau_P/\tau_E$ and $\varepsilon \equiv \tau_S/\tau_E$.

Studying various modes of the behavior of system (3) shows that the variation of the polarization P, which is a microscopic quantity, proceeds much faster than the modification of the macroscopic values of the strength E and the population S [1]. This allows us to use the adiabatic approximation $\tau_P \ll (\tau_E, \tau_S)$, in the framework of which it is possible to neglect the left-hand side of the second equation of system (3), because $\sigma \ll 1$. As a result, we obtain the relation $P = ES$ which brings about the two-parameter system

$$\begin{cases} \dot{E} = -E(1-S) + f_e, \\ \dot{S} = \varepsilon^{-1}\left[S_e - S(1+E^2)\right]. \end{cases} \tag{4}$$

Here, we have switched on the external factor, the force of which

$$f_e = -\frac{\partial V_e}{\partial E} \tag{5}$$

is determined by the potential V_e. In accordance with the recipe of the catastrophe theory, such a potential is given by three types of universal deformations [14]. In the general case,

$$V_e = AE + \frac{B}{2}E^2 + \frac{C}{3}E^3 + \frac{D}{4}E^4 + \frac{F}{5}E^5, \tag{6}$$

where A, B, C, D, and F are the parameters of the theory. For the fold catastrophe, we have $B = D = F = 0$, while the parameters A and C are arbitrary. In the case of the cusp catastrophe, $C = F = 0$; while A, B, and D may vary. At last, for the swallow-tail catastrophe, we have $D = 0$; while the parameters A, B, C, and F become arbitrary.

According to the Hopf theorem, only the action of an external potential can result in the Hopf bifurcation [12]. In this case, the standard procedure prescribes to put the coefficient of the leading term equal to unity, which can be achieved by a proper choice of the corresponding scales of dimensions. However, as we have already engaged units (2), which made the form of the Lorenz system (3) the simplest, this coefficient should be considered as arbitrary.

3. Basic Equations

Passing to the study of the possible types of solutions of the system that self-organizes into the limit cycle mode, we expose the basic points of the Hopf bifurcation algorithm [12]. With this purpose in view, we write down system (4) in the generalized form

$$\begin{cases} \dot{E} = f^{(1)}(E,S), \\ \varepsilon\dot{S} = f^{(2)}(E,S), \end{cases} \tag{7}$$

where the effective forces are represented as

$$f^{(1)}(E,S) \equiv -[A + (1+B)E + CE^2 +$$

$$+DE^3 + FE^4] + ES,$$

$$f^{(2)}(E,S) \equiv S_e - S(1+E^2). \tag{8}$$

The stationary states, which are defined by the conditions $\dot{E} = 0$ and $\dot{S} = 0$, are determined by the equations

$$(1+E_0^2)[A + (1+B)E_0 + CE_0^2 + DE_0^3 + FE_0^4] = E_0 S_e,$$

$$S_0 = S_e(1+E_0^2)^{-1}. \tag{9}$$

The dynamics of the process that self-organizes near the indicated stationary points is determined by the eigenvalue λ and the eigenvector $\vec{V}$ of the Jacobi matrix

$$M_{ij} \equiv \left(\frac{\partial f^{(i)}}{\partial x_j}\right)_{x_j = x_{j0}}; \qquad x_j \equiv \{E, S\}, \quad i,j = 1, \tag{10}$$

where the subscript 0 corresponds to the stationary state. Substituting forces (8) into definition (10) brings about the matrix elements

$$M_{11} = -M_0 + S_0, \tag{11}$$

$$M_0 \equiv (1+B) + 2CE_0 + 3DE_0^2 + 4FE_0^3;$$

$$M_{12} = E_0; \quad M_{21} = -2\varepsilon^{-1} S_0 E_0; \tag{12}$$

$$M_{22} = -\varepsilon^{-1}(1+E_0^2).$$

Then, the equation for the eigenvalues and eigenvectors

$$\sum_j M_{ij} V_j = \Lambda V_i \tag{13}$$

gives the following expressions for the Lyapunov parameter Λ, the increment λ, and the eigenfrequency ω_0:

$$\Lambda \equiv \lambda \pm \mathrm{i}\omega_0,$$

$$\lambda = \frac{1}{2}\left[(S_0 - M_0) - \varepsilon^{-1}\left(1+E_0^2\right)\right],$$

$$\omega_0 = \frac{1}{2}\sqrt{8\varepsilon^{-1} S_0 E_0^2 - \left[(S_0 - M_0) + \varepsilon^{-1}(1+E_0^2)\right]^2}. \tag{14}$$

The stationary point (E_0, S_0) is unstable, provided the condition $\lambda \geq 0$, which results in the expression

$$\varepsilon(S_0 - M_0) \geq 1 + E_0^2. \tag{15}$$

Accordingly, the characteristic frequency exists if

$$8\varepsilon S_0 E_0^2 \geq \left[\varepsilon(S_0 - M_0) + (1+E_0^2)\right]^2. \tag{16}$$

In order to define a condition of the limit cycle stability, it is necessary to rewrite the equation of motion (7), reckoning the variables E and S from their stationary values E_0 and S_0, respectively. It can be obtained on the basis of the transformation

$$\vec{X} = \vec{X}_0 + P \cdot \vec{\delta}, \tag{17}$$

where the pseudovector notations

$$\vec{X} \equiv \begin{pmatrix} E \\ S \end{pmatrix}, \quad \vec{\delta} \equiv \begin{pmatrix} E - E_0 \\ S - S_0 \end{pmatrix} \tag{18}$$

are used, and the transformation matrix

$$P \equiv \begin{pmatrix} \Re V_1 & -\Im V_1 \\ \Re V_2 & -\Im V_2 \end{pmatrix} \tag{19}$$

is constructed from the eigenvector's components

$$\vec{V} \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{20}$$

Selecting the former in the elementary form $V_1 \equiv 1$ and using Eq. (13), where the eigenfrequency ω_0 must be calculated at the bifurcation point $\lambda = 0$, we find, for the second component, that

$$V_2 = \frac{(M_0 - S_0) + \mathrm{i}\omega_c}{E_0},$$

$$\omega_c \equiv \omega_0|_{\lambda=0} = (1+E_0^2)\left[\frac{2S_e E_0^2}{(1+E_0^2)^3} - 1\right]^{1/2}. \tag{21}$$

Then, the transformation matrix (19) looks like

$$P = \begin{pmatrix} 1 & 0 \\ (M_0 - S_0)/E_0 & -\omega_c/E_0 \end{pmatrix}. \tag{22}$$

As a result, the equations of motion acquire the canonical form

$$\dot{\vec{\delta}} = \vec{F}, \qquad \vec{F} \equiv P^{-1}\vec{f}, \tag{23}$$

where the pseudovector of the canonical force

$$\vec{F} = \begin{pmatrix} F^{(1)} \\ F^{(2)} \end{pmatrix} \equiv \begin{pmatrix} f^{(1)} - f_0^{(1)} \\ f^{(2)} - f_0^{(2)} \end{pmatrix}, \tag{24}$$

which satisfies the condition [12]

$$\frac{\partial \vec{F}}{\partial \vec{\delta}} = \begin{pmatrix} 0 & -\omega_c \\ \omega_c & 0 \end{pmatrix}, \tag{25}$$

consists of the components

$$F^{(1)} = f^{(1)}, \quad F^{(2)} = \alpha f^{(1)} - \beta\varepsilon f^{(2)}; \tag{26}$$

$$\alpha \equiv \frac{M_0 - S_0}{\omega_c}, \qquad \beta \equiv \frac{E_0}{\varepsilon\omega_c}. \tag{27}$$

The stability of the limit cycle is determined by the condition $\Re\Phi < 0$ for the Floquet index [12]

$$\Phi = \frac{\mathrm{i}}{2\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{1}{2}g_{21}, \tag{28}$$

calculated at the bifurcation point. Here, the structural constants are determined by the derivatives with respect to the variables E and S, which is indicated by the corresponding subscripts:

$$g_{11} = \frac{1}{4}\left[\left(F_{EE}^{(1)} + F_{SS}^{(1)}\right) + \mathrm{i}\left(F_{EE}^{(2)} + F_{SS}^{(2)}\right)\right], \tag{29}$$

$$\begin{pmatrix} g_{02} \\ g_{20} \end{pmatrix} = \frac{1}{4}\left[\left(F_{EE}^{(1)} - F_{SS}^{(1)} \mp 2F_{ES}^{(2)}\right) + \right.$$

$$\left. +\mathrm{i}\left(F_{EE}^{(2)} - F_{SS}^{(2)} \pm 2F_{ES}^{(1)}\right)\right], \tag{30}$$

$$g_{21} = \frac{1}{8}\left\{\left[\left(F_{EEE}^{(1)} + F_{ESS}^{(1)}\right) + \left(F_{EES}^{(2)} + F_{SSS}^{(2)}\right)\right] + \right.$$

$$\left. + \mathrm{i}\left[\left(F_{EEE}^{(2)} + F_{ESS}^{(2)}\right) - \left(F_{EES}^{(1)} + F_{SSS}^{(1)}\right)\right]\right\}. \tag{31}$$

Substituting expressions (26) and (8) into Eqs. (29)–(31), we find

$$g_{11} = -\frac{1}{2}\left[\gamma + \mathrm{i}\left(\alpha\gamma - \beta S_0\right)\right], \ \gamma \equiv C + 3DE_0 + 6FE_0^2; \tag{32}$$

$$\begin{pmatrix} g_{02} \\ g_{20} \end{pmatrix} = -\frac{1}{2}\left\{\left[\gamma \pm (\alpha + 2\beta E_0)\right] + \right.$$

$$\left. +\mathrm{i}\left[\mp 1 + (\alpha\gamma - \beta S_0)\right]\right\}; \tag{33}$$

$$g_{21} = \frac{1}{4}\left[\beta - 3(1 - \mathrm{i}\alpha)\left(D + 4E_0\right)\right]. \tag{34}$$

Then, the Floquet index (28) gives the following condition for the limit cycle to be stable:

$$(\alpha\gamma - \beta S_0)(\alpha + 2\beta E_0 - 2\gamma) \le \gamma + \omega_c[3(D + 4FE_0) - \beta]. \tag{35}$$

4. Fold Catastrophe

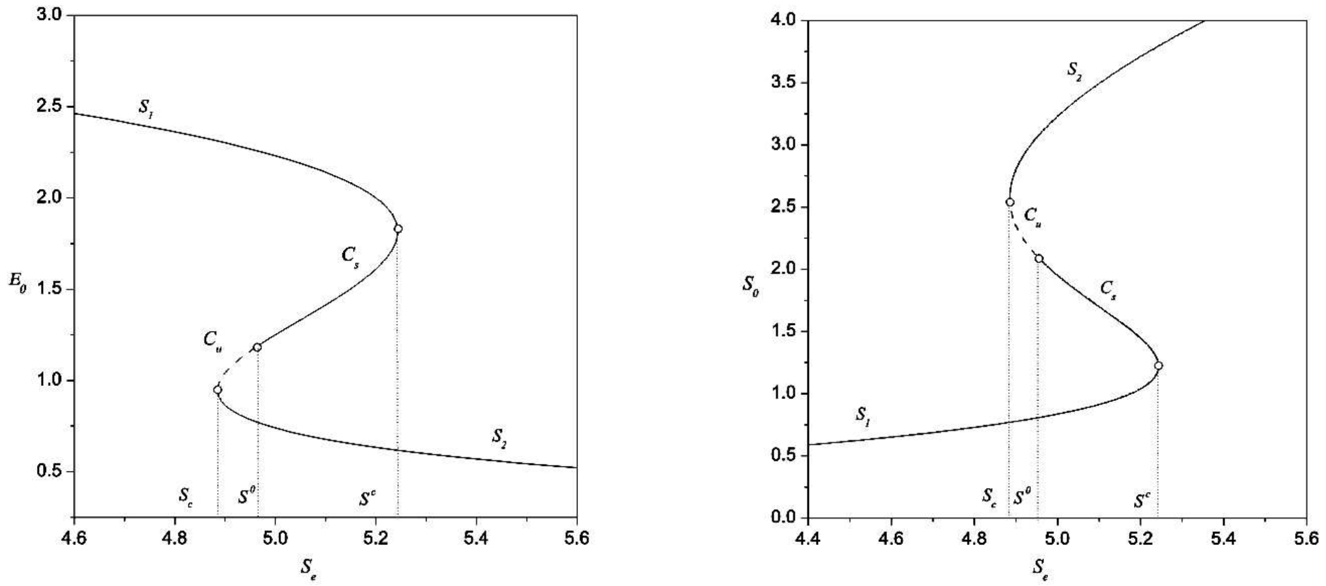
In the elementary case of the fold catastrophe, it is necessary to put $\gamma = C$ and $M_0 = 1 + 2CE_0^2$ in relationships (15) and (16), while condition (35) takes the form

$$(\alpha C - \beta S_0)(\alpha + 2\beta E_0 - 2C) \le C - \beta\omega_c. \tag{36}$$

Then system (9) possesses the stationary values of the strength E_0 and the inverse population S_0, the typical dependences of which on the external factor S_e are depicted in Fig. 1.

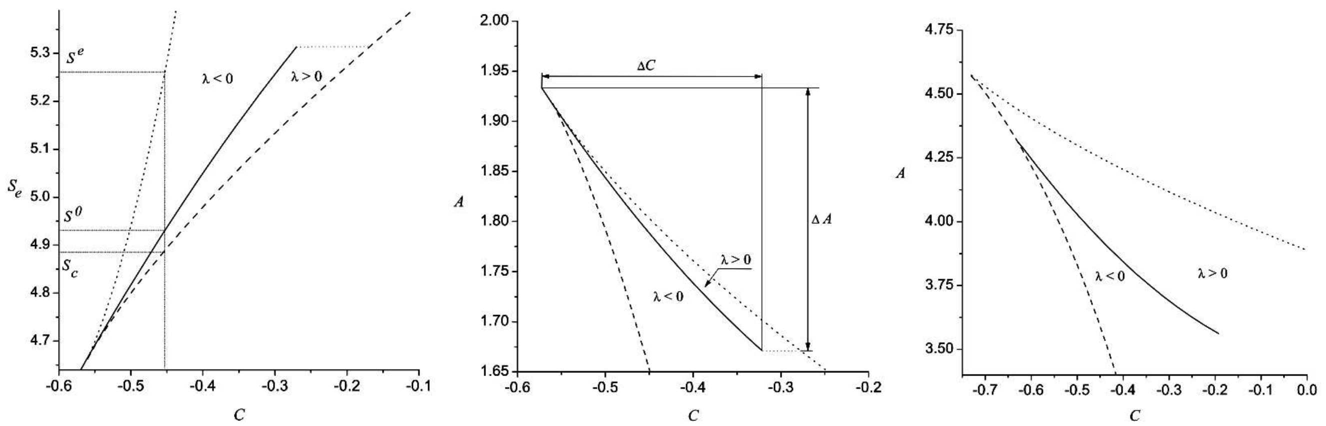
Both the presented dependences reveal a characteristic instability of the S type, but the dependence $E_0(S_e)$ manifests it in its descendant section, whereas the dependence $S_0(S_e)$ in the ascendant one. According to the bifurcation diagrams exposed in Fig. 1, if the level of external pumping is below the critical value ($S_e < S_c$), the phase portrait possesses a single saddle S_1 (Fig. 1, thin curves). As the lower critical value S_c is exceeded, there occurs a usual bifurcation of duplication, owing to which there appears the focus C represented by thick curves and a new saddle S_2 (thin curves). Within the interval $S_c < S_e < S^0$, the increment is positive ($\lambda > 0$), which points at the instability of the focus C_u, whose position is represented by the dashed lines. At an intermediate value $S_e = S^0$, the increment vanishes ($\lambda = 0$), so that the focus is transformed from the unstable, C_u, into the stable, C_s, one. Within the interval $S^0 < S_e < S^c$, λ is negative, and the focus C_s preserves its attracting character, while there occurs a reverse bifurcation at $S_e > S^c$, which results in the annihilation of the focus and the saddle S_1.

The peculiarity of the considered case of the fold catastrophe consists in its degenerate character: at the critical level of pumping $S_e = S^0$, there occurs not only the transformation of the increment λ sign, but also the zeroing of the Floquet index (28). It is typical that, in this case, the indicated condition is not fulfilled within a finite interval of the pumping level but at a single point $S_e = S^0$. This circumstance is illustrated by the phase diagrams depicted in Fig. 2, from which one can see that the conditions $\lambda = 0$ and $\Re\Phi = 0$ are obeyed along a solid curve, which changes its length depending on the parameters S_e, A, and C; the latter must become negative. The comparison of Figs. 2,*a*, 2,*b*, and 2,*c* testifies to that if the value of the parameter A is constant, the increase of the parameter C results in an extension of the region of the bifurcation duplication (bounded by a dashed and a dotted curve), although the continuous curve of the Hopf bifurcation becomes



a **b**

Fig. 1. Dependences of the stationary values of the strength E_0 (a) and the inverse population S_0 (b) on the pump parameter S_e at $A = 1.9$ and $C = -0.455$



a **b** **c**

Fig. 2. Phase diagrams of the system at $A = 1.9$ (a), $S_e = 4.7$ (b), and $S_e = 9.6$ (c)

interrupted at a certain value of S_e. As the external factor parameter S_e increases, the region of the bifurcation duplication extends and the Hopf bifurcation curve lengthens (cf. Figs. 2,b and 2,c).

Figure 3 demonstrates that, as the pumping parameter S_e grows, the lengths ΔA and ΔC of the solid curves' sections, where the condition of the Hopf bifurcation is satisfied and which are measured along the axes of variation of the parameters A and C, monotonously increase, being nonzero if $S_e > 3.705$.

Consider now the explicit view of the phase portrait of the system in various regions of its phase diagram exhibited in Fig. 2. They are the simplest in the subcritical ($S_e < S_c$) and supercritical ($S_e > S^c$) regions. From Fig. 4, one can see that both these regions are characterized by the availability of a single saddle, which corresponds to large values of the strength E_0 in the first case and to small E_0's in the second one, with the stationary value of the inverse population S_0 being small in the first case and large in the second one. More complicated is the picture in the region bounded by the dashed and dotted curves in Fig. 2. In comparison with Fig. 4, there emerge an additional saddle and a focus, which is repulsive in the $\lambda > 0$ region and attracting in the $\lambda < 0$ one (Fig. 4).

The most interesting situation is realized, provided the conditions $\lambda = 0$ and $\Re\Phi = 0$ which correspond to the solid curve in Fig. 2. Here, the focus C becomes degenerate, so that the phase trajectories become concentric closed ellipses (see Fig. 5). In a certain sense, this case can be characterized as the availability of a number of limit cycles. To demonstrate this fact, we exhibit such cycles, using their scaling-up. From Fig. 6,*b*, one can see that considering the limit cycle scaled-up does not change its portrait but only reduces the region of variation of the variables E and S.

Until now, we have considered the case of the time scale compatibility of the evolution of the order parameter and the control parameter ($\varepsilon = 1$). Nevertheless, the situation is realized in many cases where the slowest mode can be distinguished among the others in the system. This means that, while studying the systems that are the most adequate to real conditions, it is necessary to put $\varepsilon \leq 1$. The analysis carried out demonstrated that a stable limit cycle is not possible in the system at every value of the dimensionless relaxation time ε (see Fig. 7). The interval of ε, where the emergence of periodic emission becomes possible, is readily noticed in the dependence $A(\varepsilon)$ (the curves in Fig. 7 are confined in length). In addition, by comparing curves *1* to *7*, one can see that, as the amplitude C of the Q-spoiler non-linearity increases, the interval, where $\varepsilon \simeq 1$ and which corresponds to the stable periodic emission, decreases (the solid curve), and the region of the unstable emission (the dashed curve) widens. At a certain value of C (e.g., curve *4*), the

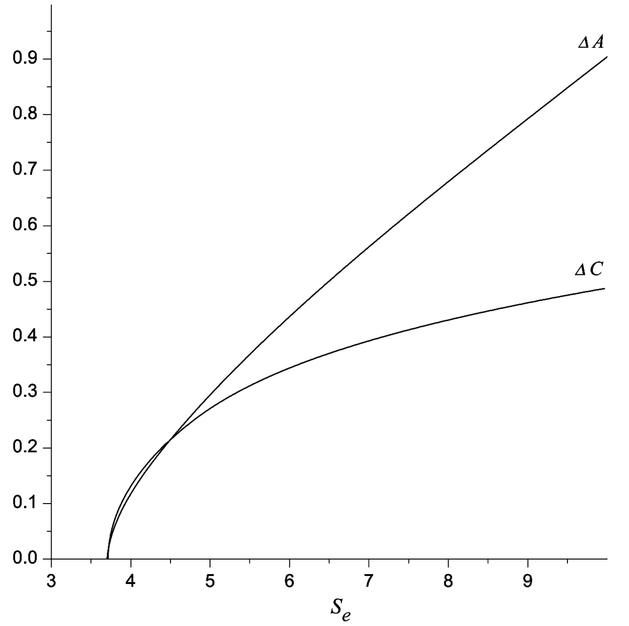


Fig. 3. Lengths of the axes of the A and C parameters' variation, where the condition of the Hopf bifurcation is satisfied

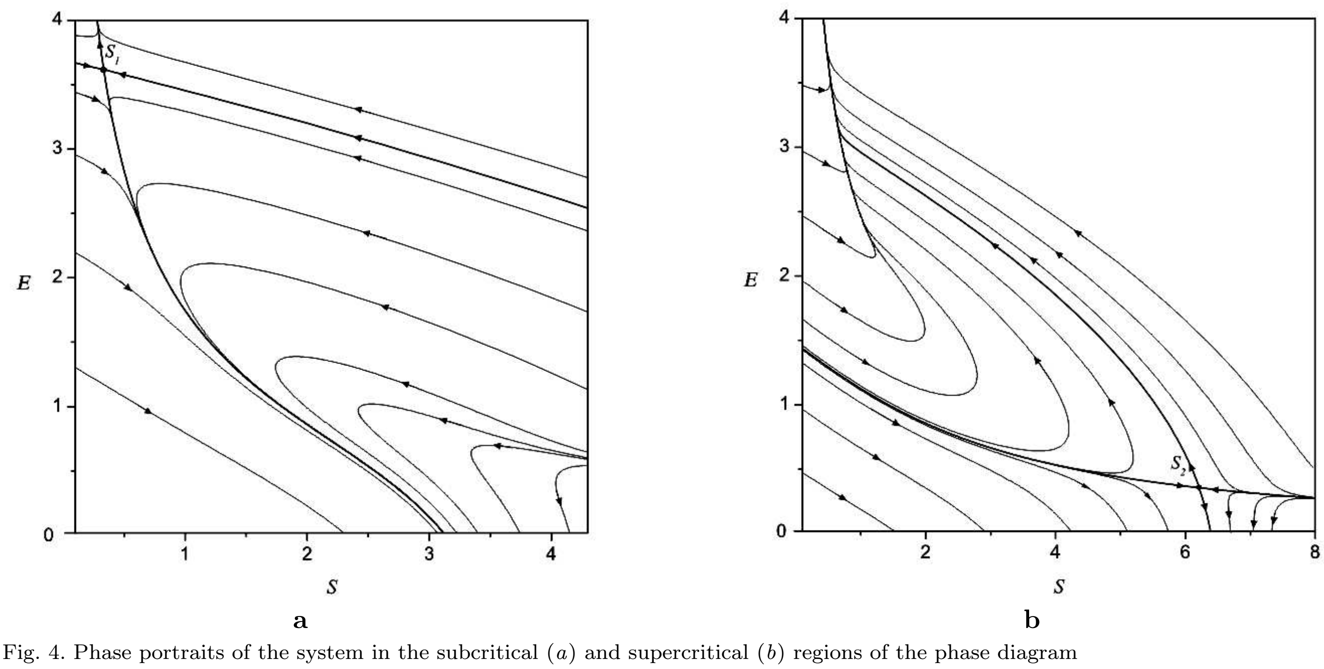


Fig. 4. Phase portraits of the system in the subcritical (a) and supercritical (b) regions of the phase diagram

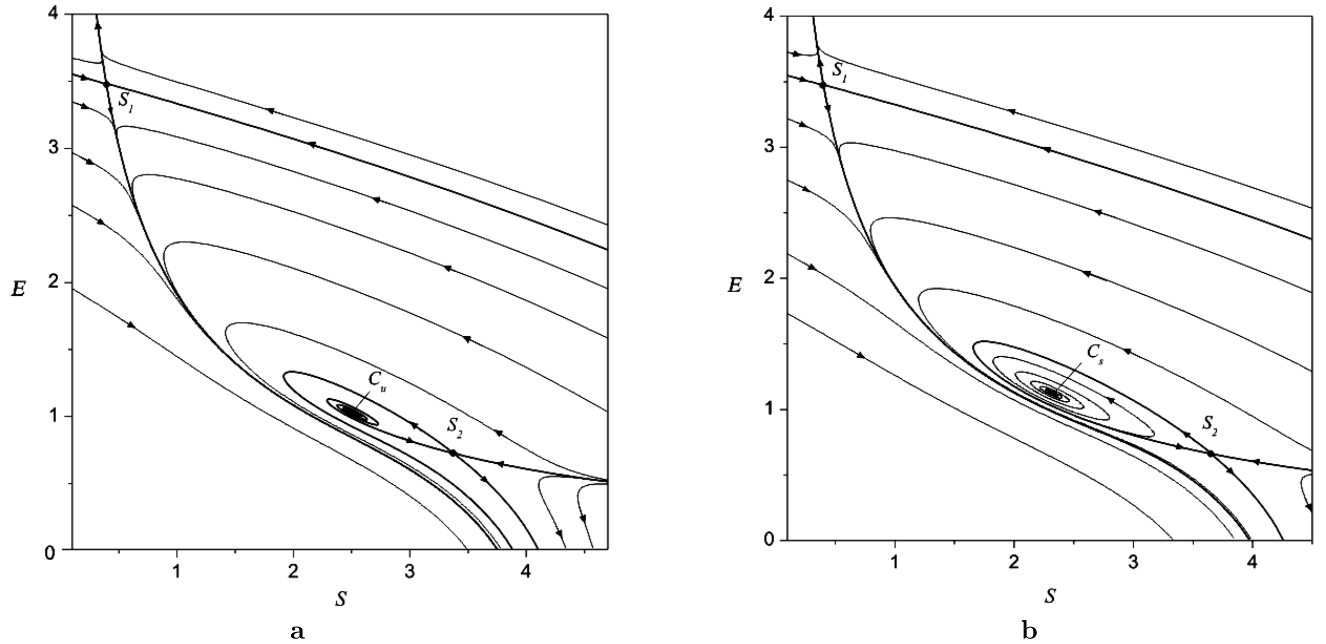


Fig. 5. Phase portraits of the system in the regions of unstable (*a,* $S_e = 5.1$) and stable (*b,* $S_e = 5.2$) foci. $A = 1.9$ and $C = -0.33$

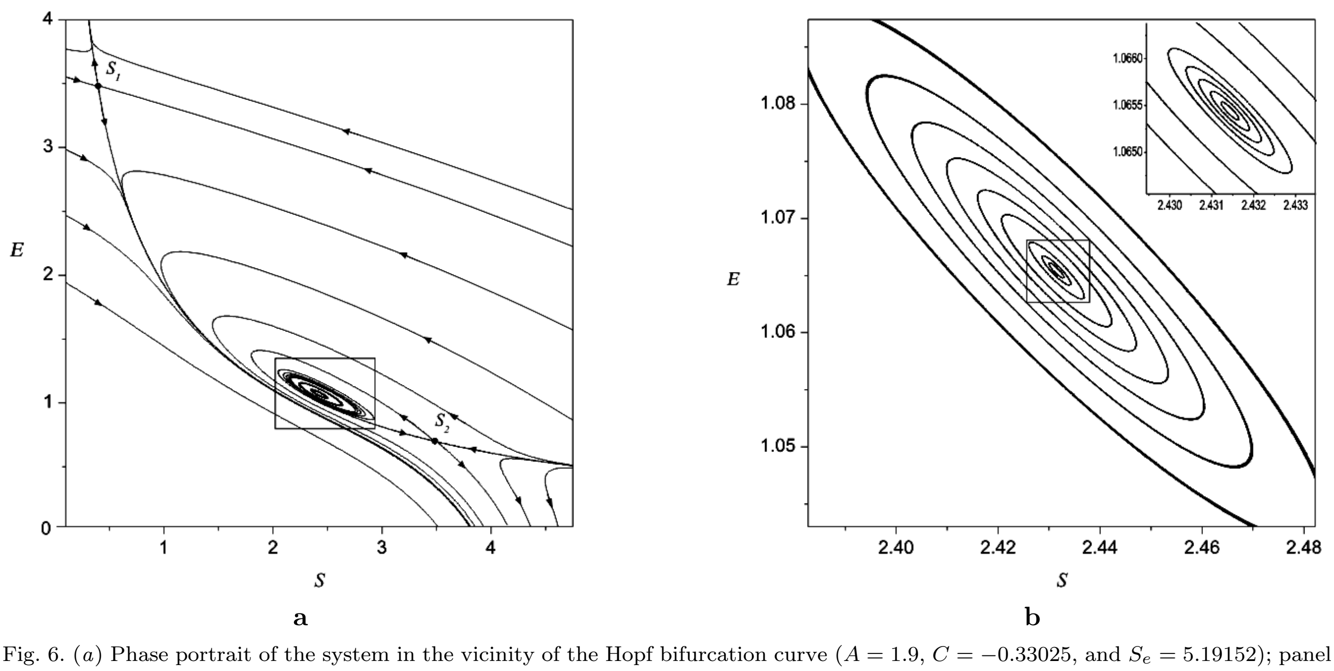


Fig. 6. (*a*) Phase portrait of the system in the vicinity of the Hopf bifurcation curve ($A = 1.9$, $C = -0.33025$, and $S_e = 5.19152$); panel *b* exhibits the scaled-up part of panel *a*

system can demonstrate only the unstable periodic emission. At positive values of C, the region, where the stable periodic emission occurs, survives only at $\varepsilon \simeq 10^{-1} \div 10^{-2}$, which means that the self-organization of the system to the oscillation motion is possible only provided that the dynamics of the whole system is subordinated to a single slow mode, namely to the electric field strength.

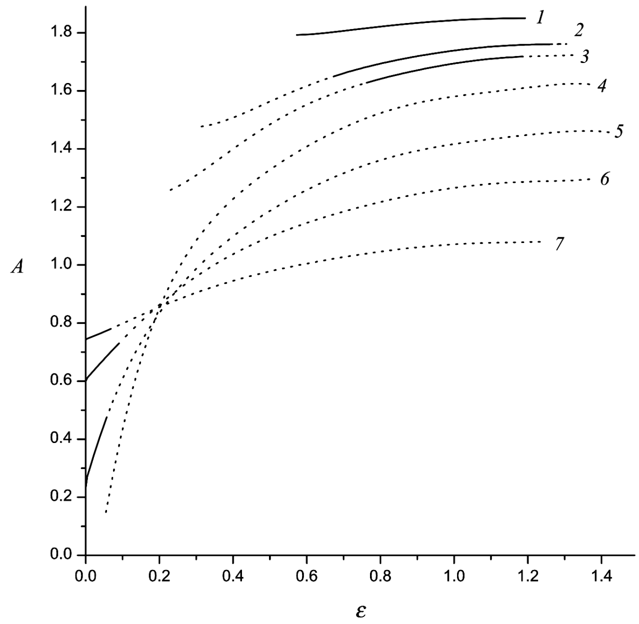


Fig. 7. Phase diagrams of the system, when the periodic emission takes place at $S_e = 4.7$ and $C = -0.5$, -0.4, -0.35, -0.2, 0.1, 0.5, and 1.2 (curves 1—7, respectively). Dotted curves determine the unstable limit cycle, while the solid ones the set of stable cycles

5. Conclusion

We have considered the simplest case of the fold catastrophe, which can be realized only provided $C < 0$. This means the unlimited increase of the field strength and the inverse population in the range of large values of E and S. From this point of view, the considered system is unstable globally. However, at the values of the parameters S_e, A, and C, which are determined by the phase diagram in Fig. 2, the local stability, whose character is governed by the conditions whether the attractive focus does exist or not (Fig. 4,a), can manifest itself. As the parameters A, C, and S_e vary, this focus transforms into the repulsive one, so that there is a family of degenerate phase trajectories of the limit-cycle type at the point of Hopf bifurcation.

A characteristic feature of the fold catastrophe consists in that its limit cycle has degenerate character. From the physical point of view, this means that, in the considered case, the autowave mode of emission modulation can arise only provided a strictly fixed choice of the laser parameters. In doing so, the variation of the initial values of the radiation field strength and the inverse population essentially changes their current values, keeping the modulation frequency to be constant.

Since the limited space of the publication allowed only the case of the fold catastrophe to be reported, it should be noted that the obvious prospect of continuation of this work comprises the researches of the cusp and swallow-tail catastrophes. One may expect that the extension of the number of parameters of those models would result in determining the finite region of existence of the stable limit cycle.

To summarize, we notice that the results obtained can find direct implications while studying the systems that self-organize under an intense external loading, provided that their kinetics is non-linear (biological, socio-economic systems, etc.).

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САМООРГАНІЗАЦІЯ НЕСТІЙКОЇ СИСТЕМИ
ЗА БІФУРКАЦІЄЮ ХОПФА

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Р е з ю м е

Розглянуто самоорганізацію синергетичної системи, що піддається збуренням типу універсальної деформації складки. Показано, що при спеціальному виборі параметрів нестійкої системи можливим стає виникнення стабільних станів унаслідок біфуркації Хопфа, результатом якої є утворення вироджених граничних циклів. Одержано відповідні біфуркаційні та фазові діаграми. Виявлено умови утворення стійкого та нестійкого фокусів.