INVARIANTS OF FAST ION MOTION IN STELLARATORS

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Approximate expressions for the invariants of motion of charged plasma particles in a stellarator are obtained in two cases: for the passing particles and the particles that always remain trapped by the dominant Fourier harmonic of the magnetic field. In the derivation, the Littlejohn's method of Lie transformations in non-canonical coordinates for Hamiltonian systems is used. The invariants obtained describe the motion of the guiding center of a particle in the absence of collisions. They can be utilized for finding the fast ion orbits and writing the drift kinetic equation. Expressions for the particle motion along the orbits, which are required to analyze the resonances between energetic particles and waves, are obtained as well.

1. Introduction

Recently, more and more attention in thermonuclear research is focusing on stellarators — magnetic confinement devices which have a toroidal magnetic field topology like tokamaks but, unlike tokamaks, have complex helical shape. The principal advantage of stellarators over tokamaks is that the necessary topology of the magnetic field is created without a toroidal plasma current. This simplifies, on the one hand, the task of designing a steady-state reactor and helps, on the other hand, to combat the instabilities arising from the toroidal plasma current, including the very dangerous crash instability.

The analysis of fast (non-thermal) ions in the device and ensuring the necessary properties of their motion is one of the most important problems in the design of reactor systems. Fast ions are created when the plasma is heated by neutral beam injection or high-frequency radio power and as the products of thermonuclear reactions. The parameters of a plasma in the largest existing stellarators (and in future reactors) are such that one may ignore collisions in many cases when describing the motion of fast ions and assume that the motion is Hamiltonian. However, because of the compex geometry of stellarators, the motion of charged particles in these devices is very complicated and, generally speaking, not integrable.

The use of adiabatic invariants is an effective means of the analysis of Hamiltonian systems. It is well known [1] that the existence of adiabatic invariants in a system is connected with the approximate periodicity of its motion, which can be interpreted as the existence of an approximate symmetry in the system. In particular, the motion of particles trapped in the local minima of the stellarator's magnetic field ("locally trapped particles") is described by the well-known longitudinal adiabatic invariant [2]

$$I_\parallel = \frac{1}{2\pi}\oint ds\, M v_\parallel, \tag{1}$$

where ds is the linear element, M is the particle's mass, $v_\parallel$ is its parallel velocity, and the integral is taken along a field line between the particle's bounce points (it seems that nobody knows when and by whom was this invariant originally proposed; the earliest papers to utilize it concerned the motion of charged particles in Earth's radiation belts). Therefore, the approximate cycle of the system's motion in this case is the motion of a particle along the field line between bounce points ("bounce motion"), and the validity condition consists in the smallness of the particle's deviation from the field line in one bounce time, which places a certain limit on the particle's energy. A different adiabatic invariant for the particle motion in stellarators, which is also applicable to locally passing particles (e.g., the particles which have a sufficient parallel velocity to pass some local minima of the magnetic field along their trajectory) was proposed in [3]:

$$I_* = \frac{1}{2\pi}\oint d\phi\, M v_\parallel \frac{B_\phi}{B} - \sigma\frac{e}{cN}\psi_p, \tag{2}$$

where e is the particle's charge, ϕ is the toroidal angle, B is the magnetic field strength and B_ϕ is its toroidal covariant component, N is the number of periods in the stellarator, ψ_p is the poloidal magnetic flux, $\sigma = \mathrm{sgn}(v_\parallel)$ for passing particles, and $\sigma = 0$ for trapped particles. The approximate cycle of motion, along which the integral is taken in invariant (2), is a piece of the line $\theta = \mathrm{const}$ (where θ is the poloidal angle) on the magnetic surface; the length of this line is one period of the device for a passing particle and the distance between bounce points for a trapped particle. This means that, in addition to the particle's drift excursion from the field line, the particle's θ excursion must be small in order for invariant (2) to be applicable, which yields the condition $\iota/N \ll 1$, where ι is the rotational transform.

The goal of this work is to obtain other adiabatic invariants for the drift particle motion in a stellarator. Unlike invariants (1) and (2), we take a two-dimensional manifold in phase space as the approximate cycle – the drift surface, upon which the trajectory of a particle would lie in a simplified magnetic field. After this, the difference between the real and the simplified magnetic field is treated as a perturbation, and corrections for the phase-space coordinates which renew the symmetry of the perturbed system are found. The invariants arising from this symmetry are the new adiabatic invariants. The factors which can break the adiabaticity of motion and can limit the applicability of the invarints will be discussed below. The advantage of these invariants is the absence of the above-mentioned limitations on the particle energy and ι. Moreover, unlike invariants (1) and (2), which characterize the particle's bounce-averaged motion, the new invariants also take into account the drift motion of a particle within one bounce. This is important, in particular, for the analysis of Alfvén instabilities driven by fast particles.

To calculate the adiabatic invariants, we will use a variant of Hamiltonian perturbation theory developed by Littlejohn in [4]. Littlejohn describes the Hamiltonian system formally with a phase space Lagrangian, which is linear in the time derivatives. The spatial coordinates and the momenta are considered to be independent variables and are varied separately. The Euler—Lagrange equations for this Lagrangian yield the usual Hamilton's equations. The advantage of this approach is the simplification of perturbation calculations and of working with noncanonical coordinates. It is very convenient for our problem, because the perturbation of the magnetic field changes the symplectic structure of the phase space, and the canonical variables of the original system will not, in general, be canonical in the perturbed system and vice versa. Following Arnold [6] and Littlejohn [4], we will use the differential-geometric formalism for our calculations, but the main results will be derived in coordinate form.

In this work, we limit ourselves to two cases: the particles which always stay locally passing and the particles which always stay locally trapped (see, e.g., the discussion of the types of particle orbits in a stellarator in [7]). Here, we do not treat the transitioning particles which switch between the locally passing and locally trapped states.

2. Model

We will assume that the configuration has closed magnetic surfaces and will use the so-called Boozer (magnetic) coordinates $(X^\psi, X^\theta, X^\phi) = (\psi, \theta, \phi)$, where ψ is a magnetic surface label (toroidal flux through the surface divided by 2π), θ and ϕ are the poloidal and toroidal angular variables, respectively [8]. The angular variables in these coordinates are chosen in such a way that the lines of the magnetic field are "straight" (that is, $d\theta/d\phi = \iota(\psi)$), and the magnetic field has the form

$$\mathbf{B} = \frac{1}{\sqrt{g}}\mathbf{e}_\phi + \frac{\iota(\psi)}{\sqrt{g}}\,\mathbf{e}_\theta = B_\psi(\psi,\theta,\phi)\nabla\psi +$$

$$+B_\theta(\psi)\nabla\theta + B_\phi(\psi)\nabla\phi, \tag{3}$$

where g is the determinant of the metric tensor, lower indices denote respective covariant components, and $\mathbf{e}_i$, $i = \psi, \theta, \phi$, are the covariant unit vectors. In the Boozer coordinates, the vector potential can be written as

$$A_\psi = 0, \quad A_\theta = \psi, \qquad A_\phi = -\psi_p = -\int\limits_0^\psi \iota d\psi. \tag{4}$$

It is worth noting that, although the use of the Boozer coordinates implies the existence of nested magnetic surfaces, the asymmetry of the magnetic field breaks, in reality, some rational surfaces and creates island chains. But since it is essential for good confinement that the islands are narrow, the use of the Boozer coordinates is justified.

We will suppose that the gyrofrequency is much larger than the other characteristic frequencies in the system and use the guiding center approximation. For our combined Hamiltonian-Lagrangian formalism, the best variant of the guiding center approximation is the one developed in [5], because we will have a Hamiltonian system in every order, and the dynamical variables satisfy relations like $\mu = M{v_\perp}^2/(2B)$ to all orders exactly. The guiding-center Lagrangian (L) is given to the first order in drifts by the equation

$$\boldsymbol{\gamma} = \frac{e}{c}\mathbf{A}\cdot \mathrm{d}\mathbf{X} - K\mathrm{d}t + \frac{Mc}{e}\mu \mathrm{d}\Theta + \frac{Mv_\parallel}{B}\mathbf{B}\cdot \mathrm{d}\mathbf{X}, \tag{5}$$

where the 1-form $\boldsymbol{\gamma} \equiv Ldt$ is referred to as the Poincaré—Cartan's integral invariant in [6] and the

Lagrangian differential form in [4]; $\mathbf{X} = (X^\psi, X^\theta, X^\phi)$, the kinetic energy K, the magnetic moment μ, and the gyrophase Θ are treated as phase space coordinates; $v_\| = \pm\sqrt{(2/M)(K-\mu B)}$ is the parallel velocity; M and e are the particle's mass and charge, respectively. In this expression, the last term contains the first-order drifts.

Since we intend to apply perturbation theory, we will separate the terms in B into a part B_0 which has some symmetry in the Boozer coordinates and which will be treated as the "non-perturbed" field, and the "perturbation" $\widetilde{B} = B - B_0$. We will also assume that the term in B_ψ in (5) is small and can be treated as a perturbation. Then form (5) splits into a term which describes an integrable system and a term which breaks the symmetry ("perturbation"). Since the motions of trapped and passing particles are different, the split will be different in these two cases.

As the "full" B will not be necessary any more, we drop the subscript 0 and assume in $v_\|$, ω_B, etc. that B equals B_0.

We also assume that the magnetic field satisfies the inequalities

$$\frac{\Delta}{\iota}\frac{\mathrm{d}\iota}{\mathrm{d}\psi} \ll 1, \quad \frac{\Delta}{B}\frac{\mathrm{d}B}{\mathrm{d}\psi} \ll 1, \quad B_\psi\Delta \ll B_\theta, B_\phi, \tag{6}$$

where Δ is the characteristic change of ψ along the particle orbit. Specifically, in optimized stellarators of the Wendelstein line, where the shear of magnetic field is very weak, $(\psi_a/\iota)\mathrm{d}\iota/\mathrm{d}\psi \ll 1$ (ψ_a is the value of ψ at the plasma edge), these conditions hold even for very wide orbits ($\Delta \sim \psi_a$).

3. Passing Particles

First, we consider the passing particles. For them, none of the harmonics is large enough to limit the motion, so we will treat all of them as a perturbation. Then $B = \bar{B}$, where $\bar{B}$ is the average magnetic field on the axis. Separate those terms in the full Lagrangian (5) which are of the zeroth order in the magnetic field:

$$\boldsymbol{\gamma} = \frac{e}{c}(X^\psi \mathrm{d}X^\theta - I\mathrm{d}X^\phi) - K\mathrm{d}t + \frac{Mc}{e}\mu\mathrm{d}\Theta +$$

$$+\frac{Mv_\|}{B}(B_\theta \mathrm{d}X^\theta + B_\phi \mathrm{d}X^\phi), \tag{7}$$

where $I \equiv \int_0^\psi \iota d\psi$.

This Lagrangian is obviously already converted to the action-angle form, with the actions being $\frac{e}{c}X^\psi + \frac{Mv_\|}{B}B_\theta$, $-\frac{e}{c}I + \frac{Mv_\|}{B}B_\phi$ and μ, and the conjugate angles — θ, ϕ, and Θ, respectively. We will not consider the evolution of the gyrophase and treat the magnetic moment as a parameter. Denoting the actions as I_θ and I_ϕ, we use the Lagrangian

$$\boldsymbol{\gamma} = I_\theta \mathrm{d}X^\theta + I_\phi \mathrm{d}X^\phi - K\mathrm{d}t \tag{8}$$

as the basis to determine a movement (transformation) of the coordinates which would convert the sum of the unperturbed Lagrangian $\boldsymbol{\gamma}_0$ and the perturbation

$$\widetilde{\boldsymbol{\gamma}} = \frac{Mv_\|}{B}\Bigg[B_\psi \mathrm{d}X^\psi -$$

$$-\frac{\widetilde{B}}{B}\frac{2K-\mu B}{2K-2\mu B}(B_\theta \mathrm{d}X^\theta + B_\phi \mathrm{d}X^\phi)\Bigg] \tag{9}$$

to the form of the original unperturbed Lagrangian. As noted in [4], the generator vector field of such a transformation $\mathbf{H}$ may be determined from the equation

$$\boldsymbol{\gamma} = \boldsymbol{\gamma} + \widetilde{\boldsymbol{\gamma}} + \mathrm{d}\boldsymbol{\gamma}(\mathbf{H}) - \mathrm{d}S, \tag{10}$$

where S is an unknown scalar (analogous to the generating function in Hamiltonian formalism).

We will look for a transformation which will only modify the coordinates. This will give us at once the orbits of particles in the perturbed field: the "perturbation-corrected" coordinate functions in the phase space are determined through the unperturbed coordinates from the equation

$$X_{1i} = X_i + \mathbf{H}(X_i). \tag{11}$$

Therefore, in the perturbed field, the integral of motion equivalent to ψ will be the function $\psi + H_\psi$. A simple calculation yields

$$\tilde{\psi} = \psi + \frac{v_\|}{\omega_B}\frac{2K-\mu B}{2K-2\mu B}\times$$

$$\times\sum_{m,n}\frac{\widetilde{B}_{mn}}{B}\frac{mB_\phi + nB_\theta}{m\iota - n}e^{im\theta - in\phi} \tag{12}$$

for H_ψ. The corrections H_ϕ and H_θ to the angular variables may be determined from (10). Since the canonical angles increase linearly in time, these corrections allow one to describe the time dependence of Boozer's angles. This is important in determining wave-particle resonances. The expressions for H_ϕ and H_θ are rather bulky; therefore, they are given in the Appendix.

4. Trapped Particles

Consider now a particle trapped by a particular harmonic of the magnetic field $B_* = B_{m_0n_0}\cos(m_0\theta - n_0\phi)$. We cannot consider this harmonic to be a perturbation; therefore, we will put it into the unperturbed field. Let $\theta_1 = m_0\theta - n_0\phi$ and $\phi_1 = n_0\theta + m_0\phi$ be new angular variables which are more convenient. Because of this ansatz, $B = \bar{B} + B_*$ is a function only of θ_1.

Equation (5) will now be

$$\boldsymbol{\gamma} = \left(\frac{e}{c}A_{\theta_1} + \frac{Mv_\parallel}{B}B_{\theta_1}\right)\mathrm{d}X^{\theta_1} +$$

$$+\left(\frac{e}{c}A_{\phi_1} + \frac{Mv_\parallel}{B}B_{\phi_1}\right)\mathrm{d}X^{\phi_1} - K\mathrm{d}t, \tag{13}$$

whence we have at once the action

$$I_2 = \frac{e}{c}A_\phi + \frac{Mv_\parallel}{B}B_\phi \tag{14}$$

(below, we drop subscripts 1 in θ and ϕ).

The further analytic advancement is possible through the expansion

$$I_2 \simeq \frac{e}{c}\left.A_\phi\right|_{\psi=\psi_b} + \frac{e}{c}A'_\phi(\psi-\psi_b) + \frac{Mv_\parallel}{B}B_\phi \tag{15}$$

which stands in stellarators even for fairly large $\delta\psi \sim \psi$. As usual, the action I_1 is found by integration over the angular variable:

$$I_1 \approx \frac{B_\theta A'_\phi - B_\phi A'_\theta}{2\pi}\oint\frac{Mv_\parallel}{B}\frac{d\theta}{A'_\phi}. \tag{16}$$

Neglecting the angle dependence of B in the denominator, we obtain

$$I_1 \approx \frac{B_\theta A'_\phi - B_\phi A'_\theta}{2\pi}\frac{Mv_{\parallel\max}}{B}\times$$

$$\times\frac{8}{A'_\phi}\left[\mathbf{E} - (1-\kappa^2)\mathbf{K}\right], \tag{17}$$

where ${v_\parallel}^2 = {v_{\parallel\max}}^2\kappa^{-2}(\kappa^2 - \sin^2\frac{\theta}{2})$, $\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals of the first and second kinds, respectively, and $\kappa^2 = (\mu\bar{B} + \mu B_{m_0n_0} - K)/(2\mu B_{m_0n_0})$. For the conjugate angle variable ξ_1, we have

$$\kappa^{-1}\sin\frac{\theta}{2} = \mathrm{sn}\left(\frac{2\mathbf{K}}{\pi}\xi_1\right) = \mathrm{sn}\,\xi, \tag{18}$$

and the parallel velocity $v_\parallel = v_{\parallel\max}\,\mathrm{cn}\,\xi$, where sn and cn are the elliptic sine and cosine. The conjugate angle ξ_2 to the action I_2 is the angle ϕ with a periodical correction χ determined from

$$\frac{\partial\chi}{\partial\xi_1} = \frac{2\kappa\mathbf{K}}{\pi}\frac{A'_\theta}{A'_\phi}\,\mathrm{cn}\,\xi. \tag{19}$$

Consider now the perturbed field. Decomposing $\boldsymbol{\gamma}$ in components by the coordinate 1-forms $\boldsymbol{\gamma}_1 = F_i dX^i$, substituting into (10), and collecting terms, we obtain the system

$$\frac{\partial S}{\partial t} = -\mu\left(\frac{\partial B}{\partial\psi_b}H^{\psi_b} + \frac{\partial B}{\partial\xi_1}H^{\xi_1}\right),$$

$$\frac{\partial S}{\partial K} = -H^{\xi_1}\frac{\partial I_1}{\partial K},$$

$$\frac{\partial S}{\partial\psi_b} = -H^{\xi_2}\frac{dI_2}{d\psi_b} - H^{\xi_1}\frac{\partial I_1}{\partial\psi_b},$$

$$\frac{\partial S}{\partial\xi_1} = H^{\psi_b}\frac{\partial I_1}{\partial\psi_b} + F_{\xi_1},$$

$$\frac{\partial S}{\partial\xi_2} = H^{\psi_b}\frac{dI_2}{\partial\psi_b} + F_{\xi_2}. \tag{20}$$

Fourier-transforming in the canonical angles ξ_1 and ξ_2 and excluding S from (20), we obtain the equation

$$H^{\psi_b}{}_{(mn)} = -\frac{mF_{\xi_2(mn)} - nF_{\xi_1(mn)}}{m\frac{dI_2}{d\psi_b} - n\frac{\partial I_1}{\partial\psi_b}}, \tag{21}$$

where (m,n) are the Fourier harmonic numbers in the canonical angles ξ_1 and ξ_2, respectively. For the needed components of the perturbation F_i, we have

$$F_{\xi_1} = -\frac{Mv_{\parallel\max}}{B^2}\left(B_\theta\frac{\partial\theta}{\partial\xi_1} + B_\phi\frac{\partial\phi}{\partial\xi_1}\right)\widetilde{B}\,\mathrm{cn}\xi,$$

$$F_{\xi_2} = -\frac{Mv_{\parallel\max}}{B^2}\left(B_\theta\frac{\partial\theta}{\partial\xi_2} + B_\phi\frac{\partial\phi}{\partial\xi_2}\right)\widetilde{B}\,\mathrm{cn}\xi. \tag{22}$$

For strongly trapped particles ($\kappa \ll 1$), expression (21) can be reduced to the form

$$H^{\psi_b} = \frac{\omega_b}{\omega_B}\frac{v_{\parallel\max}}{A'_\phi}\sum_{\mu,m,n}\frac{\widetilde{B}_{\mu n}}{B}\exp(im\xi_1 - in\xi_2)\times$$

$$\times \frac{m^2 B_\phi J_m - n\kappa(2B_\theta - \iota_1 B_\phi)(\frac{m^2}{x} J_m - J'_m)}{x(m\omega_b - n\omega_p)}, \qquad (23)$$

where $\iota_1 = A'_\theta / A'_\phi$, $x = \kappa(2\mu + \iota_1 n)$, the m-th order Bessel function J_m and it's derivative J'_m are taken at x, and

$$m\xi_1 - n\xi_2 \approx m \arcsin(\theta/2\kappa) - n(\phi + \iota_1\theta/2).$$

Using Eq. (15) to express ψ_b as a function of ψ, we deduce that K, μ, and

$$\widetilde{\psi}_b = \psi_b + H^{\psi_b} = \psi + \frac{v_\parallel B_\phi}{\omega_{B0} A'_\phi} + H^{\psi_b} \qquad (24)$$

will be the integrals of motion in the perturbed case. The corrections to the angular variables, H_θ and H_ϕ, can be determined from the same system (20) and are presented in the Appendix.

5. Conclusions

The invariants given by Eqs. (12) and (24) give the guiding center orbit in the absence of collisions. Therefore, they can be used to analyze the prompt losses of fast ions (the products of fusion reactions or suprathermal ions produced by various plasma heating methods), i.e., the losses occuring on the orbital motion timescales. Moreover, these invariants are the "natural" variables to use in the drift-kinetic equations, when the effect of collisions on longer timescales is being considered. However, there are some limitations on expressions (12) and (24) which stem from their perturbative derivation.

First of all, these invariants describe the particle motion if the particle is not too close to the resonant drift surfaces, i.e., those drift surfaces, on which the ratio of the poloidal and toroidal motion frequencies is rational. This is obvious from the resonant denominators in Eqs. (12) and (24) which vanish at these surfaces. We mentioned above that a perturbation will alter the phase space topology in the vicinity of such surfaces, by creating magnetic islands and the areas of stochastic motion.

Secondly, one may expect that our expressions will not work closely to the region of transitioning particles, where some of those harmonics which we treated as perturbations begin to affect the orbit topology. Invariant (24) cannot be used, in particular, in devices which have no dominant harmonic in their magnetic field, as almost all particles will be transitioning.

It is also interesting to compare invariants (12) and (23) with the earlier invariants (2) and (1). One can show that the terms with $n = 0$ in Eqs. (12) and (23), which represent the motion independent of the bounce phase, approximately correspond to the invariant [3] and the longitudinal adiabatic invariant, respectively. From Eq. (12), one can see that these terms usually dominate in devices with large $N = 5 - 10$. Indeed, as n is a multiple of the number of periods, N, and ι is typically less than unity, the denominators of the $n \neq 0$ terms are always large in such devices. On the other hand, in low-N (compact) devices, the contribution of the terms with $n \neq 0$ can be significant. Therefore, the new expressions allow a more precise description of such devices. The advantage of invariants (2) and (1) is that they are valid for all locally trapped and locally passing particles, including the respective states of transitioning particles.

Finally, let us discuss the applicability of our results in areas other than fusion theory. The principal requirement for their applicability is that the influence of collisions on particle motion be negligible. This means a high enough plasma temperature, a low plasma density, and a sufficiently high energy of particles, to which our results are applied. These conditions are frequently fulfilled in space plasmas (as we mentioned earlier, adiabatic invariants have been used for a long time to describe the motion of particles in Earth's radiation belts). The magnetic fields with a toroidal topology also occur in space; the structures that erupt from Sun's magnetosphere [9] are one example. The invariants derived in this paper can be used to analyze the influence of toroidal asymmetry on the particle motion in such structures. The topology of the magnetic fields in the radiation belts and magnetospheres of planets and stars is different (the field lines pass through the central body, and charged particles are effectively confined in a magnetic bottle). For this reason, it is impossible to use the derived invariants directly in these cases, but the perturbation method presented here could be applied to derive other invariants, starting with an axially symmetric magnetic field configuration. One might expect such invariants to be more precise than the longitudinal adiabatic invariant (1) for particles with large orbit width.

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APPENDIX
Temporal evolution of angular variables

For passing particles, the corrections to the angular variables are given by the following expressions:

$$\hat{H}^{\theta} = -\omega_{\theta}\frac{\partial \hat{S}}{\partial K} + J\frac{\partial I_{\phi}}{\partial K} \quad \frac{\partial \hat{S}}{\partial \psi} - \frac{v_{\|}}{\omega_B}B_{\psi}\Bigg) \tag{D1}$$

$$\hat{H}^{\phi} = -\omega_{\phi}\frac{\partial \hat{S}}{\partial K} - J\frac{\partial I_{\theta}}{\partial K} \quad \frac{\partial \hat{S}}{\partial \psi} - \frac{v_{\|}}{\omega_B}B_{\psi}\Bigg), \tag{D2}$$

where $\omega_{\theta} = J\frac{\partial I_{\phi}}{\partial \psi}$ and $\omega_{\phi} = -J\frac{\partial I_{\theta}}{\partial \psi}$ are the motion frequencies,

$$\hat{S} = i\frac{v_{\|}}{\omega_B}\,\frac{2K-\mu B}{2K-2\mu B}\,\frac{\hat{B}}{B}\,\frac{\omega_{\theta}B_{\theta}-\omega_{\phi}B_{\phi}}{m\omega_{\theta}-n\omega_{\phi}},$$

$$J^{-1} = \frac{\partial I_{\phi}}{\partial \psi}\,\frac{\partial I_{\theta}}{\partial K} - \frac{\partial I_{\theta}}{\partial \psi}\,\frac{\partial I_{\phi}}{\partial K}.$$

For trapped particles,

$$H^{\xi_1} = - \quad \frac{\partial I_1}{\partial K} \quad ^{-1} Q\left[\frac{\partial \hat{F_{\xi_1}}}{\partial K}\,\frac{dI_2}{d\psi_b} - \frac{\partial \hat{F_{\xi_2}}}{\partial K}\,\frac{\partial I_1}{\partial \psi_b} - \right.$$

$$\left. - \quad \hat{F_{\xi_2}} - in\hat{S} \quad \frac{\partial^2 I_1}{\partial \psi_b \partial K}\right], \tag{D3}$$

$$H^{\xi_2} = - \quad \frac{dI_2}{d\psi_b} \quad ^{-1}\left\{Q\left[\frac{\partial \hat{F_{\xi_1}}}{\partial \psi_b}\,\frac{dI_2}{d\psi_b} - \right.\right.$$

$$-\frac{\partial \hat{F_{\xi_2}}}{\partial \psi_b}\,\frac{\partial I_1}{\partial \psi_b} - \quad \hat{F_{\xi_2}} - in\hat{S} \quad \frac{\partial^2 I_1}{\partial^2 \psi_b} +$$

$$\left.\left. + \quad \hat{F_{\xi_1}} - im\hat{S} \quad \frac{d^2 I_2}{d^2 \psi_b}\right] + \frac{\partial I_1}{\partial \psi_b}\,H^{\xi_1}\right\}, \tag{D4}$$

where

$$\hat{S} = Q \quad \hat{F_{\xi_1}}\frac{dI_2}{d\psi_b} - \hat{F_{\xi_2}}\frac{\partial I_1}{\partial \psi_b} \quad , \quad Q = m\frac{dI_2}{d\psi_b} - n\frac{\partial I_1}{\partial \psi_b}.$$

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ІНВАРІАНТИ РУХУ ШВИДКИХ ІОНІВ У СТЕЛАРАТОРАХ

А.В. Тихий, Ю.В. Яковенко

Р е з ю м е

Отримано наближені вирази для інваріантів руху заряджених частинок плазми у стелараторі для двох випадків: для пролітних частинок та для частинок, що завжди залишаються локально захопленими головною фур'є-гармонікою магнітного поля. При цьому використано метод перетворень Лі у неканонічних координатах для гамільтонових систем. Отримані інваріанти описують рух ведучого центра частинки за відсутності зіткнень і можуть бути використані для знаходження орбіт швидких іонів та запису дрейфово-кінетичного рівняння. Знайдено також вирази для руху частинки по орбіті, що необхідно для аналізу резонансів швидких іонів з хвилями.