$1/\omega$-SPECTRUM OF ACOUSTIC HEAT PHONONS

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This paper is dedicated to Victor G. Bar'yakhtar on the occasion of his 75th birthday

It is shown that nonlinear heat waves in solids, whose potential energy is defined by a displacement of particles from the equilibrium position (the Einstein model) or the derivatives of displacements of particles with respect to coordinates (the Debye model), possess the $1/\omega$ spectrum, only if the temperature of the system exceeds the critical one. It is shown that such a spectrum is caused by the presence of a double feedback in the system. The experimentally observed $1/\omega$-spectrum of photons scattered by acoustic heat waves in solids is explained.

1. Introduction

The so-called $1/\omega$-noise or flicker-noise, whose intensity increases with decrease in the frequency at small frequencies [1—3], was discovered in 1925 [4] as the flickering effect of the current of the electron emission in electronic tubes. Then such a noise was measured in carbon resistors, semiconductor devices, biological and geological systems, etc. This noise is a universal phenomenon, but its nature is unclear in many aspects up to now [5—9]. A lot of models pretending to the explanation of the $1/\omega$-noise in electric circuits is proposed, but none of them is not commonly accepted. No physically substantial models of the appearance of the $1/\omega$-noise in phonon systems were considered, despite the direct experimental evidence for its existence [10]. The most known mathematical model of the $1/\omega$-noise is based on the summation of Lorentzians [8,9]

$$\langle (emf)^2 \rangle_\omega = \int\limits_0^{+\infty} g(\tau) \frac{\tau}{1+\tau^2\omega^2} d\tau \propto \frac{1}{\omega},$$

which is equivalent to the assumption about the presence of the spectrum of Gauss distributions with a weight function $g(\tau)$ in the system. Here, the left-hand side is the noise-induced electromotive force in an electric circuit. The problem consists in that the required asymptotic behavior of the weight function $g(\tau) \sim \tau^{-1}$ is difficult to be realized as $\tau \to \infty$ [1,9]. Therefore, such a theory can hardly describe this phenomenon widely spread in the nature. We mention also the attempts to discover the $1/\omega$-noise by means of the study of nonlinear differential equations with numerical methods [11]. It is established almost surely that the $1/\omega$ -noise has both the thermodynamic and kinetic bases. The experimental indication of its thermodynamic nature is given in [10, 12].

Below, we advance one of mechanisms of the appearance of the $1/\omega$-noise under equilibrium conditions as a consequence of the feedback. Namely, we will prove that such a noise appears in oscillators which compose a solid according to the Einstein model of heat capacity if only the coefficient of elasticity of these oscillators undergoes the action of a sufficiently powerful random Gauss perturbation which is correlated in time. The proof does not require any hypotheses and is based on the basic equations of dynamics and on the validity of the Gibbs distribution. Then we will show that the quite satisfactory results follow from the so-called "one-loop" approximation admitting the analysis of the problem on the algebraic level. Just in this approximation, we study more realistic models and compare the derived results with experimental data.

Consider an elastic wave in a solid which is defined by displacements of particles from their equilibrium positions. For the sake of simplicity, we will consider that the displacement of a particle is a scalar quantity $\varphi(\mathbf{r}, t)$, where $\mathbf{r}$ and t is the coordinate and time of a particle of the solid which is considered continuous and homogeneous. We take the crystal structure into account by introducing the minimum (Debye) wavelength. The most general form of the description of waves satisfying the superposition principle with periodic conditions on the boundaries of a cube $V = L_x L_y L_z$ looks as follows:

$$\varphi(\mathbf{r},t) = \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \Big(\alpha_{\mathbf{k}} \exp(i\mathbf{kr} - i\omega_{\mathbf{k}}t) +$$

$$+ \alpha^*_{\mathbf{k}} \exp(-i\mathbf{kr} + i\omega_{\mathbf{k}}t) \Big),$$

where $\gamma_{\mathbf{k}}$, $\alpha_{\mathbf{k}}$, and $\omega_{\mathbf{k}}$ are some constants defined by the wave vector $\mathbf{k}$. The transition to the quantum description of a system is realized, similarly to quantum electrodynamics, by the change of the constants $\alpha_{\mathbf{k}}$ and $\alpha^*_{\mathbf{k}}$ by the operators

$$\alpha_{\mathbf{k}} \to \hat{\alpha}_{\mathbf{k}}, \quad \alpha^*_{\mathbf{k}} \to \hat{\alpha}^+_{\mathbf{k}}, \quad \hat{\alpha}_{\mathbf{k}} \hat{\alpha}^+_{\mathbf{k}'} - \hat{\alpha}^+_{\mathbf{k}'} \hat{\alpha}_{\mathbf{k}} = \delta_{\mathbf{kk}'},$$

where $\delta_{\mathbf{kk}'}$ is the Kronecker delta. In the representation of Heisenberg, the scalar function $\varphi(\mathbf{r}, t)$ becomes the operator

$$\breve{\varphi}(\mathbf{r},t) = \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \Big(\hat{\alpha}_{\mathbf{k}} \exp(i\mathbf{kr} - i\omega_{\mathbf{k}}t) +$$

$$+ \hat{\alpha}^+_{\mathbf{k}} \exp(-i\mathbf{kr} + i\omega_{\mathbf{k}}t) \Big). \quad (1)$$

2. The Einstein Model

At first, we consider the Einstein model of solids, in which the potential energy of particles is defined by their displacement from the equilibrium position. Despite the well-known drawbacks, e.g. the absence of the group velocity of sound, this model admits a sufficiently full mathematical investigation and allows one to reveal the main properties of the system and to clarify the problems related to such an investigation.

2.1. Hamiltonian of the system

We take the Hamiltonian of the system in the form

$$\hat{H} = \int \frac{\rho}{2} \left(\frac{d\breve{\varphi}(\mathbf{r},t)}{dt} \right)^2 d\mathbf{r} + \frac{\chi}{2} \int \breve{\varphi}^2(\mathbf{r},t) d\mathbf{r}. \quad (2)$$

Here, ρ is the substance density, and χ is the coefficient of elasticity. The substitution of (1) in (2) gives

$$\hat{H} = \sum_{\mathbf{k}} \hbar\omega_k \left(\hat{\alpha}^+_{\mathbf{k}} \hat{\alpha}_{\mathbf{k}} + \frac{1}{2} \right), \quad (3)$$

Moreover, in this model, we have

$$\gamma_{\mathbf{k}} = \sqrt{\frac{\hbar}{2V\sqrt{\chi\rho}}}, \quad \omega_{\mathbf{k}} = \omega_0 = \sqrt{\frac{\chi}{\rho}}.$$

In the Einstein model, $\omega_{\mathbf{k}}$ loses the dependence on $\mathbf{k}$. Therefore, it is convenient to change the notation $\omega_{\mathbf{k}}$ by ω_0. At $t = t'$, we have

$$\Big[\breve{\varphi}(x), \breve{\varphi}(x')\Big] = 0, \quad \left[\breve{\varphi}(x), \frac{d\breve{\varphi}(x')}{dt'}\right] = i\frac{\hbar}{\rho}\delta(\mathbf{r} - \mathbf{r}'). \quad (4)$$

where x denotes the totality $x = \{\mathbf{r}, t\}$.

The purpose of the present work consists in the study of the correlator $\langle\breve{\varphi}(x)\breve{\varphi}(x')\rangle$ under conditions of thermodynamic equilibrium. The broken brackets stand for the averaging over the Gibbs distribution in both the quantum and statistical senses. In the case of free waves and the operator $\breve{\varphi}(x)$ given in the explicit form (1), this problem is trivially solved. It is convenient to investigate the Fourier transform of the correlator under consideration. By averaging the bilinear combination of operators (1) over the eigenfunctions of the operator $\hat{H}$ and the Gibbs distribution, we get

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega} = \int \langle\breve{\varphi}(x)\breve{\varphi}(0)\rangle \exp\left(-i\mathbf{kr} + i\omega t\right) d\mathbf{r}dt =$$

$$= 2\pi\gamma^2_{\mathbf{k}} V[\delta(\omega - \omega_0)\left(1 + N(\omega)\right)\rangle + \delta(\omega + \omega_0) N(-\omega)],$$

$$N(\omega) = \left(\exp\left(\frac{\hbar\omega}{T}\right) - 1 \right)^{-1}, \quad (5)$$

where T is the temperature. Here, we took into account the fact that the correlator $\langle\breve{\varphi}(x)\breve{\varphi}(x')\rangle$ depends on the difference of the arguments for the homogeneous systems under conditions of thermodynamic equilibrium.

2.2. The fluctuation-dissipation theorem

If the principle of superposition is broken in a solid due to the nonlinear interaction of heat waves one with another and if the explicit formula for the operator $\breve{\varphi}(x)$ is not available, then the required construction $\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}$ can be derived with the help of the fluctuation-dissipation theorem (FDT) first given in works [13,14]. The mathematical formulation of FDT used below does not depend on the specific form of the interaction Hamiltonian [15] and, therefore, can be established on the basis of the formulas for waves noninteracting one with another. With this purpose, we introduce the retarded Green function

$$G_r(\mathbf{k},\omega) = -\frac{i}{\hbar}\int\langle[\breve{\varphi}(x)\breve{\varphi}(0)]\rangle\vartheta(t)\times$$

$$\times\exp\left(-i\mathbf{kr}+i\omega t\right)d\mathbf{r}dt = \frac{1}{\rho}\frac{1}{\omega^2-\omega_0^2+2i0\omega} \qquad (6)$$

and the advanced one

$$G_a(\mathbf{k},\omega) = \frac{i}{\hbar}\int\langle[\breve{\varphi}(x)\breve{\varphi}(0)]\rangle\vartheta(-t)\times$$

$$\times\exp\left(-i\mathbf{kr}+i\omega t\right)d\mathbf{r}dt = \frac{1}{\rho}\frac{1}{\omega^2-\omega_0^2-2i0\omega}. \qquad (7)$$

By $\vartheta(t)$, we denote the Heaviside step-function. It is worth noting that

$$G_r(\mathbf{k},\omega) = G_a^*(\mathbf{k},\omega) = G_r^*(\mathbf{k},-\omega).$$

The comparison of equalities (5)—(7) with regard for the relation $1+N(\omega) = -N(-\omega)$ allows us to write

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega} = i\hbar\left(1+N(\omega)\right)[G_r(\mathbf{k},\omega)-G_a(\mathbf{k},\omega)]. \qquad (8)$$

This equality that establishes the connection between the correlation function $\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}$ and the Green functions (6) and (7) presents a mathematical representation of FDT and is, in fact, an identity. The conditions for this identity to be valid are the Gibbs distribution and the possibility to describe the system in terms of some Hamiltonian. It is worth noting that the operators $\breve{\varphi}(x)$ and $\breve{\varphi}(x')$ at $t \neq t'$ do not commute each with other. For this reason, the representation of FDT for the correlator $\langle\breve{\varphi}(x')\breve{\varphi}(x)\rangle$, which differs from the above-considered one by the permutation of arguments, takes the different form

$$\int\langle\breve{\varphi}(0)\breve{\varphi}(x)\rangle\exp\left(-i\mathbf{kr}+i\omega t\right)d\mathbf{r}dt =$$

$$= i\hbar N(\omega)[G_r(\mathbf{k},\omega)-G_a(\mathbf{k},\omega)]. \qquad (9)$$

For classical frequencies $\hbar\omega << T$, equalities (8) and (9) coincide:

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega} = i\frac{T}{\omega}[G_r(\mathbf{k},\omega)-G_a(\mathbf{k},\omega)]. \qquad (10)$$

2.3. Phonon waves with nonlinear interaction

Consider the interaction of sonic waves. In the Einstein model, the potential energy of waves depends on a displacement of particles from the equilibrium position. The account of cubic terms in the expansion of the potential energy in $\breve{\varphi}(x)$ in the Schrödinger representation leads to the Hamiltonian

$$\hat{H}_f = \hat{H} + \int\hat{\varphi}(\mathbf{r})f(\mathbf{r},t)d\mathbf{r},$$

$$\hat{H} = \sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}\left(\hat{\alpha}_{\mathbf{k}}^+\hat{\alpha}_{\mathbf{k}}+\frac{1}{2}\right)+\frac{g}{3}\int\hat{\varphi}^3(\mathbf{r})d\mathbf{r}. \qquad (11)$$

In the Schrödinger representation,

$$\hat{\varphi}(\mathbf{r}) = \sum_{\mathbf{k}}\gamma_{\mathbf{k}}\left(\hat{\alpha}_{\mathbf{k}}\exp(i\mathbf{kr})+\hat{\alpha}_{\mathbf{k}}^+\exp(-i\mathbf{kr})\right).$$

Into the Hamiltonian $\hat{H}_f$, we introduced a classical function $f(\mathbf{r},t)$. The sense of the introduction of this auxiliary function will be clear in what follows. In the final formulas, we will set $f(\mathbf{r},t) = 0$. Our goal to calculate the correlator $\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}$ will be attained if, according to (8)-(10), we will succeed to calculate the functions $G_{r,a}(\mathbf{k},\omega)$. The indirect way to calculate these functions consists in the following. From the Schrödinger representation, we pass to the Heisenberg one,

$$\breve{\varphi}_f(\mathbf{r},t) = \hat{U}^+\hat{\varphi}(\hat{r})\hat{U} \qquad (12)$$

with the help of the operator $\hat{U}$ satisfying the equation

$$i\hbar\frac{d\hat{U}}{dt} = \hat{H}_f\hat{U}, \qquad (13)$$

the operation of differentiation of operators with respect to time

$$i\hbar\frac{d\breve{\varphi}_f}{dt} = \left[\breve{\varphi}_f,\hat{H}_f\right], \quad i\hbar\frac{d^2\breve{\varphi}_f}{dt^2} = \left[\frac{d\breve{\varphi}_f}{dt},\hat{H}_f\right]$$

and the commutation relations (4). We get

$$-\frac{d^2\breve{\varphi}_f(x)}{dt^2} = \frac{\chi}{\rho}\breve{\varphi}_f(x)+\frac{g}{\rho}\breve{\varphi}_f^2(x)+\frac{f(x)}{\rho}. \qquad (14)$$

The Hamiltonian $\breve{H}$ can be expressed in terms of the Heisenberg operators $\breve{\varphi}_f(x)$ as

$$\breve{H}_f = \frac{\rho}{2}\int\left(\frac{d\breve{\varphi}_f(x)}{dt}\right)^2 d\mathbf{r} + \frac{\chi}{2}\int\breve{\varphi}_f^2(x)d\mathbf{r}+$$
$$+\frac{g}{3}\int\breve{\varphi}_f^3(x)d\mathbf{r} + \int\breve{\varphi}_f(x)f(x)d\mathbf{r}. \quad (15)$$

The operator $\breve{\varphi}_f(x)$ can be constructed by solving Eq. (14) with the additional conditions (4) and taking into account that $\breve{\varphi}_f(x) \to \hat{\varphi}(r)$ as $t \to -\infty$. But we may do this with another means. This operator can be calculated directly from definition (13). Equation (13) can be rewritten in the integral form as

$$\breve{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} + \frac{1}{i\hbar}\int\limits_{-\infty}^{t} e^{-\frac{i}{\hbar}\hat{H}(t-t')}\times$$
$$\times\int f(\mathbf{r}',t')\breve{\varphi}_f(\mathbf{r}',t')\hat{U}(t')d\mathbf{r}'dt'.$$

The substitution of this expression in (12) yields the Kubo formula [16]

$$\breve{\varphi}_f(x) = \breve{\varphi}(x) - \frac{i}{\hbar}\int\limits_{-\infty}^{t}\Big[\breve{\varphi}(x)\breve{\varphi}(x')\Big]f(x')dx',$$
$$\breve{\varphi}(x) = e^{\frac{i}{\hbar}\hat{H}t}\hat{\varphi}(\mathbf{r})e^{-\frac{i}{\hbar}\hat{H}t}, \quad dx' = d\mathbf{r}'dt'. \quad (16)$$

By averaging both sides of equality (16) over the ensemble of systems, comparing the derived expression with definition (6), and taking the relation $\langle\breve{\varphi}(x)\rangle = 0$ into account, we get

$$\langle\breve{\varphi}_f(x)\rangle = \int G_r(x,x')f(x')dx. \quad (17)$$

Expression (17) has quite formal character, since, according to (16), the function G_r is not determined explicitly. But formula (17) indicates the means for the explicit derivation of G_r as the coefficient of proportionality between $\langle\breve{\varphi}\rangle$ and f, if we will manage to calculate $\langle\breve{\varphi}\rangle$ in any other way. Such a problem is solved below with the help of Eq. (14). However, we will solve preliminarily the other auxiliary problem.

2.4. Heat waves in an external Gauss field

Instead of Eq. (14), we consider the equation

$$-\frac{d^2\breve{\varphi}_f(x)}{dt^2} = \omega_0^2\breve{\varphi}_f(x) + \frac{g}{\rho}\breve{\xi}(x)\breve{\varphi}_f(x) + \frac{f(x)}{\rho}, \quad (18)$$

where $\xi(x)$ describes some random external Gauss field which acts on the system and is stationary in time. In other words, we are interesting in the thermal noise in the system, whose coefficient of elasticity undergoes the action of random fluctuations. Such a problem admits a complete study by analytic methods. Hamiltonian (15) now includes both the term describing the interaction of the fields $\breve{\xi}(x)$ and $\breve{\varphi}(x)$ and the Hamiltonian $\breve{H}_\xi$ responsible for the free field $\breve{\xi}(x)$. We need the existence of the joint Hamiltonian in order to have possibility to use FDT (8)-(10) in what follows. We rewrite Eq. (18) in the integral form as

$$\breve{\varphi}_f(x) - \breve{\varphi}(x) = \varphi_0(x) + \frac{g}{\rho}\int D_r^0(x,x_1)\breve{\xi}(x_1)\times$$
$$\times\Big[\breve{\varphi}_f(x_1) - \breve{\varphi}(x_1)\Big]dx_1, \quad (19)$$

where

$$-\frac{d^2\varphi_0(x)}{dt^2} - \omega_0^2\varphi_0(x) = \frac{f(x)}{\rho},$$

$$-\frac{d^2D_r^0(x,x')}{dt^2} - \omega_0^2D_r^0(x,x') = \delta(x-x'),$$

$$\delta(x-x') = \delta(t-t')\delta(\mathbf{r}-\mathbf{r}'),$$

and $\breve{\varphi}_f(x) \to \breve{\varphi}(x)$ as $f(x) \to 0$. We will solve Eq. (19) by the method of iterations:

$$\breve{\varphi}_f(x) - \breve{\varphi}(x) = \varphi_0(x)+$$
$$+\frac{g}{\rho}\int D_r^0(x-x_1)\breve{\xi}(x_1)\varphi_0(x_1)dx_1+$$
$$+\frac{g^2}{\rho^2}\int D_r^0(x-x_1)\breve{\xi}(x_1)D_r^0(x_1-x_2)\breve{\xi}(x_2)\times$$
$$\times\varphi_0(x_2)dx_1dx_2 + \ldots .$$

We will perform the operation of averaging in this series, by setting $\langle\breve{\xi}(x)\rangle = \langle\breve{\varphi}(x)\rangle = 0$. We get

$$\langle\breve{\varphi}_f(x)\rangle = \varphi_0(x) + \int D_r^0(x-x_1)\langle\breve{\xi}(x_1)\breve{\xi}(x_2)\rangle\times$$
$$\times D_r^0(x_1-x_2)\varphi_0(x_2)dx_1dx_2 + \ldots \quad (20)$$

It should be taken into account that, due to the Gauss character of the $\breve{\xi}(x)$, the correlators of higher

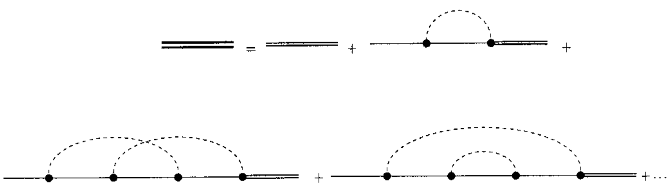

Fig. 1

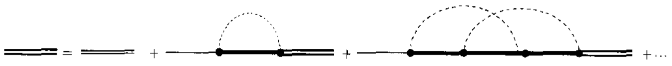

Fig. 2

orders $\langle \breve{\xi}\breve{\xi} \cdots \breve{\xi}\breve{\xi} \rangle$ are representable in the form of a sum of products of the correlators $\langle \breve{\xi}\breve{\xi} \rangle$ with all possible pairings. For example,

$$\langle \breve{\xi}(x_1)\breve{\xi}(x_2)\breve{\xi}(x_3)\breve{\xi}(x_4)\rangle =$$

$$= \langle \breve{\xi}(x_1)\breve{\xi}(x_2)\rangle\langle \breve{\xi}(x_3)\breve{\xi}(x_4)\rangle +$$

$$+\langle \breve{\xi}(x_1)\breve{\xi}(x_3)\rangle\langle \breve{\xi}(x_2)\breve{\xi}(x_4)\rangle +$$

$$+\langle \breve{\xi}(x_1)\breve{\xi}(x_4)\rangle\langle \breve{\xi}(x_2)\breve{\xi}(x_3)\rangle.$$

It is convenient now to clarify the structure of the iterative series with the help of Feynman diagrams.

We associate a solid line with the function D_r^0, a dashed line with the function $\langle \breve{\xi}(x)\breve{\xi}(x')\rangle$, a thin double line with the function φ_0, and a bold double line with the function $\langle \breve{\varphi}_f \rangle$. Let the constant g/ρ be associated with a point. The expansion in series (20) corresponds to the plot in Fig. 1. After the summation of reducible [17] diagrams, expansion (20) takes the form given in Fig. 2. In this series, the bold line corresponds to the sum of diagrams that is plotted in Fig. 3 and denoted as D_r in formulas. The series presented in Figs. 2 and 3 are transformed into the following Dyson integral equations:

$$\langle \breve{\varphi}_f(x)\rangle = \varphi_0(x) +$$

$$+\int D_r^0(x-x_1)\pi_r(x_1-x_2)\langle \breve{\varphi}_f(x_2)\rangle dx_1 dx_2, \qquad (21)$$

$$D_r(x-x') = D_r^0(x-x') +$$

$$+\int D_r^0(x-x_1)\pi_r(x_1-x_2)D_r(x_2-x')dx_1 dx_2, \qquad (22)$$

in which the operator π_r is associated with the graphical series presented in Fig. 4. This series represents the sum

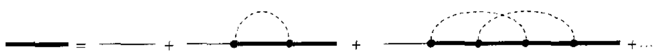

Fig. 3

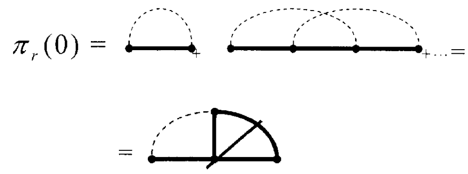

Fig. 4

of irreducible [17] diagrams constructed from bold continuous lines. In the one-loop approximation corresponding to the account of only the first term of this series, we get

$$\pi_r(x_1-x_2) = \frac{g^2}{\rho^2} D_r(x_1-x_2)\langle \breve{\xi}(x_1-x_2)\breve{\xi}(0)\rangle. \qquad (23)$$

We take explicitly into account the dependence of the functions on the difference of their arguments, which is true for spatially homogeneous systems. After the Fourier transformation

$$\langle \breve{\varphi}_f \rangle_{\mathbf{k}\omega} = \int e^{-i\mathbf{kr}+i\omega t}\langle \breve{\varphi}_f(x)\rangle dx$$

Eqs. (21) and (22) are solved in the explicit form. With regard for the equalities

$$D_r^0(\mathbf{k},\omega) = \frac{1}{\omega^2-\omega_0^2+2i0\omega},$$

$$\varphi_0(\mathbf{k},\omega) = \frac{f(\mathbf{k},\omega)}{\omega^2-\omega_0^2+2i0\omega},$$

these solutions take the form

$$\langle \breve{\varphi}_f \rangle_{\mathbf{k}\omega} = \frac{1}{\rho}\frac{f(\mathbf{k},\omega)}{\omega^2-\omega_0^2+2i0\omega} = \frac{1}{\rho}D_r(\mathbf{k},\omega)f(\mathbf{k},\omega), \qquad (24)$$

$$\langle D_r(\mathbf{k},\omega)\rangle = \frac{1}{\omega^2-\omega_0^2-\pi_r(\mathbf{k},\omega)}. \qquad (25)$$

The comparison of (24) to (17) shows that

$$G_r(\mathbf{k},\omega) = \frac{1}{\rho}D_r(\mathbf{k},\omega).$$

Thus, the function $D_r(\mathbf{k},\omega)$ defines completely the function $G_r(\mathbf{k},\omega)$ entering into the right-hand side of the mathematical representation of FDT.

Let us return to the study of the series presented in Fig. 4. First of all, we note that the function $D_r^0(\mathbf{k},\omega)$

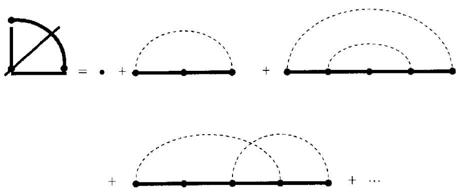

Fig. 5

does not contain the argument **k** in the model under consideration. This means that

$$D_r^0(x-x') = D_r^0(t-t')\delta(\mathbf{r}-\mathbf{r}'),$$

$$D_r^0(t-t') = \int e^{i\omega(t-t')} D_r^0(\mathbf{k},\omega)\frac{d\omega}{2\pi}.$$

Since the thin continuous line in the Feynman graphs which corresponds to the function D_r^0 is continuous and pierces all graphs, the structure of required functions turns out to be as follows:

$$D_r(x,x') = \delta(\mathbf{r}-\mathbf{r}')D_r(t-t'),$$

$$\pi_r(x,x') = \delta(\mathbf{r}-\mathbf{r}')\pi_r(t-t').$$

In this case in the model under consideration, the correlators $\langle\breve{\xi}(x)\breve{\xi}(x')\rangle$ in the Feynman series reveal themselves as the functions at spatially coinciding points. Therefore, without loss of generality, we can consider this correlator to be dependent only on the difference of times:

$$\langle\breve{\xi}(\mathbf{r},t)\breve{\xi}(\mathbf{r},t')\rangle = \eta(t-t').$$

Now, the polarization operator presented in Fig. 4 can be written analytically as

$$\pi_r(t,t') = \frac{g}{\rho}\int D_r(t-t_1)\eta(t_1-t_2)\Gamma(t_1,t_2,t')dt_1dt_2. \tag{26}$$

The vertex function Γ corresponds to the series of diagrams given in Fig. 5. It is worth noting the circumstance that the terms of the series for Γ, beginning from the second one, originate from terms of the series for π_r expressed through D_r^0 by means of the successive change of each thin continuous line corresponding to the function D_r^0 by a combination of two thin lines separated by a point, which corresponds to the product $D_r^0(g/\rho)D_r^0$. After the Fourier transformation,

$$\Gamma(t_1,t_2,t') = \int \Gamma(\omega_1,\omega_2,\omega_3)\times$$

$$\times\exp\left(-i\omega_1 t_1 - i\omega_2 t_2 + i\omega_3 t'\right)\frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi)^3}$$

we get, according to (26),

$$\pi_r(\omega,\omega') = \frac{g}{\rho}\int \delta(\omega-\omega_1-\omega_2)\times$$

$$\times D_r(\omega_1)\eta(\omega_2)\Gamma(\omega_1,\omega_2,\omega')\frac{d\omega_1 d\omega_2}{2\pi}. \tag{27}$$

We now make assumption as for the correlator $\eta(t-t')$. Let this correlator exponentially decay with increase in t with a constant γ:

$$\eta(t-t') = \eta\exp\left(-\gamma|t-t'|\right),$$

Here, η is some constant. Then the Fourier transform

$$\eta(\omega) = i\eta\left(\frac{1}{\omega+i\gamma} - \frac{1}{\omega-i\gamma}\right)$$

possesses a clearly pronounced maximum at small γ at the point $\omega = 0$, which allows us to take out the integrand in (27) at the point $\omega_2 = 0$ beyond the integral symbol and to represent this expression as

$$\pi_r(\omega,\omega') = \eta\frac{g}{\rho}\int \delta(\omega-\omega_1)D_r(\omega_1)\Gamma(\omega_1,0,\omega')d\omega_1. \tag{28}$$

The comparison of this equality with relation (23) derived in the one-loop approximation shows that $\Gamma(\omega_1,0,\omega') = 2\pi\delta(\omega_1-\omega')g/\rho$ in the one-loop approximation. The next terms of the series $\Gamma(\omega_1,0,\omega')$ are shown in Fig. 5. After the Fourier transformation, the δ-function depending on the algebraic sum of three frequencies appears at each elementary vertex of this series represented by a point. At the points separating two functions D_r^0, one of the frequencies of the δ-function is zero because $\omega_2 = 0$ in the function Γ. For this reason, the arguments of two functions $D_r^0(\omega)$ separated by a point turn out to be the same. But

$$\left(D_r^0(\omega)\right)^2 = \frac{d}{d\omega_0^2}\frac{1}{\omega^2-\omega_0^2+2i0\omega} = \frac{d}{d\omega_0^2}D_r^0(\omega).$$

Now, the comparison of the series shown in Figs. 5 and 4 indicates that

$$\Gamma(\omega_1,0,\omega') = 2\pi\frac{g}{\rho}\delta(\omega_1-\omega') + \frac{g}{\rho}\frac{d}{d\omega_0^2}\pi_r(\omega_1,\omega'). \tag{29}$$

This connection of the vertex function Γ and the operator π_r has the same algebraic nature as the Ward identity in electrodynamics [18] which connects also a mass operator with a vertex function, for which one of its arguments is zero.

We introduce the notation

$$\pi_r(\omega,\omega')=2\pi\delta(\omega-\omega')\pi_r(\omega)$$

and substitute (29) in (28). For the operator $\pi_r(\omega)$ under consideration, we get the equation

$$\pi_r(\omega)=\eta\frac{g^2}{\rho^2}D_r(\omega)\left[1+\frac{d}{d\omega_0^2}\pi_r(\omega)\right]. \tag{30}$$

Despite the fact that Eq. (30) is proved with the help of divergent Feynman series, this equation is valid for any g. The exactness of this equation is conditioned by that of the Ward identity in electrodynamics. In view of (25), we may write Eq. (30) as

$$\pi_r(\omega)=-\eta\frac{g^2}{\rho^2}\frac{d}{d\omega_0^2}\ln\left(\omega^2-\omega_0^2-\pi_r(\omega)\right)=$$

$$=-\eta\frac{g^2}{\rho^2}\frac{d}{d\omega_0^2}\ln D_r^{-1}(\omega).$$

We will be interested in the asymptotic behavior of the functions at small frequencies. Setting the frequency $\omega=0$ in (30), we obtain

$$\pi_r(0)=\frac{g^2}{\rho^2}\frac{\eta}{-\omega_0^2-\pi_r(0)}\left[1+\frac{d}{d\omega_0^2}\pi_r(0)\right]. \tag{31}$$

We begin to study this equation in the one-loop approximation. The accuracy of the results derived in such a way will be estimated later on. In other words, we keep only the first term in the series in Fig. 4, which is equivalent to the neglect of the second term in the square brackets of (31). The quadratic equation for $\pi_r(0)$ arisen in such a way has the obvious solution

$$\pi_r(0)=-\frac{\omega_0^2}{2}+\sqrt{\frac{\omega_0^4}{4}-\frac{g^2\eta}{\rho^2}}.$$

The sign before the root is chosen so that $\pi_r(0)\to 0$ as $g\to 0$. If the inequality

$$\omega_0^2>\omega_{0c}^2=2\sqrt{\frac{g^2\eta}{\rho^2}}$$

holds, the operator $\pi_r(0)$ is real, and, according to FDT, none noise arises in the system at small frequencies. At the point $\omega_0=\omega_{0c}$, a distinctive "phase transition" arises, and the operator $\pi_r(0)$ acquires the imaginary part

$$\pi_r(0)=-\frac{\omega_0^2}{2}-i\,\mathrm{sgn}\omega\sqrt{\frac{g^2\eta}{\rho^2}-\frac{\omega_0^4}{4}}. \tag{32}$$

The sign before the root is chosen so that the right-hand sides in (8)—(10) turn out positive values. The sign of $\mathrm{Im}\pi_r(0)$, by the definition

$$\mathrm{Im}\pi_r(\omega)=\int\limits_0^\infty \pi_r(t)\sin\omega t dt \tag{33}$$

varies вместе with variation in the sign of the frequency ω. The substitution of (32) in (8) testifies to the appearance of a noise in the system in the classical limit $\hbar\omega << T$:

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}=\frac{2T}{\omega}\mathrm{sgn}\omega\frac{\sqrt{\frac{g^2\eta}{\rho^2}-\frac{\omega_0^4}{4}}}{\eta\frac{g^2}{\rho}},\quad \omega\to 0 \tag{34}$$

whose characteristic peculiarity is the $1/\omega$ behavior. According to (10) and (32), this peculiarity appears due to the finite imaginary part of the operator $\pi_r(0)$, as seen from the solution of the nonlinear equation (31). The imaginary part cannot be derived by solving Eq. (31) by the iterative method. We may say that the $1/\omega$ peculiarity appears due to the "phase transition" conditioned, according to Eq. (31), by the feedback by noise in the system. We note that correlator (34) does not depend on the variable $\mathbf{k}$ in the accepted model.

The presence of the finite imaginary part of the operator $\pi_r(\omega)$ as $\omega\to 0$ is possible, according to (33), only in the case where $\pi_r(t)$ possesses the $\pi_r(t)\sim t^{-1}$ asymptote at large times. The same asymptote at large times is characteristic of the Green function, $D_r(t)\sim t^{-1}$, for the same reason. At the same time, it is clear that the $1/\omega$ peculiarity of the correlator $\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}$ cannot extend up to arbitrarily small frequencies. This would break the boundedness of the integral

$$\int\limits_0^\infty\langle\breve{\varphi}'\breve{\varphi}\rangle_{\mathbf{k}\omega}d\omega.$$

and would lead to the infinite power in the fluctuating system. The arisen complication is caused by the used approximation upon the transition from (27) to (28), which is true for $\gamma\to 0$. Since, the correlator $\langle\breve{\varphi}(x)\breve{\varphi}(x')\rangle$ decays exponentially with increase in the argument

$|t - t'|$ for a finite γ, it turns out that $\mathrm{Im}\pi_r(\omega) \to 0$ as $\omega \to 0$ according to (23) but does not remain to be a finite value. We note that the $1/\omega$ peculiarity of the noise arises only upon a finite $\mathrm{Im}\pi_r(0)$, which occurs under the neglect of the constant γ. Hence, the $1/\omega$ spectrum exists only in the frequency interval

$$\gamma << \omega << \omega_0. \tag{35}$$

We now go beyond the scope of the one-loop approximation and consider the term omitted earlier in (31). In other words, we will show that, under certain conditions, the full equation (31) possesses a solution, whose imaginary part remains to be finite as $\omega \to 0$, which proves the presence of the $1/\omega$-noise. We introduce a new variable

$$\pi_r(0) = -\frac{\omega_0^2}{2} + y(\omega_0),$$

Then

$$-\frac{\omega_0^4}{4} + y^2(\omega_0) + \frac{g^2\eta}{\rho^2}\left(\frac{1}{2} + \frac{dy(\omega_0)}{d\omega_0^2}\right) = 0.$$

In the dimensionless variables, this equation looks

$$-\frac{v^2}{4} + \Phi^2(v) + \frac{1}{2} + \frac{d\Phi(v)}{dv} = 0,$$

$$v = \omega_0^2\sqrt{\frac{\rho^2}{g^2\eta}}, \quad \Phi(v) = y(\omega_0)\sqrt{\frac{\rho^2}{g^2\eta}}. \tag{36}$$

If $g \to \infty$, then the argument $v \to 0$. Consider large values of the parameter g, when the inequality

$$\nu = \omega_0^2\sqrt{\frac{\rho^2}{g^2\eta}} << 1 \tag{37}$$

is satisfied. In this case, the solution of Eq. (36) should be searched in the form

$$\Phi(v) = \lambda_0 + \lambda_1 v + \lambda_2 v^2 + \lambda_3 v^3 + \lambda_4 v^4, \tag{38}$$

where the constants $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, and λ_4 should be determined. By substituting (38) in (36) and equating the coefficients of the identical degrees of the variable v, we get the unique solution

$$\Phi(v) = \frac{v^3}{12} - \frac{isgn\omega}{\sqrt{2}}\left(1 - \frac{\nu^4}{24}\right).$$

for $\lambda_0 \neq 0$. In this case, the operator

$$\pi_r(0) = -\frac{\omega_0^2}{2}\left(1 - \frac{\omega_0^4\rho^2}{6g^2\eta}\right) -$$

$$-i\mathrm{sgn}\omega\sqrt{\frac{g^2\eta}{2\rho^2}}\left(1 - \frac{\omega_0^8\rho^4}{24g^4\eta^2}\right). \tag{39}$$

The existence of such an asymptote, which testifies to the boundedness of $Im\pi_r(0)$ as $\omega \to 0$ under conditions where inequality (37) holds, proves the presence of the $1/\omega$-noise in the frequency interval (35) irrespective of the one-loop approximation. If inequality (37) holds, the one-loop approximation leads, in its turn, to the relation

$$\pi_r(0) = -\frac{\omega_0^2}{2} - i\mathrm{sgn}\omega\sqrt{\frac{g^2\eta}{\rho^2}}. \tag{40}$$

The comparison of the asymptotic result (40) derived in the one-loop approximation to the asymptotic behavior of the exact solution (39) indicates that their real parts are practically identical with regard for inequality (37). But the imaginary part in the one-loop approximation is overestimated by a factor of $\sqrt{2}$ for the proper combination of letters. If the inverse inequality

$$\nu = \omega_0^2\sqrt{\frac{\rho^2}{g^2\eta}} >> 1 \tag{41}$$

is valid, the one-loop approximation gives the proper result, because $\pi_r(0)$ leads to the first term of perturbation theory. Thus, the accuracy of the one-loop approximation is quite sufficient in order to judge the presence or absence of the $1/\omega$-noise in the system upon the primary acquaintance with it. In the interval intermediate between inequalities (37) and (41), the one-loop approximation gives the interpolational result and properly predicts the presence of a "phase transition". However, the quantitative relations derived upon its description in this interval should be considered with the precaution.

2.5. Double Feedback

The above-performed analysis allows us to assert that one should use no external noise-based field $\breve{\xi}(x)$ in order to generate the $1/\omega$-noise. Under the conditions of thermodynamic equilibrium in physical systems described, in particular, by Hamiltonian (3), the standard thermal fluctuations arise. The amplitude of fluctuations increases with temperature. If these fluctuations are used as a field $\breve{\xi}(x)$, i.e. they are allowed to act on the coefficient of elasticity χ, then we may expect the appearance of fluctuations with the $1/\omega$ spectrum in such a nonlinear system with the feedback described by Eq. (14). Since a single feedback has been already studied above according to (31) in connection with the investigation of the $1/\omega$-noise in systems described by the linear equation (18), we say now about a double feedback.

Consider the nonlinear equation (14) in more details. Assume that the problem is solved and its solution is known. We denote it as $\breve{\psi}(x)$. Rewrite Eq. (14) as

$$-\frac{d^2\breve{\varphi}_f(x)}{dt^2} = \omega_0^2\breve{\varphi}_f(x) + \frac{g}{\rho}\breve{\psi}(x)\breve{\varphi}_f(x) + \frac{f(x)}{\rho}. \quad (42)$$

Since the operator $\breve{\psi}(t)$ is known, we get the above-studied linear equation (18) for the operator $\breve{\varphi}_f(x)$. The operator $\breve{\psi}(t)$ serves, of course, as its solution. We make a show that we do not know it and begin to study Eq. (42). By restricting ourselves by the one-loop approximation, we get

$$\pi_r(t-t') = \frac{g^2}{\rho^2}D_r(t-t')\langle\breve{\psi}(\mathbf{r},t)\breve{\psi}(\mathbf{r},t')\rangle. \quad (43)$$

in accordance with (23). Instead of the operators $\breve{\psi}$ in (43), we will again write the operator $\breve{\varphi}$, bearing in mind that the results derived in such a way will correspond to the nonlinear equation (14) for $f(\mathbf{r},t) = 0$. In this case, the question about the commutation relations does not arise, because we study the classical region of frequencies. After the Fourier transformation, the operator π_r becomes

$$\pi_r(\omega) = \frac{g^2}{\rho^2}\int D_r(\omega-\omega_1)\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega_1}\frac{d\omega_1 d\mathbf{k}}{(2\pi)^4}, \quad (44)$$

Since the operator $\pi_r(0)$ defines the required correlator $\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}$ and, according to (43), depends also on this correlator, the consistency condition arises. We are interested in the solution with the $1/\omega$ spectrum for $\omega \to 0$ and will seek for the required correlator in the form

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega} = \frac{2\pi A}{|\omega|}\vartheta\left(\omega_0 - |\omega|\right) +$$

$$+2\pi B[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)], \quad B = \frac{T}{2\chi}, \quad (45)$$

in accordance with (5) and (34), where A is the unknown constant. Taking the $1/|\omega|$ peculiarity in (45) into account, we can take out the function D_r in (44) beyond the integral symbol at the point $\omega_1 = 0$. We get

$$\pi_r(\omega) = \frac{g^2}{\rho^2}D_r(\omega)\left[\int\limits_{\gamma}^{\omega_0}\frac{2A}{\omega}d\omega + \right.$$

$$+\frac{Bg^2/\rho^2}{(\omega-\omega_0)^2 - \omega_0^2 - \pi_r(\omega-\omega_0)} +$$

$$\left. +\frac{Bg^2/\rho^2}{(\omega+\omega_0)^2 - \omega_0^2 - \pi_r(\omega+\omega_0)}\right]\frac{k_{\mathrm{D}}^3}{6\pi^2}. \quad (46)$$

By integrating over the variable $\mathbf{k}$, we took into account the boundary of the Debye sphere k_{D}. In view of (10) and the parity of the correlator $\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}$, the interval of the integration over ω in (46) is the semiaxis ($\gamma \div \infty$). But if γ is a small value, the integrand at $t = 0$ mainly contributes in the region of small frequencies due to the $1/\omega$ peculiarity. For this reason, the upper integration limit can be cut by the frequency ω_0, which agrees with (35). The lower integration limit cannot be zero. But the model under consideration has no parameters, from which we can construct the lower integration limit with the dimension of frequency. This means that, in the accepted model, the $1/\omega$ peculiarity spreads up to arbitrarily small frequencies. The limitation from the side of small frequencies can arise only at the expense of the interaction of the studied system with external fields providing the lower integration limit in (46) in real situations, in accordance with the above-presented consideration.

Equality (46) is a system of coupling equations. The later this chain will be broken, the more exact the result will be obtained. We are interested in $\pi_r(0)$. According to (46),

$$\pi_r(0) = \left(\frac{g^2}{\rho^2}\frac{2A\ln\frac{\omega_0}{\gamma}}{\omega_0^2 - \pi_r(0)} - \frac{Bg^2/\rho^2}{\pi_r(-\omega_0)} - \frac{Bg^2/\rho^2}{\pi_r(\omega_0)}\right)\frac{k_{\mathrm{D}}^3}{6\pi^2}. \quad (47)$$

We now write the equations for $\pi_r(\pm\omega_0)$ and restrict ourselves by the diagonal approximation in these equations, which is quite admissible as $\gamma \to 0$. Then

$$\pi_r^2(\pm\omega_0) = -A\frac{g^2k_{\mathrm{D}}^3}{3\pi^2\rho^2}\ln\frac{\omega_0}{\gamma}.$$

The function $\mathrm{Im}\pi_r(\omega)$ is odd according to (33), which leads to the mutual cancellation of two last terms in (47). This means that we may neglect the influence of a noise in the resonance region of frequencies on the formation of the $1/\omega$-noise in the first approximation. We now get

$$\pi_r(0) = -\frac{\omega_0^2}{2} - i\mathrm{sgn}\omega\sqrt{\frac{k_\mathrm{D}^3 g^2}{3\pi^2\rho^2}A\ln\frac{\omega_0}{\gamma} - \frac{\omega_0^4}{4}}, \qquad (48)$$

and formulas (10) and (48) yield

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega} = \frac{2T}{\omega}\mathrm{sgn}\omega\frac{\sqrt{A\frac{k_\mathrm{D}^3 g^2}{3\pi^2\rho^2}\ln\frac{\omega_0}{\gamma} - \frac{\omega_0^4}{4}}}{A\frac{k_\mathrm{D}^3 g^2}{3\pi^2\rho}\ln\frac{\omega_0}{\gamma}}. \qquad (49)$$

The comparison of (45) and (49) shows that

$$\left(A\frac{k_\mathrm{D}^3 g^2}{3\pi^2\rho^2}\ln\frac{\omega_0}{\gamma}\right)^2 =$$

$$= \frac{Tg^2\omega_0^2 k_\mathrm{D}^3}{3\pi^3\rho^3}\sqrt{A\frac{k_\mathrm{D}^3 g^2}{3\pi^2\omega_0^4\rho^2}\ln\frac{\omega_0}{\gamma} - \frac{1}{4}}\ln\frac{\omega_0}{\gamma}.$$

The equation of consistency arisen in such a way can be rewritten in the dimensionless quantities as

$$z^2 = \zeta\sqrt{z - \frac{1}{4}}, \quad z = A\frac{g^2 k_\mathrm{D}^3}{3\pi^2\rho^2\omega_0^4}\ln\frac{\omega_0}{\gamma},$$

$$\zeta = \frac{k_\mathrm{D}^3 g^2 T}{3\pi^3\rho^3\omega_0^6}\ln\frac{\omega_0}{\gamma}.$$

This algebraic equation has solution only if the parameter ζ is greater than some critical value ζ_c which defines, in its turn, the critical temperature T_c. At the critical point, the curves z^4 and $\zeta^2(z - 1/4)$ touch each other. Therefore, at this point, their derivatives are identical, which allows us to get $\zeta_c = 2/\sqrt{27}$, $z_c = 1/3$, and

$$T_c = \frac{2\pi^3\rho^3\omega_0^6}{\sqrt{3}k_\mathrm{D}^3 g^2}\left(\ln\frac{\omega_0}{\gamma}\right)^{-1}.$$

For $T > T_c$, the equation defining z has two solutions. The existence of one of them is reliable, because it falls into the region where the one-loop approximation holds. The other solution falls into the interpolational region and serves as an object of the further studies.

Instead of formula (49), we now have

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega} = z\frac{6\pi^3\omega_0^4}{k_\mathrm{D}^3 g^2\omega}\rho^2\mathrm{sgn}\omega\left(\ln\frac{\omega_0}{\gamma}\right)^{-1}$$

We have proved the appearance of the $1/\omega$-noise in a system upon the study of the Einstein model in a Gauss random field with the help of Eq. (31), whereas we restrict ourselves by the one-loop approximation upon the study of the Einstein model with a nonlinear interaction. The one-loop approximation allows us to perform all calculations in the analytic form. Nevertheless, the following problem remains unsolved. Upon the summation of reducible diagrams to deduce formula (23), we made assumption about the Gauss character of fluctuations, which allows us to simplify the correlators of higher orders. But nothing yields the Gauss character of the field $\breve{\psi}(x)$. This calls into question the validity of the derivation of formula (23), which concerns Eq. (42). To eliminate such an objection, we will show that the validity of formulas in the one-loop approximation, in particular of formula (23), is independent of the form of statistical properties of the field $\breve{\psi}(x)$.

2.6. Method of Functional Derivatives

The solution of Eq. (42) can be represented as

$$\breve{\varphi}_f(x) = \breve{\varphi}(x) + \frac{1}{\rho}\int\breve{D}_r(x,x')f(x')dx', \qquad (50)$$

where $\breve{\varphi}(x)$ stands for a solution of the corresponding homogeneous equation. The operator function $\breve{D}_r(x,x')$ can be derived from

$$\frac{d^2 D_r(x,x')}{dt^2} + \omega_0^2\breve{D}_r(x,x') +$$

$$+\frac{g}{\rho}\breve{\psi}(x)\breve{D}_r(x,x') = -\delta(x-x'). \qquad (51)$$

By averaging (50) over the ensemble of systems and comparing the derived equality with equality (17), we get that the function $\langle\breve{D}_r(x-x')\rangle$ derived in such a way is related to the Green function entering into the right-hand side of the mathematical representation of FDT in the following manner:

$$G_r(x,x') = \frac{1}{\rho}\langle\breve{D}_r(x,x')\rangle.$$

We now average Eq. (51) over the statistical ensemble of systems

$$\frac{d^2\langle\breve{D}_r(x,x')\rangle}{dt^2} + \omega_0^2\langle\breve{D}_r(x,x')\rangle +$$

$$+\frac{g}{\rho}\langle\breve{\psi}(x)\breve{D}_r(x,x')\rangle = -\delta(x-x') \qquad (52)$$

and then will find $\langle \breve{D}_r \rangle$. To solve Eq. (52), we introduce the auxiliary functional

$$\breve{S} = \exp\left(-i\int \mu(x)\breve{\psi}(x)dx\right),$$

where $\mu(x)$ is some smooth classical function. We multiply Eq. (51) by $\breve{S}$ from the right and carry out the summation over the ensemble of systems. For the auxiliary function

$$D_r(x,x'|\mu) = \frac{\langle \breve{D}_r(x,x')\breve{S}\rangle}{\langle \breve{S}\rangle},$$

we have

$$\frac{\delta D_r(x,x'|\mu)}{\delta\mu(z)} = -i\frac{\langle \breve{\psi}(z)\breve{D}_r(x,x')\breve{S}\rangle}{\langle \breve{S}\rangle} +$$

$$+iD_r(x,x'|\mu)\frac{\langle \breve{\psi}(z)\breve{S}\rangle}{\langle \breve{S}\rangle}.$$

Then, according to Eq. (52), we get

$$\frac{d^2 D_r(x,x'|\mu)}{dt^2} + \omega_0^2 D_r(x,x'|\mu) + i\frac{g}{\rho}\frac{\delta D_r(x,x'|\mu)}{\delta\mu(x)} +$$

$$+\frac{g}{\rho}\frac{\langle \breve{\psi}(x)\breve{S}\rangle}{\langle \breve{S}\rangle}D_r(x,x'|\mu) = -\delta(x-x'). \tag{53}$$

It is obvious that $D_r(x,x'|\mu) = \langle \breve{D}_r(x-x')\rangle$ for $\mu(x) = 0$. In addition, we are interested in the noise in the system as $\omega \to 0$. This region of classical physics allows us not to trace the commutation relations of the operators $\breve{\psi}(x)$, by considering them as commuting one with another.

Various methods of solution of the equations with variational derivatives are available. Taking $\delta D_r(x,x'|\mu)/\delta\mu(z)$ as the unknown function, we can deduce an equation for it, by varying (53) over $\mu(z)$:

$$\frac{d^2}{dt^2}\frac{\delta D_r(x,x'|\mu)}{\delta\mu(z)} + \omega_0^2\frac{\delta D_r(x,x'|\mu)}{\delta\mu(z)} + i\frac{g}{\rho}\frac{\delta^2 D_r(x,x'|\mu)}{\delta\mu(x)\delta\mu(z)} +$$

$$+\frac{g}{\rho}\frac{\langle \breve{\psi}(t)\rangle}{\langle \breve{S}\rangle}\frac{\delta D_r(x,x'|\mu)}{\delta\mu(z)} + \frac{g}{\rho}\left[-i\frac{\langle \breve{\psi}(x)\breve{\psi}(z)\breve{S}\rangle}{\langle \breve{S}\rangle} +\right.$$

$$\left.+i\frac{\langle \breve{\psi}(x)\breve{S}\rangle\langle \breve{\psi}(z)\breve{S}\rangle}{\langle \breve{S}\rangle^2}\right]D_r(x,x'|\mu) = 0. \tag{54}$$

But we are faced now with the second variational derivative of $D_r(x,x'|\mu)$. For it, we can deduce an equation, by varying (54). Thus, we get the unclosed chain of equations. The later such a chain will be broken, the more exact the result will be derived. We restrict ourselves by two equations (53) and (54). This means that the more exact solutions derived with regard for omitted equations will only improve our result. We note that this is the additional substantiation of the validity of using the one-loop approximation which is independent of the previous analysis. Just this circumstance explains the closeness of formulas (39) and (40).

With the help of (53), Eq. (54) can be resolved as follows:

$$\frac{\delta D_r(x,x'|\mu)}{\delta\mu(z)} = -i\frac{g}{\rho}\int D_r(x,x_1|\mu)\times$$

$$\times\left[\frac{\langle \breve{\psi}(x_1)\breve{\psi}(z)\breve{S}\rangle}{\langle \breve{S}\rangle} - \frac{\langle \breve{\psi}(x_1)\breve{S}\rangle\langle \breve{\psi}(z)\breve{S}\rangle}{\langle \breve{S}\rangle^2}\right]\times$$

$$\times D_r(x_1,x'|\mu)dx_1. \tag{55}$$

If Eq. (54) would not contain variational derivatives, then (55) would be its exact solution. The substitution of (55) in (54) demonstrates that one variational derivative remains not compensated. The approximate equality (55) corresponds to the one-loop approximation. Indeed, setting $\rho = 0$, substituting $\breve{\psi}$ by $\breve{\varphi}$, and taking into account that $\langle \breve{\varphi}(x)\rangle = 0$, formulas (55) and (53) yield a closed integro-differential equation for the required $\langle D_r(x-x')\rangle$ which is equivalent to the integral equation (22). In this case, the operator $\pi_r(t-t')$ coincides with the operator defined by (43). We have arrived at the previous result. But the method of variational derivatives has some advantage. This method yields that the form of formulas in the one-loop approximation does not depend on the form of a statistics for the distribution of a studied quantity, which justifies the application of formula (23) to Eq. (42).

Thus, we have shown that, the attainment of some threshold temperature T_c in the Einstein model of solids leads to the appearance of the thermal $1/\omega$-noise. In this case, we made no assumptions about the presence of Gauss distributions.

3. Debye Model

Contrary to the Einstein model, the potential energy of real solids is defined by the derivatives of the displacements of particles with respect to coordinates. Thus, the written Hamiltonian properly describes acoustic waves with linear dispersion relation.

3.1. Theory of Linear Acoustic Waves

We consider a solid to be isotropic. As above, we restrict ourselves by the scalar description of a displacement $\breve{\varphi}(x)$. Such a model is called the Debye model. In this model, the linear waves are described by the Hamiltonian

$$\hat{H} = \frac{\rho}{2}\int\left(\frac{\partial\breve{\varphi}(x)}{\partial t}\right)^2 d\mathbf{r} + \frac{\chi}{2}\int\frac{\partial\breve{\varphi}(x)}{\partial r_i}\frac{\partial\breve{\varphi}(x)}{\partial r_i}d\mathbf{r}. \quad (56)$$

The repeated indices i should be summed. The substitution of (1) in (56) allows us to write the Hamiltonian in the form (3), if we set

$$\gamma_k = \sqrt{\frac{\hbar}{2kV\sqrt{\rho\chi}}}, \quad \omega_k = k\sqrt{\frac{\chi}{\rho}}.$$

In this case, the commutation relations (4) preserve their form. Using the operation of differentiation of operators with respect to time, we obtain

$$\nabla^2\breve{\varphi}(x) - \frac{1}{v^2}\frac{\partial^2\breve{\varphi}(x)}{\partial t^2} = 0, \quad v = \sqrt{\frac{\chi}{\rho}}.$$

Here, v is the sound velocity. Thus, Hamiltonian (56) describes the propagation of acoustic waves in a solid with the dispersion relation $\omega_k = vk$. As above, we are interested in the correlator $\langle\breve{\varphi}\breve{\varphi}\rangle$ describing the noise of acoustic waves with nonlinear interaction. Preliminarily, we will solve an auxiliary problem.

3.2. Thermal fluctuations of acoustic waves in the external field

We assume that the coefficient of elasticity χ undergoes the action of the external random Gauss field $\breve{\xi}(x)$. Such a field can be formed, in particular, by phonons of different nature which are present in a solid. Instead of Hamiltonians (3) and (56), we consider a more general expression

$$\hat{H}_f = \sum_{\mathbf{k}}\hbar\omega_k\left(\hat{\alpha}_{\mathbf{k}}\breve{\alpha}_{\mathbf{k}} + \frac{1}{2}\right) +$$

$$+\frac{1}{2}\int\breve{\xi}(x)\frac{\partial\breve{\varphi}_f(x)}{\partial r_i}\frac{\partial\breve{\varphi}_f(x)}{\partial r_i}d\mathbf{r} + \int f(x)\breve{\varphi}_f(x)d\mathbf{r} + \hat{H}_\xi. \quad (57)$$

As above in (11), we have introduced an auxiliary function in the Hamiltonian $f(x)$, and $\hat{H}_\xi$ stands for the Hamiltonian of the free external field $\breve{\xi}(x)$. Hamiltonian (57) in the Heisenberg representation corresponds to the wave equation

$$\nabla^2\breve{\varphi}_f(x) - \frac{1}{v^2}\frac{\partial^2\breve{\varphi}_f(x)}{\partial t^2} + \frac{1}{\chi}\frac{\partial}{\partial r_i}\breve{\xi}(x)\frac{\partial}{\partial r_i}\breve{\varphi}_f(x) = \frac{f(x)}{\chi}. \quad (58)$$

We rewrite this equation in the integral form as

$$\breve{\varphi}_f(x) - \breve{\varphi}(x) = \varphi_0(x) - \frac{1}{\chi}\int D_r^0(x, x_1)\times$$

$$\times\frac{\partial}{\partial r_{1i}}\breve{\xi}(x_1)\frac{\partial}{\partial r_{1i}}\Big[\breve{\varphi}_f(x_1) - \breve{\varphi}(x_1)\Big]dx_1, \quad (59)$$

where

$$\nabla^2\varphi_0(x) - \frac{1}{v^2}\frac{\partial^2\varphi_0(x)}{\partial t^2} = \frac{f(x)}{\chi},$$

$$\nabla^2 D_r^0(x, x') - \frac{1}{v^2}\frac{\partial^2 D_r^0(x, x')}{\partial t^2} = \delta(x - x'). \quad (60)$$

By $\breve{\varphi}(x)$, we denote the solution of Eq. (58) for $f(x) = 0$. We solve Eq. (59) by the iterative method and carry out the integration by parts:

$$\breve{\varphi}_f(x) - \breve{\varphi}(x) = \varphi_0(x) + \frac{1}{\chi}\int\frac{\partial}{\partial r_{1i}}D_r^0(x, x_1)\times$$

$$\times\breve{\xi}(x_1)\frac{\partial}{\partial r_{1i}}\varphi_0(x_1)dx_1 + \frac{1}{\chi^2}\int\frac{\partial}{\partial r_{1i}}D_r^0(x, x_1)\breve{\xi}(x_1)\times$$

$$\times\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}D_r^0(x_1, x_2)\cdot\breve{\xi}(x_2)\frac{\partial}{\partial r_{2j}}\varphi_0(x_2)dx_1dx_2 + \ldots$$

To this iterative series, we apply the operation of averaging. Since $\langle\breve{\xi}(x)\rangle = 0$, we get

$$\langle\breve{\varphi}_f(x)\rangle = \varphi_0(x) + \frac{1}{\chi^2}\times$$

$$\times\int\frac{\partial}{\partial r_{1i}}D_r^0(x, x_1)\langle\breve{\xi}(x_1)\breve{\xi}(x_2)\rangle\times$$

$$\times\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}D_r^0(x_1,x_2)\frac{\partial}{\partial r_{2j}}\varphi_0(x_2)dx_1dx_2+\ldots \quad (61)$$

By using again the Gauss properties of the distribution of $\breve{\xi}(x)$, we carry out the summation of reducible diagrams in (61). In the one-loop approximation, we obtain

$$\langle\breve{\varphi}_f(x)\rangle=\varphi_0(x)+\frac{1}{\chi^2}\times$$
$$\times\int\frac{\partial}{\partial r_{1i}}D_r^0(x,x_1)\langle\breve{\xi}(x_1)\breve{\xi}(x_2)\rangle\times$$
$$\times\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}D_r(x_1,x_2)\frac{\partial}{\partial r_{2j}}\langle\breve{\varphi}(x_2)\rangle dx_1dx_2, \quad (62)$$

where the full Green function D_r can be derived from the equation

$$D_r(x,x')=D_r^0(x,x')+\frac{1}{\chi^2}\times$$
$$\times\int\frac{\partial}{\partial r_{1i}}D_r^0(x,x_1)\langle\breve{\xi}(x_1)\breve{\xi}(x_2)\rangle\times$$
$$\times\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}}D_r(x_1,x_2)\frac{\partial}{\partial r_{2j}}D_r(x_2,x')dx_1dx_2. \quad (63)$$

If the random field $\breve{\xi}(x)$ is stationary in time and spatially homogeneous, the functions $\langle\breve{\xi}(x)\breve{\xi}(x')\rangle$ and $D_r(x,x')$ depend only on the difference of their arguments. Equations (62) and (63) can be easily solved by applying the Fourier transformation

$$\langle\breve{\varphi}\rangle_{\mathbf{k}\omega}=\frac{\varphi_0(\mathbf{k},\omega)}{1-D_r^0(\mathbf{k},\omega)\pi_r(\mathbf{k},\omega)},$$
$$D_r(\mathbf{k},\omega)=\frac{D_r^0(\mathbf{k},\omega)}{1-D_r^0(\mathbf{k},\omega)\pi_r(\mathbf{k},\omega)}, \quad (64)$$

where

$$\pi_r(\mathbf{k},\omega)=\frac{1}{\chi^2}\int\langle\breve{\xi}\breve{\xi}\rangle_{\mathbf{k}'\omega'}\left(k^2-(\mathbf{k}\mathbf{k}')\right)^2\times$$
$$\times D_r(\mathbf{k}-\mathbf{k}',\omega-\omega')\frac{d\mathbf{k}'d\omega'}{(2\pi)^4}. \quad (65)$$

According to (60), we have

$$\varphi_0(\mathbf{k},\omega)=D_r^0(\mathbf{k},\omega)f(\mathbf{k},\omega)$$

and

$$D_r^0(\mathbf{k},\omega)=\frac{1}{\frac{\omega^2}{v^2}-k^2+2i0\omega},$$

It follows now from (64) that

$$D_r(\mathbf{k},\omega)=\frac{1}{\frac{\omega^2}{v^2}-k^2-\pi_r(\mathbf{k},\omega)},$$
$$\langle\breve{\varphi}\rangle_{\mathbf{k}\omega}=\frac{1}{\chi}D_r(\mathbf{k},\omega)f(\mathbf{k},\omega). \quad (66)$$

The comparison of (66) and (17) shows that the function $G_r(\mathbf{k},\omega)$ entering the mathematical representation is connected with the функцией $D_r(\mathbf{k},\omega)$ through the relation

$$G_r(\mathbf{k},\omega)=\frac{1}{\chi}D_r(\mathbf{k},\omega).$$

Thus, our problem consists in the derivation of $D_r(\mathbf{k},\omega)$. To do this, it is necessary to calculate operator (65) according to (64). The system of equations (64) and (65) allows us to derive the operator $\pi_r(\mathbf{k},\omega)$ and the function $D_r(\mathbf{k},\omega)$. This system of equations reminds the system of equations (22)—(23). But if the system of equations (22)—(23) was transformed in an algebraic one for any dependence of the correlator $\langle\breve{\xi}(x_1)\breve{\xi}(x_2)\rangle$ on the difference of spatial coordinates, the system of equations (64)—(65) can be transformed in an algebraic one only if the correlator of the external random field is independent of the difference of spatial coordinates and is only a function of time:

$$\langle\breve{\xi}(x_1)\breve{\xi}(x_2)\rangle=\eta(t_1-t_2).$$

In this case,

$$\langle\breve{\xi}\breve{\xi}\rangle_{\mathbf{k}'\omega'}=(2\pi)^3\,\delta(\mathbf{k}')\eta(\omega')$$

and the integral over $\mathbf{k}'$ in (65) disappears. As for the function $\eta(\omega)$, we represent it, as above, as

$$\eta(\omega)=i\eta\left(\frac{1}{\omega+i\gamma}-\frac{1}{\omega-i\gamma}\right).$$

As $\gamma\to 0$, this function has a clearly pronounced maximum as $\omega\to 0$. Therefore, Eq. (65) is transformed in an algebraic one:

$$\pi_r(\mathbf{k},\omega)=\frac{\eta}{\chi^2}\frac{k^4}{\frac{\omega^2}{v^2}-k^2-\pi_r(\mathbf{k},\omega)}.$$

This equation is easily solved. Let $\omega=0$, and let the condition $\frac{\eta}{\chi^2}>\frac{1}{4}$ hold. Then we get $\pi_r(\mathbf{k},0)=-\kappa^2\left(\frac{1}{2}+i\,\mathrm{sgn}\omega\sqrt{\frac{\eta}{\chi^2}-\frac{1}{4}}\right)$. This expression together with (10) and (64) show the appearance of the $1/\omega$-noise in the system for $\hbar\omega << T$. This noise is described by the formula

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega} = \frac{2T\chi}{\omega k^2\eta}\sqrt{\frac{\eta}{\chi^2}-\frac{1}{4}}, \quad \omega\to 0. \tag{67}$$

This formula yields that, in the coordinate space,

$$\langle\breve{\varphi}\breve{\varphi}\rangle_{r\omega} = \int e^{i\mathbf{kr}}\langle\breve{\varphi}\breve{\varphi}\rangle_{\mathbf{k}\omega}\frac{d\mathbf{k}}{(2\pi)^3} =$$
$$= \frac{T\chi}{\pi\omega r\eta}\sqrt{\frac{\eta}{\chi^2}-\frac{1}{4}} \quad \text{при} \quad k_{\mathrm{D}}r >> 1.$$

Here, we took into account that the integral over k is taken only up to the limit k_{D} defined by the Debye wavelength in a solid.

3.3. Nonlinear acoustic waves

Consider now the nonlinear theory of acoustic waves. In the model with scalar field $\breve{\varphi}(x)$, the construction $\breve{\varphi}(\partial\breve{\varphi}/\partial r_i)(\partial\varphi/\partial r_i)$ can serve as the nonlinear scalar term in the Hamiltonian. Such a construction corresponds to the propagation of acoustic waves in the medium, whose coefficient of elasticity depends on the amplitude of the propagating field. We will use such a model which is simple in the mathematical aspect and demonstrates, nevertheless, the main properties of the system and the problems concerning the nonlinear interaction of waves.

We consider that the Hamiltonian of the system looks as

$$\hat{H} = \frac{\rho}{2}\int\left(\frac{\partial\varphi_f}{\partial t}\right)^2 d\mathbf{r} + \frac{\chi}{2}\int\frac{\partial\breve{\varphi}_f}{\partial r_i}\frac{\partial\breve{\varphi}_f}{\partial r_i}d\mathbf{r} +$$
$$\frac{g}{2}\int\breve{\varphi}\frac{\partial\breve{\varphi}_f}{\partial r_i}\frac{\partial\breve{\varphi}_f}{\partial r_i}d\mathbf{r} + \int f\breve{\varphi}d\mathbf{r}.$$

The commutation relations of the operators $\breve{\varphi}$ and $\partial\breve{\varphi}/\partial t$ are invariant relative to the form of the interaction Hamiltonian and therefore are defined again by equality (4).

The differentiation of the operators with respect to time gives the equation

$$\nabla^2\breve{\varphi}_f - \frac{1}{v^2}\frac{\partial^2\breve{\varphi}_f}{\partial t^2} - \frac{g}{2\chi}\frac{\partial\breve{\varphi}_f}{\partial r_i}\frac{\partial\varphi_f}{\partial r_i} + \frac{g}{\chi}\frac{\partial}{\partial r_i}\breve{\varphi}_f\frac{\partial\breve{\varphi}_f}{\partial r_i} = \frac{f}{\chi}. \tag{68}$$

Suppose that this equation is solved. We denote its solution by $\breve{\psi}(x)$. Then Eq. (68) can be represented as a nonlinear equation

$$\nabla^2\breve{\varphi}_f - \frac{1}{v^2}\frac{\partial^2\breve{\varphi}_f}{\partial t^2} - \frac{g}{2\chi}\frac{\partial\breve{\psi}}{\partial r_i}\frac{\partial\varphi_f}{\partial r_i} + \frac{g}{\chi}\frac{\partial}{\partial r_i}\breve{\psi}\frac{\partial\breve{\varphi}_f}{\partial r_i} = \frac{f}{\chi}. \tag{69}$$

The solution of this equation is, of course, the operator $\breve{\psi}(x)$. We rewrite (69) in the integral form as

$$\breve{\varphi}_f(x) - \breve{\varphi}(x) = \varphi_0(x) + \frac{g}{2\chi}\times$$
$$\times\int D_r^0(x,x_1)\left(\frac{\partial\breve{\psi}(x_1)}{\partial r_{1i}} - 2\frac{\partial}{\partial r_{1i}}\breve{\psi}(x_1)\right)\times$$
$$\times\frac{\partial}{\partial r_{1i}}\Big[\breve{\varphi}_f(x_1) - \breve{\varphi}(x_1)\Big]dx_1, \tag{70}$$

where $\breve{\varphi}(x)$ is the solution of Eq. (69) for $f(x)=0$, and we get

$$\nabla^2\breve{\varphi}_0 - \frac{1}{v^2}\frac{\partial^2\varphi_0}{\partial t^2} = \frac{f}{\chi},$$
$$\nabla^2 D_r^0(x,x') - \frac{1}{v^2}\frac{\partial^2 D_r^0(x,x')}{\partial t^2} = \delta(x-x'). \tag{71}$$

Then we solve Eq. (70) by the method of iterations and average the expression derived in such a way over quantum states and over the Gibbs distribution. By summing the reducible diagrams in the one-loop approximation, we obtain

$$\langle\breve{\varphi}_f\rangle = \varphi_0 + \frac{g^2}{4\chi^2}\int D_r^0(x,x_1)\times$$
$$\times\left\langle\left(\frac{\partial\breve{\psi}(x_1)}{\partial r_{1i}} - 2\frac{\partial}{\partial r_{1i}}\breve{\psi}(x_1)\right)\frac{\partial}{\partial r_{1i}}D_r(x_1,x_2)\times\right.$$
$$\left.\times\left(\frac{\partial\breve{\psi}(x_2)}{\partial r_{2j}} - 2\frac{\partial}{\partial r_{2j}}\breve{\psi}(x_2)\right)\right\rangle\frac{\partial}{\partial r_{2j}}\langle\breve{\varphi}(x_2)\rangle dx_1dx_2. \tag{72}$$

In the one-loop approximation, the function $D_r(x,x')$ depending only on the difference of its arguments satisfy the equation

$$D_r(x-x') = D_r^0(x-x') +$$
$$+\int D_r^0(x-x_1)\pi_r(x_1-x_2)D_r(x_2-x')dx_1dx_2. \tag{73}$$

For the operator π_r, we get

$$\pi_r(\mathbf{k},\omega) = \frac{\mathrm{g}^2}{4\chi^2}\int\Big[4\mathrm{k}^4 + 2\mathrm{k}^2(\mathrm{k}')^2 - (\mathbf{kk}')\big(8\mathrm{k}^2 + (\mathrm{k}')^2 -$$

$$-3(\mathbf{kk}'))\Big]\langle\breve\psi\breve\psi\rangle_{\mathbf{k}'\omega'}D_r(\mathbf{k}-\mathbf{k}',\omega-\omega')\frac{d\mathbf{k}'d\omega'}{(2\pi)^4}. \quad (74)$$

by applying the Fourier transformation. With the help of the operator π_r, Eq. (72) can be rewritten in the standard form

$$\langle\breve\varphi_f(x)\rangle=\varphi_0(x)+$$

$$+\int D_r^0(x-x_1)\pi_r(x_1-x_2)\langle\breve\varphi(x_2)\rangle dx_1dx_2.$$

Equations (71), (72) and (73) lead again, after applying the Fourier transformation, to formulas (64) and (66) which yield

$$G_r(\mathbf{k},\omega)=\frac{1}{\chi}D_r(\mathbf{k},\omega),$$

$$D_r(\mathbf{k},\omega)=\frac{1}{\frac{\omega^2}{v^2}-k^2-\pi_r(\mathbf{k},\omega)}. \quad (75)$$

The system of equations (74), (75) is sufficient for the determination of the required $\pi_r(\mathbf{k},\omega)$ and $D_r(\mathbf{k},\omega)$. Solving this system in the analytic form seems to be impossible. But we can make estimate and clarify the general character of the solution in the following way. Since the operators $\breve\psi(x)$ and $\breve\varphi_f(x)$ coincide, we write the correlator $\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}'\omega'}$ instead of the correlator $\langle\breve\psi\breve\psi\rangle_{\mathbf{k}'\omega'}$ in (74) for $f(\mathbf{r},t)=0$. We are interested in the correlator $\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}'\omega'}$ which has the $1/\omega$ peculiarity. Therefore, we can take out the integrand in integral (74) at the point $\omega'=0$ beyond the integral symbol. The integration over $\mathbf{k}'$ in integral (74) is performed in the scope of the Debye sphere. The maximum value of $k'=|\mathbf{k}'|$ is $k_{\rm D}$. Consider the solution of Eqs. (74), (75) in the region of maximum values of $k\approx k_{\rm D}$. Since the variable of integration, k', is always less than $k_{\rm D}$ in integral (74), we neglect it as compared to k in all functions of the integrand, besides $\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}'\omega'}$ which has the $(k')^{-2}$ peculiarity according to (67). We get

$$\pi_r(\mathbf{k},\omega)=\frac{g^2k^4}{\chi^2}D_r(\mathbf{k},\omega)\int\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}'\omega'}\frac{d\mathbf{k}'d\omega'}{(2\pi)^4}. \quad (76)$$

Of course, we obtain only estimating formulas on this way which are easily studied by the algebraic methods. These formulas will lead us to the $1/\omega$ spectrum, what is our purpose. But, in order to get the $1/\omega$ spectrum, we should not obligatorily transform the system of equations (74)-(75) in an algebraic one. For its appearance, it is only necessary that their solution $\pi_r(\mathbf{k},\omega)$ have a finite imaginary component for $\omega\to 0$. There are no doubts that the region of existence of such solutions is significantly greater than the region of solutions which admits the algebraic analysis. From (75), (76), and (10), we get

$$\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}\omega}=\frac{2T}{\omega k^2}\mathrm{sgn}\omega\frac{\sqrt{\frac{g^2}{\chi^2}\int\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}'\omega'}\frac{d\mathbf{k}'d\omega'}{(2\pi)^4}-\frac{1}{4}}}{\frac{g^2}{\chi}\int\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}'\omega'}\frac{d\mathbf{k}'d\omega'}{(2\pi)^4}}. \quad (77)$$

The unknown integrals in (77) can be derived from the equation of consistency which follows from (77) after the integration of its both sides over $\mathbf{k}$ and ω. This equation takes the form

$$z^2=\zeta\sqrt{z-\frac{1}{4}},$$

if we use the dimensionless quantities

$$z=\frac{g^2}{\chi^2}\int\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}'\omega'}\frac{d\mathbf{k}'d\omega'}{(2\pi)^4},$$

$$\zeta=\frac{Tg^2k_{\rm D}}{\pi^3\chi^3}\left(\ln\frac{vk_{\rm D}}{\gamma}-1\right).$$

We took into account that the integration over frequencies is performed in the interval$(\gamma\div vk)$, and the region of integration over $\mathbf{k}$ is limited by the Debye value, $k_{\rm D}$:

$$\frac{1}{(2\pi)^4}\int\frac{d\mathbf{k}d\omega}{\omega k^2}=\frac{k_{\rm D}}{2\pi^3}\left(\ln\frac{vk_{\rm D}}{\gamma}-1\right).$$

The required formula can be written now as

$$\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}\omega}=\frac{2\pi^3\chi^2}{g^2k_{\rm D}}\left(\ln\frac{\mathbf{v}k_{\rm D}}{\gamma}-1\right)^{-1}\mathrm{sgn}\omega\frac{z}{\omega k^2}. \quad (78)$$

This yields that the $1/\omega$-noise arises only at temperatures above the critical one,

$$T_c=\frac{2\pi^3\chi^3}{g^2k_{\rm D}\sqrt{27}}\left(\ln\frac{vk_{\rm D}}{\gamma}-1\right)^{-1}.$$

Formula (78) yields that

$$\langle\breve\varphi\breve\varphi\rangle_{\mathbf{r}\omega}=\int e^{i\mathbf{kr}}\langle\breve\varphi\breve\varphi\rangle_{\mathbf{k}\omega}\frac{d\mathbf{k}}{(2\pi)^3}=$$

$$=\frac{\pi^2\chi^2}{g^2\omega}\left(\ln\frac{vk_{\rm D}}{\gamma}-1\right)^{-1}\mathrm{sgn}\omega\frac{z}{k_{\rm D}r}\quad\text{for}\quad k_{\rm D}r>>1.$$

4. Scattering of Photons by Phonons

The presence of the thermal $1/\omega$-spectrum of acoustic waves was experimentally verified. Let an electromagnetic wave, whose positive-frequency component of the electric field intensity has the form $\mathcal{E}_0 \exp(i\mathbf{k}_0\mathbf{r} - ick_0t)$, where c is the light velocity, be scattered by the fluctuations $\breve{\varphi}(\mathbf{r}, t)$ of acoustic phonons occupying a volume V. The inelastically scattered electromagnetic signal in the Born and dipole approximations has the form

$$\breve{\mathcal{E}}(\mathbf{r}, t) \sim \mathcal{E}_0 \frac{\exp(ik_0r - ick_0t)}{r} \int\limits_V \breve{\varphi}(\mathbf{r}', t) \exp(-i\mathbf{q}\mathbf{r}') d\mathbf{r}',$$

$$\mathbf{q} = k_0(\mathbf{n} - \mathbf{n}_0), \quad |\mathbf{q}| = 2k_0 \sin\frac{\vartheta}{2}.$$

at large distances r from the scatterer. Here, $\mathbf{n}$ and $\mathbf{n}_0$ are unit vectors in the directions of the scattered and incident waves, respectively, ϑ is the angle between these vectors. Upon studying the interference of amplitudes on a receiver of the emission, we have

$$\int\limits_{-\infty}^{\infty} \langle \breve{\mathcal{E}}(\mathbf{r}, t+\tau) \overset{+}{\breve{\mathcal{E}}}(\mathbf{r}, t) \rangle \exp(i\omega\tau) d\tau \sim \langle \breve{\varphi}\breve{\varphi} \rangle_{\mathbf{q}, \omega - ck_0} V.$$

According to formula (45) that is valid for Debye waves with the change $\omega_0 \to vk$, the registered signal has maxima at the frequencies $\omega = ck_0$ and $\omega == ck_0 \pm vq = k_0(c \pm 2v \sin\vartheta/2)$. All three maxima are well distinguished experimentally [10,19]. According to the presented theory, the central maximum has the $1/(\omega - ck_0)$ peculiarity due to the inelastic scattering of electromagnetic waves, which should be taken into account in the future experimental investigations.

Upon the observation of the interference of intensities, the registered signal looks as

$$\int\limits_{-\infty}^{\infty} \langle \overbrace{\breve{\mathcal{E}}(\mathbf{r}, t+\tau) \overbrace{\overset{+}{\breve{\mathcal{E}}}(\mathbf{r}, t+\tau) \breve{\mathcal{E}}(\mathbf{r}, t)} \overset{+}{\breve{\mathcal{E}}}}(\mathbf{r}, t) \rangle \times$$

$$\times \exp(i\omega\tau) d\tau \sim V^2 \int \langle \breve{\varphi}\breve{\varphi} \rangle_{\mathbf{q}, \omega - \omega'} \langle \breve{\varphi}\breve{\varphi} \rangle_{\mathbf{q}, \omega'} d\omega' \sim$$

$$\sim V^2 \langle \breve{\varphi}\breve{\varphi} \rangle_{\mathbf{q}\omega} \int \langle \breve{\varphi}\breve{\varphi} \rangle_{\mathbf{q}\omega'} d\omega'. \quad (79)$$

The horizontal brackets show the scheme of the uncoupling of correlators upon the estimating calculation of the noise effects.. We used the circumstance that the function $\langle \breve{\varphi}\breve{\varphi} \rangle_{\mathbf{q}\omega'}$ has a sharp maximum as $\omega' \to 0$ and took out the integrand beyond the integral symbol. It follows from (79) that the registered signal arising as a result of the interference of intensities has the $1/\omega$-peculiarity as $\omega \to 0$ in accordance with (45) and (78) which was experimentally discovered in [10]. The above-presented formalism explains the reason of its appearance. If the number of the phonon modes participating in the scattering of photons exceeds 1 and equals $\Delta\Omega$, we must multiply expression (79) by this value. For this reason, the signal, which was observed in [10] and is defined by expression (79) divided by the squared intensity, turns out to be proportional to $(\Delta\Omega)^{-1}$. Just such a dependence was observed in [10]. We may conclude that the presence of the fluctuations of the number of phonons in modes, which is described by fourth-order correlators, should not be assumed to explain the experimental regularities, as was proposed in [10]. According to (79), it is sufficient to restrict oneself by the study of the fluctuations of amplitudes in these modes, i.e. by the study of the quadratic forms of the operators of fields.

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$1/\omega$-СПЕКТР ТЕПЛОВИХ АКУСТИЧНИХ ФОНОНІВ

Б.А. Векленко

Р е з ю м е

Показано, що нелінійні теплові хвилі у твердих тілах, потенціальна енергія яких визначається зміщенням частинок з положення рівноваги (модель Ейнштейна) або похідними по координатах від зміщення частинок (модель Дебая), мають спектр $1/\omega$, якщо тільки температура системи перевищує критичну. Показано, що такий спектр зумовлений наявністю в системі подвійного зворотного зв'язку. Пояснено експериментально спостережений $1/\omega$-спектр фотонів, розсіяних на теплових акустичних хвилях твердих тіл.