THE su(1,1)-MODELS OF QUANTUM OSCILLATOR

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Models of a quantum oscillator on the base of the discrete series representations of the Lie algebra su(1,1) are constructed. The position and momentum operators in these models coincide with the operators J_2 and J_1 of these representations, respectively. As for the standard quantum harmonic oscillator, the position and momentum operators in the models have continuous simple spectra, covering the real line. The eigenfunctions of these operators are explicitly found. It is shown that the usual quantum harmonic oscillator is a limit of the oscillators constructed in the paper, that is, the last oscillators can be considered as deformations of the quantum harmonic oscillator.

1. Introduction

The su(1,1)-model of a quantum oscillator is a model that obeys the dynamics of a harmonic oscillator, with the position and momentum operators and Hamiltonian being the functions of elements of the Lie algebra su(1,1). The aim of this paper is to develop the theory of such oscillators using the discrete series representations of the Lie algebra su(1,1).

There exist many algebraic constructions which can be used for building up different models of quantum oscillators. For most of them, it is difficult to construct the theory of such an oscillator: spectra of observables, explicit form of eigenfunctions of observables, description of time evolution, etc. Only for some of such models, one can develop the corresponding theory. In 1989, there was proposed (see [1] and [2]) the so-called q-oscillator which is a q-deformed analog of the usual quantum harmonic oscillator. The theory of this oscillator was elaborated in detail. There exist physical problems for which the q-oscillator is more useful than the quantum harmonic oscillator (see, for example, [3] and [4]). Unlike the quantum field theory constructed on the base of standard quantum harmonic oscillators, a quantum field theory constructed on the base of a q-oscillator is free of some divergences. The q-oscillator has many useful properties which are absent for the standard quantum harmonic oscillator (see, for example, [5] and [6]).

However, for the q-oscillator, the basic relations

$$[H,Q] = -\mathrm{i}P, \quad [H,P] = \mathrm{i}Q \tag{1}$$

are violated. This is why the q-oscillator is not attractive for many physicists.

For this reason, there were made much efforts to construct useful models of quantum oscillators which do not violate relations (1). There were formulated the postulates which have to be satisfied on the construction of models of quantum oscillators [7]. These postulates are

1. There exists an essentially self-adjoint (Hermitian) *position* operator denoted as Q, whose spectrum Spec Q is the set of positions of the system.

2. There exists a self-adjoint *Hamiltonian* operator, H, which generates time evolution through the Newton—Lie, or equivalent Hamilton—Lie, equations

$$[H,[H,Q]] = Q \quad \Longleftrightarrow \quad \begin{cases} [H,Q] & =: & -\mathrm{i}P, \\ [H,P] & = & \mathrm{i}Q, \end{cases} \tag{2}$$

where $[\,\cdot\,,\,\cdot\,]$ is the commutator. The first Hamilton equation in (2) defines the *momentum* operator P, while the second one contains the harmonic oscillator dynamics. The set of momentum values of the system is the spectrum Spec P of P.

3. The three operators, Q, P and H closed into an *associative algebra* satisfy the Jacobi identity

$$[P,[H,Q]] + [Q,[P,H]] + [H,[Q,P]] = 0. \tag{3}$$

The second and third postulates determine that $[Q,P]$ must commute with H, which implies that it can only be of the form $[Q,P] = \mathrm{i}\,f(H)$, where f is some function of H (including constants) and the i is placed to make $f(H)$ self-adjoint, but do not otherwise specify this basic commutator further. For a constant $f(H) = \hbar\hat{1}$, one recovers the standard oscillator algebra $H_4 = \mathrm{span}\,\{H,Q,P,\hat{1}\}$ which contains the basic Heisenberg—Weyl subalgebra $W_1 = \mathrm{span}\,\{Q,P,\hat{1}\}$ of quantum mechanics. In paper [8], the authors examined the cases which correspond, in the unitary irreducible representations of spin $j = \frac{1}{2}N$ ($N \in \{0,1,\dots\}$ fixed), to the linear function $f(H) = H - (j+\frac{1}{2})\hat{1} =: J_3$, and so the operators close into the Lie algebra $\mathrm{so}(3) \equiv \mathrm{su}(2) = \mathrm{span}\,\{Q,P,J_3\}$. In paper [9], the above postulates are used to construct the so-called finite q-oscillator, for which (contrary to the Biedenharn—Macfarlane q-oscillator mentioned above) the relations $[H,Q]=-\mathrm{i}P$ and $[H,P]=\mathrm{i}Q$ are fulfilled.

The quantum oscillators constructed in [8] and [9] are characterized by the property that the position and momentum operators have finite spectra. This property is useful for quantum optics. In the present paper, we use the Lie algebra su$(1,1)$ in order to construct, by means of the above postulates, models of the quantum oscillator which would have continuous spectra of the position and momentum operators.

For deriving properties of our oscillators, we use the theory of special functions and orthogonal polynomials. Namely, using the connection between self-adjoint operators (in our case, they are the position and momentum operators) and orthogonal polynomials (see [6] for the description of this connection), we are able to find spectra of the position and momentum operators and to derive an explicit form of their eigenfunctions. We also describe explicitly the evolution operator in the coordinate space.

We will show at the end of the paper that our oscillators parametrized by a positive number l give the usual quantum harmonic oscillator in the limit $l\to\infty$. This means that our oscillators can be considered as a deformation of the quantum harmonic oscillator. It may occur that this deformation in some cases can be more useful than the Biedenharn—Macfarlane q-oscillator.

We consider that our oscillators can be useful for the application to quantum systems in a curved space-time (related to the motion group SU(1,1)≡ $\mathrm{SO}_0(2,1)$) and to quantum systems with the group SU(1,1)≡ $\mathrm{SO}_0(2,1)$ describing their dynamical symmetry. These oscillators can be considered (along with the well-known q-oscillator) as new deformations of the standard quantum harmonic oscillator. We believe that there exist the specific quantum mechanical systems described by our oscillators. A work in this direction will be continued.

2. Discrete Series Representations of su(1, 1)

The Lie algebra su$(1,1)$ has the generators J_0,J_1,J_2 which satisfy the commutation relations

$$[J_0,J_1]=\mathrm{i}J_2\,,\quad [J_1,J_2]=-\mathrm{i}J_0\,,\quad [J_2,J_0]=\mathrm{i}J_1. \tag{4}$$

Instead of the generators J_1,J_2, sometimes the generators $J_\pm=J_1\pm\mathrm{i}J_2$ are used. They obey the relations

$$[J_0,J_\pm]=\pm J_\pm\,,\quad [J_-,J_+]=2J_0 \tag{5}$$

which follows from (4).

The Lie algebra $\mathrm{su}_{1,1}\sim\mathrm{so}_{2,1}$ has several series of unitary irreducible representations, that is, representations which satisfy the relations $J_0^*=J_0$ and $J_+^*=J_-$ (see, for example, [10], Chapter 6). These relations mean that the corresponding representations of the Lie group SU(1,1)$\sim\mathrm{SO}_0(2,1)$ are unitary. For our construction, we need the so-called positive discrete series of irreducible representations. They are given by a positive number l and are denoted by T_l, respectively. Below we will conduct our consideration only for integer or half-integer l. In order to generalize this consideration, we need only to replace factorials containing l by the corresponding Γ-functions.

In order to realize the representations T_l, we consider the space $\mathcal{P}$ of all polynomials in one variable y. Fixing l, we introduce a scalar product in $\mathcal{P}$, considering that the monomials

$$e_n^l(y)=a_n^l y^n\,,\quad a_n^l=\left(\frac{(2l+n-1)!}{n!}\right)^{1/2}, \tag{6}$$

$n=0,1,2,3,\cdots$,

are orthonormal, that is, $\langle e_m^l,e_n^l\rangle=\delta_{mn}$. Closing the space $\mathcal{P}$ with respect to this scalar product, we obtain a Hilbert space which will be denoted as $\mathcal{H}_l$. Clearly, it depends on a value of l. A detailed characterization of this Hilbert space (as a space of analytic functions) and the corresponding description of the scalar product can be found in [10], Chapter 6.

An explicit realization of the representation operators J_i, $i=0,1,2$, of the representation T_l in terms of the first-order differential operators is given as

$$J_0=y\frac{d}{dy}+l\,,\quad J_1=\frac{1}{2}(1+y^2)\frac{d}{dy}+ly\,,$$

$$J_2=\frac{\mathrm{i}}{2}(1-y^2)\frac{d}{dy}-\mathrm{i}ly \tag{7}$$

(see [10], Chapter 6). Acting by these operators upon the basis elements $e_n^l\equiv e_n^l(y)$, $n=0,1,2\cdots$, we find that

$$J_0\,e_n^l=(l+n)\,e_n^l,\quad J_+\,e_n^l=\sqrt{(2l+n)(n+1)}\,e_{n+1}^l, \tag{8}$$

$$J_-\,e_n^l=\sqrt{(2l+n-1)n}\,e_{n-1}^l\,. \tag{9}$$

It is easy to check that these operators satisfy the unitarity conditions $J_0^*=J_0$, $J_+^*=J_-$.

3. Description of the su(1, 1)-models

In order to describe the models of the quantum oscillator based on irreducible representations of the Lie algebra su(1, 1), we fix a non-negative integer or half-integer l and consider the discrete series representation T_l from the previous section. If J_0, J_1, J_2 are operators of this representation, we define the Hamiltonian H and the position and momentum operators Q and P as

$$H = J_0 - l + 1/2, \quad Q = J_2, \quad P = J_1. \tag{10}$$

Then, due to (4), Q, P, and H satisfy the commutation relations

$$[H, Q] = -\mathrm{i}P, \quad [H, P] = \mathrm{i}Q, \quad [Q, P] = \mathrm{i}(H + l - 1/2). \tag{11}$$

Clearly, these operators satisfy postulates 1—3 of Introduction. The evolution of our system over time is the harmonic motion with

$$e^{\mathrm{i}\tau H} \begin{pmatrix} Q \\ P \end{pmatrix} e^{-\mathrm{i}\tau H} = \begin{pmatrix} Q(\tau) \\ P(\tau) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\tau \sin\tau \\ -\sin\tau \cos\tau \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}.$$

This is a group $U(1)$ of inner automirphisms of the Lie algebra su(1, 1) and of rotations of the phase-space surface. We have

$$\exp \mathrm{i}\tau H = \exp \mathrm{i}\tau(J_0 - l + 1/2) = e^{-\mathrm{i}(l-1/2)\tau} \exp \mathrm{i}\tau J_0. \tag{12}$$

The explicit form of the time evolution in the coordinate space will be derived below.

Remark that the basis e^l_n, $n = 0, 1, 2, \cdots$, of the Hilbert space $\mathcal{H}_l$ introduced in the previous section consists of eigenfunctions of the Hamiltonian H:

$$He^l_n = (n + 1/2)e^l_n, \quad n = 0, 1, 2, \cdots,$$

that is, the spectrum of H coincides with the spectrum of the Hamiltonian of the standard quantum harmonic oscillator.

Thus, to each number l ($l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$), there corresponds a model of the quantum oscillator. To different values of l, there correspond non-equivalent models.

4. Spectrum and Eigenfunctions of the Momentum Operator

Since $P = J_1$, the momentum operator P in the basis of the Hamiltonian eigenfunctions e^l_n, $n = 0, 1, 2, \cdots$, has the form

$$Pe^l_n = \frac{1}{2}\left[\sqrt{(2l+n)(n+1)}\, e^l_{n+1} + \sqrt{(2l+n-1)n} e^l_{n-1}\right].$$

We wish to find the spectrum and eigenfunctions of this operator. Let $\psi_p(y)$ be an eigenfunction of P corresponding to the eigenvalue p, $P\psi_p(y) = p\psi_p(y)$. Then

$$\psi_p(y) = \sum_{n=0}^{\infty} h_n(p) e^l_n(y), \tag{13}$$

where $h_n(p)$ are coefficients depending on eigenvalues p.

In order to find an explicit form of eigenfunctions $\psi_p(y)$, we substitute the expression (13) for $\psi_p(y)$ into the equation $P\psi_p(y) = p\psi_p(y)$:

$$\frac{1}{2}\sum_{n=0}^{\infty} h_n(p) \left[\sqrt{(2l + n)(n + 1)}\, e^l_{n+1} + \right.$$

$$\left. + \sqrt{(2l + n - 1)n}\, e^l_{n-1}\right] = p \sum_{n=0}^{\infty} h_n(p) e^l_n.$$

Equating the coefficients of a fixed basis element e^l_n on both sides of this equality, we obtain a recurrence relation for the coefficients $h_n(p)$:

$$2ph_n(p) = \sqrt{(2l + n)(n + 1)}\, h_{n+1}(p) +$$

$$+ \sqrt{(2l + n - 1)n}\, h_{n-1}(p). \tag{14}$$

It is clear from (13) that the coefficients $h_n(p)$ are uniquely determined up to a common constant. We have $h_{-1}(p) = 0$. Setting $h_0(p) = 1$, we see that $h_n(p)$, $n = 1, 2, \cdots$, are calculated uniquely. Moreover, relation (14) shows that $h_n(p)$ are polynomials in p.

In order to solve the recurrence relation (14), we make the substitution

$$h_n(p) = (n!(2l + n - 1)!^{-1})^{1/2} h'_n(p).$$

Then relation (14) turns into

$$2ph'_n(p) = (n + 1)\, h'_{n+1}(p) + (2l + n - 1)\, h'_{n-1}(p). \tag{15}$$

Comparing this relation with the recurrence relation

$$(n+1)P^{(\lambda)}_{n+1}(z; \phi) - 2[z \sin\phi + (n + \lambda)\cos\phi] P^{(\lambda)}_n(z; \phi) +$$

$$+(n+2\lambda-1)P_{n-1}^{(\lambda)}(z;\phi)=0$$

(see formula (1.7.3) in [11]) for the Meixner—Pollaczek polynomials

$$P_n^{(\lambda)}(z;\phi)=\frac{(2\lambda)_n}{n!}e^{\mathrm{i}n\phi}{}_2F_1(-n,\ \lambda+\mathrm{i}z;\ 2\lambda;\ 1-e^{-2\mathrm{i}\phi})$$

at $\phi=\pi/2$, we find that

$$h'_n(p)=P_n^{(l)}(p;\pi/2)=\frac{(2l)_n}{n!}\mathrm{i}^n{}_2F_1(-n,l+\mathrm{i}p;\ 2l;\ 2),$$

where ${}_2F_1$ is the Gauss hypergeometric function of a polynomial type. For the coefficients in (13), we get

$$h_n(p)=(n!(2l+n-1)!^{-1})^{1/2}P_n^{(l)}(p;\pi/2). \tag{16}$$

Thus, we can state that eigenfunctions of the momentum operator P are of the form

$$\psi_p(y)=\sum_{n=0}^{\infty}\left(\frac{n!}{(2l+n-1)!}\right)^{1/2}P_n^{(l)}(p;\pi/2)e_n^l(y)=$$

$$=\sum_{n=0}^{\infty}P_n^{(l)}(p;\pi/2)y^n \tag{17}$$

with the Meixner—Pollaczek polynomials $P_n^{(l)}(p,\pi/2)$, where we have taken into account expression (6) for the basis elements.

We can sum up expression (17) for the eigenfunctions $\psi_p(y)$. Namely, taking into account formula (1.7.11) in [11], we finally find that the eigenfunctions are of the form

$$\psi_p(y)=(1+\mathrm{i}y)^{-l-\mathrm{i}p}(1-\mathrm{i}y)^{-l+\mathrm{i}p}. \tag{18}$$

We use formula (17) in order to find the spectrum of the momentum operator P. To this end, we take into account the following. It is well known that the operator P coinciding with the operator J_1 of the discrete series representation T_l is self-adjoint. Moreover, P is representable in the basis $\{e_n^l\}$ by a Jacobi matrix, that is, by a matrix of the form

$$M=\begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots\\ a_0 & b_1 & a_1 & 0 & 0 & \cdots\\ 0 & a_1 & b_2 & a_2 & 0 & \cdots\\ 0 & 0 & a_2 & b_3 & a_3 & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},\qquad a_i\neq 0.$$

There exists a theory which allows to find the spectra of operators of such a type ([12], Chapter VII; a short description of this theory see in [6]). To apply this theory, we note that the eigenfunctions $\psi_p(y)$ are expressed in terms of the basis elements e_n^l by formula (13) with coefficients (16) which are polynomials. According to the results of Chapter VII in [12], these polynomials are orthogonal with respect to some measure $d\mu(p)$. (This measure is unique, up to a constant, since the operator P is self-adjoint; see [6].) The set (a subset of $\mathbb{R}$), on which the polynomials are orthogonal, coincides with the spectrum of the operator P, and $d\mu(p)$ determines the spectral measure of this operator. Moreover, the spectrum of P is simple.

Thus, to find the spectrum of the momentum operator P, we note that the Meixner—Pollaczek polynomials $P_n^{(l)}(p;\pi/2)$ are orthogonal and the orthogonality relation is of the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Gamma(l+\mathrm{i}p)|^2P_m^{(l)}(p;\pi/2)P_n^{(l)}(p;\pi/2)dp=$$

$$=\Gamma(n+2l)(2^{2l}n!)^{-1}\delta_{mn} \tag{19}$$

(see formula (1.7.2) in [11]). This means that the spectrum of P coincides with the whole real line:

$$\mathrm{Spec}\ P=\mathbb{R}.$$

Thus, the spectrum is continuous and simple. The continuity of the spectrum means that the eigenfunctions $\psi_p(y)$ are not elements of the Hilbert space $\mathcal{H}_l$. They form a continuous basis of $\mathcal{H}_l$ (similar to the basis $\{e^{\mathrm{i}px}\}$ of the Hilbert space $L^2(\mathbb{R})$).

Eigenfunctions of P are determined up to constants. In order to normalize the eigenfunctions $\psi_p(y)$, we take into account the orthogonality relation (19) for Meixner—Pollaczek polynomials. Since these polynomials correspond to the determinate moment problem (see, for example, [6] for the description of this correspondence), the set $P_n^{(l)}(p,\pi/2)$, $n=0,1,2,\cdots$, is complete in the Hilbert space $L^2(\mathbb{R},d\mu(p))$ with respect to the orthogonality measure $d\mu(p)$ for these polynomials. This means that

$$\sum_{n=0}^{\infty}\frac{2^{2l}n!}{2\pi\Gamma(n+2l)}|\Gamma(l+\mathrm{i}p)|^2P_n^{(l)}(p,\pi/2)P_n^{(l)}(p',\pi/2)=$$

$$=\delta(p-p').$$

Then, by (17),

$$\langle\psi_p(y),\psi_{p'}(y)\rangle=\sum_{n=0}^{\infty}\frac{n!}{(2l+n-1)!}P_n^{(l)}(p,\pi/2)\times$$

$$\times P_n^{(l)}(p',\pi/2)=\frac{2\pi\delta(p-p')}{2^{2l}|\Gamma(l+\mathrm{i}p)|^2}.$$

Therefore, the functions

$$\tilde\psi_p(y)=(2\pi)^{-1/2}2^l|\Gamma(l+\mathrm{i}p)|\psi_p(y),\quad p\in\mathbb{R},$$

are normalized, that is, $\langle\tilde\psi_p(y),\tilde\psi_{p'}(y)\rangle=\delta(p-p')$.

5. Spectrum and Eigenfunctions of the Position Operator

The position operator Q in the basis e_n^l, $n=0,1,2,\cdots$ has the form

$$Qe_n^l=\frac{1}{2\mathrm{i}}\left[\sqrt{(2l+n)(n+1)}\,e_{n+1}^l-\sqrt{(2l+n-1)n}\,e_{n-1}^l\right].$$

By changing the basis $\{e_n^l\}$ by the basis $\{\hat e_n^l\}$, where $\hat e_n^l=\mathrm{i}^{-n}e_n^l$, we see that the position operator Q is given in the last basis by the same formula as the momentum operator is given in the basis $\{e_n^l\}$. This means that the spectrum of the operator Q coincides with the spectrum of P, that is,

Spec $Q=\mathbb{R}$.

We have to find eigenfunctions of the position operator. They can be found (by using the basis $\{\hat e_n^l\}$) in the same way as in the case of the momentum operator. For this reason, we expose only the results.

Let $\phi_x(y)$ be an eigenfunction of Q corresponding to the eigenvalue x, $Q\phi_x(y)=x\phi_p(y)$. Then

$$\phi_x(y)=\sum_{n=0}^{\infty}\tilde h_n(x)e_n^l,\tag{20}$$

where, as before, $e_n^l(y)=a_n^ly^n$ and $\tilde h_n(x)$ are coefficients depending of eigenvalues x.

Repeating the reasoning of the previous section, we derive a three-term linear recurrence relation for the polynomials $\tilde h_n(x)$ and find that

$$\tilde h_n(x)=\mathrm{i}^{-n}h_n(x)=$$

$$=\mathrm{i}^{-n}\left(\frac{n!}{(2l+n-1)!}\right)^{1/2}P_n^{(l)}(x;\pi/2),\tag{21}$$

where $P_n^{(l)}(x;\pi/2)$ is the Meixner—Pollaczek polynomial from Section 4. Thus,

$$\phi_x(y)=\sum_{n=0}^{\infty}\mathrm{i}^{-n}\left(\frac{n!}{(2l+n-1)!}\right)^{1/2}P_n^{(l)}(x;\pi/2)e_n^l(y)=$$

$$=\sum_{n=0}^{\infty}\mathrm{i}^{-n}P_n^{(l)}(x;\pi/2)y^n.\tag{22}$$

Taking into account formula (1.7.11) in [11], we find that eigenfunctions of the position operator Q are of the form

$$\phi_x(y)=(1+y)^{-l-\mathrm{i}x}(1-y)^{-l+\mathrm{i}x}.\tag{23}$$

The functions

$$\tilde\phi_x(y)=\frac{2^l|\Gamma(l+\mathrm{i}x)|}{\sqrt{2\pi}}\phi_x(y)\tag{24}$$

are normalized, that is, they satisfy the normalization condition $\langle\tilde\phi_x(y),\tilde\phi_{x'}(y)\rangle_{\mathcal{H}_l}=\delta(x-x')$.

6. Coordinate Realization of an Oscillator

In Section 3, we have constructed a realization of our oscillator (depending on a value of l) on the space of functions in the supplementary variable y. It is natural to have its realization on the space of functions in the coordinate x and on the space of functions in the momentum p.

Let $L^2(\mathbb{R};\mu)$ be the space of squared integrable functions $f(x)$ (where x is the coordinate of the oscillator) with respect to the measure μ from formula (19), that is, the scalar product in $L^2(\mathbb{R};\mu)$ is given by

$$\langle f(x),g(x)\rangle=\frac{2^{2l}}{2\pi}\int\limits_{-\infty}^{\infty}f(x)\overline{g(x)}|\Gamma(l+\mathrm{i}x)|^2dx.\tag{25}$$

It follows from (19) that polynomials (21) constitute the orthonormal basis of $L^2(\mathbb{R};\mu)$.

We construct a one-to-one linear isometry Ω from the Hilbert space $\mathcal{H}_l$, considered in Section 2, onto the Hilbert space $L^2(\mathbb{R};\mu)$ given by the formula

$$\Omega:\ \mathcal{H}_l\ni g(y)\to f(x)=\langle g(y),\phi_x(y)\rangle_{\mathcal{H}_l}\in L^2(\mathbb{R};\mu),\tag{26}$$

where $\phi_x(y)$ are eigenfunctions (23) of Q. It follows from (20) that

$$\mathcal{H}_l\ni e_n^l(y)\to\langle e_n^l(y),\phi_x(y)\rangle_{\mathcal{H}_l}=\tilde h_n(x),$$

that is, Ω maps the orthonormal basis $\{e_n^l(y)\}$ of $\mathcal{H}_l$ onto the orthonormal basis $\{\tilde{h}_n(x)\}$ of $L^2(\mathbb{R};\mu)$. This means that Ω is, indeed, a one-to-one isometry.

The operator Q acts on $L^2(\mathbb{R};\mu)$ as the multiplication operator,

$$Qf(x) = xf(x).$$

Indeed, according to (26) if $\Omega g(y) = f(x) = \langle g(y), \phi_x(y)\rangle_{\mathcal{H}_l}$, we have

$$Qg(y) \to Qf(x) = \langle Qg(y), \phi_x(y)\rangle_{\mathcal{H}_l} =$$

$$= \langle g(y), Q\phi_x(y)\rangle_{\mathcal{H}_l} = \langle g(y), x\phi_x(y)\rangle_{\mathcal{H}_l} = xf(x).$$

Unfortunately, we could not find an acceptable differential form for the operator P.

We can find how Q acts upon the basis elements $\tilde{h}_n(x)$, $n = 0, 1, 2, \cdots$, of the Hilbert space $L^2(\mathbb{R};\mu)$. According to the recurrence relation for polynomials (21) (which follows from the recurrence relation for the polynomials $P_n^{(l)}(z, \pi/2)$), we have

$$Q\tilde{h}_n(x) = x\tilde{h}_n(x) = \frac{\mathrm{i}}{2}\left[\sqrt{(2l+n)(n+1)}\tilde{h}_{n+1}(x) - \right.$$

$$\left. -\sqrt{(2l+n-1)n}\,\tilde{h}_{n-1}(x)\right].$$

Clearly,

$$H\,\tilde{h}_n(x) = (n + 1/2)\,\tilde{h}_n(x).$$

7. Evolution Operator in the Coordinate Space

According to (12), the evolution operator $\exp \mathrm{i}\tau H$ acts upon the basis elements e_n^l, $n = 0, 1, 2, \cdots$, of the Hilbert space $\mathcal{H}_l$ as

$$(\exp \mathrm{i}\tau H)e_n^l = e^{-\mathrm{i}(l-1/2)\tau} e^{\mathrm{i}(l+n)\tau} e_n^l = e^{\mathrm{i}(n+1/2)\tau}\, e_n^l.$$

We wish to find how this operator acts on the coordinate space, that is, on the Hilbert space $L^2(\mathbb{R};\mu)$. If the isometry Ω from (26) maps a function $g(y) \in \mathcal{H}_l$ onto a function $f(x) \in L^2(\mathbb{R};\mu)$, then, to the function $(\exp \mathrm{i}\tau H)g(y) \in \mathcal{H}_l$, there corresponds the function

$$(\exp \mathrm{i}\tau H)f(x) = \langle(\exp \mathrm{i}\tau H)g(y), \phi_x(y)\rangle_{\mathcal{H}_l} =$$

$$= \langle g(y), \exp(-\mathrm{i}\tau H)\phi_x(y)\rangle_{\mathcal{H}_l} =$$

$$= \sum_{n=0}^{\infty} \langle g(y), e_n^l\rangle_{\mathcal{H}_l} \langle e_n^l, \exp(-\mathrm{i}\tau H)\phi_x(y)\rangle_{\mathcal{H}_l} =$$

$$= \sum_{n=0}^{\infty} \langle g(y), e_n^l\rangle_{\mathcal{H}_l} \langle \exp(\mathrm{i}\tau H)e_n^l, \phi_x(y)\rangle_{\mathcal{H}_l} =$$

$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \langle g(y), \tilde{\phi}_{x'}(y)\rangle_{\mathcal{H}_l} \langle \tilde{\phi}_{x'}(y), e_n^l\rangle_{\mathcal{H}_l} dx' \times$$

$$\times \exp(\mathrm{i}\tau(n + 1/2))\langle e_n^l, \phi_x(y)\rangle_{\mathcal{H}_l} =$$

$$= \frac{2^{2l}}{2\pi} \int_{-\infty}^{\infty} f(x') K^\tau(x, x') |\Gamma(l + \mathrm{i}x')|^2 dx',$$

where the kernel $K^\tau(x, x')$ coincides with

$$K^\tau(x, x') =$$

$$= \sum_{n=0}^{\infty} \langle \phi_{x'}(y), e_n^l\rangle_{\mathcal{H}_l} \langle e_n^l, \phi_x(y)\rangle_{\mathcal{H}_l} \exp(\mathrm{i}\tau(n + 1/2)).$$

Taking into account the expression for $\langle e_n^l, \phi_x(y)\rangle_{\mathcal{H}_l}$, we find that

$$K^\tau(x, x') = \sum_{n=0}^{\infty} \exp(\mathrm{i}\tau(n + 1/2))\tilde{h}_n(x)\overline{\tilde{h}_n(x')} =$$

$$= e^{\mathrm{i}\tau/2} \sum_{n=0}^{\infty} e^{\mathrm{i}n\tau} \frac{(2l+n-1)!}{n!(2l-1)!^2}\, {}_2F_1(-n, l + \mathrm{i}x;\ 2l;\ 2) \times$$

$$\times {}_2F_1(-n, l - \mathrm{i}x';\ 2l;\ 2),$$

where we have taken into account that $\tilde{h}_n(x) = \frac{(2l+n-1)!^{1/2}}{n!^{1/2}(2l-1)!}\, {}_2F_1(-n, l + \mathrm{i}x;\ 2l;\ 2)$. Due to formula (12) of Section 2.5.2 in [13], we finally obtain

$$K^\tau(x, x') = c' e^{\mathrm{i}\tau/2}(1 + 2e^{\mathrm{i}\tau})^{-2l - \mathrm{i}(x - x')}(1 - e^{\mathrm{i}\tau})^{\mathrm{i}(x - x')} \times$$

$$\times {}_2F_1\left(l + \mathrm{i}x, l - \mathrm{i}x';\ 2l;\ \frac{4e^{\mathrm{i}\tau}}{(1 + 2e^{\mathrm{i}\tau})^2}\right), \qquad (27)$$

where $c' = (2l - 1)!^{-2}$. Thus, the evolution operator $\exp \mathrm{i}\tau H$ is given by the formula

$$(\exp \mathrm{i}\tau H)f(x) = \frac{2^{2l}}{2\pi} \int_{-\infty}^{\infty} K^\tau(x, x') f(x') |\Gamma(l + \mathrm{i}x')|^2 dx',$$

where the kernel $K^\tau(x,x')$ is given by (27). Since $e^{i\tau H}e^{i\tau' H}=e^{i(\tau+\tau')H}$, this kernel satisfies the relation

$$\frac{2^{2l}}{2\pi}\int_{-\infty}^{\infty}K^\tau(x,x')\overline{K^\tau(x',x'')}\,|\Gamma(l+ix')|^2dx'=$$

$$=K^{\tau+\tau'}(x,x''),$$

which gives the corresponding relation for the hypergeometric function ${}_2F_1$ in (27).

8. Momentum Realization of the Oscillator

In Section 6, we have constructed a realization of our oscillator on the space of functions in the coordinate x. In this section, we realize the oscillator on the space of functions in the momentum p.

Let $\tilde{L}^2(\mathbb{R};\mu)$ be the space of square integrable functions $f(p)$ (where p is the momentum of the oscillator) with respect to the measure μ from formula (19), that is, the scalar product in $\tilde{L}^2(\mathbb{R};\mu)$ is given by

$$\langle f(p),g(p)\rangle=\frac{2^{2l}}{2\pi}\int_{-\infty}^{\infty}f(p)\overline{g(p)}|\Gamma(l+ip)|^2dp.$$

Polynomials (16) constitute an orthonormal basis of the Hilbert space $\tilde{L}^2(\mathbb{R};\mu)$.

We construct a one-to-one isometry $\tilde{\Omega}$ from the Hilbert space $\mathcal{H}_l$ considered in Section 2 onto the Hilbert space $\tilde{L}^2(\mathbb{R};\mu)$ given by the formula

$$\tilde{\Omega}:\quad \mathcal{H}_l\ni g(y)\to\tilde{f}(p):=\langle g(y),\psi_p(y)\rangle_{\mathcal{H}_l}\in\tilde{L}^2(\mathbb{R};\mu),\tag{28}$$

where $\psi_p(y)$ are eigenfunctions (18) of the momentum operator P. It follows from (13) that

$$\mathcal{H}_l\ni e_n^l(y)\to\langle e_n^l(y),\psi_p(y)\rangle_{\mathcal{H}_l}=h_n(p),$$

that is, $\tilde{\Omega}$ maps the orthonormal basis $\{e_n^l(y)\}$ of $\mathcal{H}_l$ onto the orthonormal basis $\{h_n(p)\}$ of $\tilde{L}^2(\mathbb{R};\mu)$.

As in the case of the position operator Q, we get that the operator P acts on $\tilde{L}^2(\mathbb{R};\mu)$ as the multiplication operator,

$$Pf(p)=pf(p).$$

We can find how P acts upon the basis elements $h_n(p)$, $n=0,1,2,\cdots$, of the Hilbert space $\tilde{L}^2(\mathbb{R};\mu)$. According to the recurrence relation for polynomials (16) (which follows from the recurrence relation for the polynomials $P_n^{(l)}(z,\pi/2)$), we have

$$Ph_n(p)=ph_n(p)=\frac{1}{2}\left[\sqrt{(2l+n)(n+1)}h_{n+1}(p)+\right.$$

$$\left.+\sqrt{(2l+n-1)n}\,h_{n-1}(p)\right].$$

We also have $H\,h_n(p)=(n+1/2)\,h_n(p)$.

9. An Analog of the Fourier Transformation

Let us first consider what we have in the case of the standard quantum harmonic oscillator. This oscillator is determined by the relation

$$aa^+-a^+a=1.$$

For the position and momentum operators Q_A and P_A, we have

$$\mathcal{Q}=\frac{1}{\sqrt{2}}(a^++a),\quad \mathcal{P}=\frac{i}{\sqrt{2}}(a^+-a).$$

The Hilbert space of states $\mathcal{H}$ is spanned by the orthonormal vectors $|n\rangle$, $n=0,1,2,\cdots$. For eigenvectors of $\mathcal{Q}$ and $\mathcal{P}$, we have

$$\mathcal{Q}|x\rangle=x|x\rangle,\quad \mathcal{P}|p\rangle=p|p\rangle$$

and Spec $\mathcal{Q}=\mathbb{R}$, Spec $\mathcal{P}=\mathbb{R}$.

For $h\in\mathcal{H}$, we have

$$\langle h,x\rangle_{\mathcal{H}}=h(x),\qquad \langle h,p\rangle_{\mathcal{H}}=\tilde{h}(p).\tag{29}$$

In this way, we obtain a realization of $\mathcal{H}$ as a space of functions in the coordinate or as a space of functions in the momentum. Then the functions $h(x)$ and $\tilde{h}(p)$ from (29) are related with each other by the usual Fourier transformation:

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{h}(p)e^{ipx}dp.$$

Our aim in this section is to find what is an analog of the Fourier transformation for our oscillators. Let $g(y)$ be a function of the Hilbert space $\mathcal{H}_l$ of Section 2. Then

$$\Omega:g(y)\to f(x)\in L^2(\mathbb{R},\mu),\quad \tilde{\Omega}:g(y)\to\tilde{f}(p)\in\tilde{L}^2(\mathbb{R},\mu).$$

We have to find how the functions $f(x)$ and $\tilde{f}(p)$ are connected with each other. By (26) and (28), one has

$$\tilde{f}(p)=\langle g(y),\psi_p(y)\rangle_{\mathcal{H}_l}=$$

$$= \int_{-\infty}^{\infty} \langle g(y), \tilde{\phi}_x(y)\rangle_{\mathcal{H}_l} \langle \tilde{\phi}_x(y), \psi_p(y)\rangle_{\mathcal{H}_l} dx =$$

$$= \frac{2^{2l}}{2\pi} \int_{-\infty}^{\infty} |\Gamma(l+\mathrm{i}x)|^2 f(x) \langle \phi_x(y), \psi_p(y)\rangle_{\mathcal{H}_l} dx =$$

$$= \frac{2^{2l}}{2\pi} \int_{-\infty}^{\infty} f(x) F(x,p) |\Gamma(l+\mathrm{i}x)|^2 dx,$$

where the kernel $F(x,p)$ coincides with

$$F(x,p) = \langle \phi_x(y), \psi_p(y)\rangle_{\mathcal{H}_l} =$$

$$= \sum_{n=0}^{\infty} \frac{n!\mathrm{i}^{-n}}{(2l+n-1)!} P_n^{(l)}(x,\pi/2) \overline{P_n^{(l)}(p,\pi/2)} =$$

$$= \sum_{n=0}^{\infty} \frac{\mathrm{i}^{-n}(2l+n-1)!}{(2l-1)!^2 n!} {}_2F_1(-n, l+\mathrm{i}x;\ 2l;\ 2) \times$$

$$\times {}_2F_1(-n, l-\mathrm{i}p;\ 2l;\ 2).$$

Taking into account formula (12) of Section 2.5.2 in [13], we finally obtain

$$F(x,p) = \frac{(1+2\mathrm{i})^{-2l}}{(2l-1)!^2} (1+2\mathrm{i})^{\mathrm{i}(p-x)} (1-\mathrm{i})^{\mathrm{i}(x-p)} \times$$

$$\times {}_2F_1\left(l+\mathrm{i}x, l-\mathrm{i}p;\ 2l;\ \frac{4\mathrm{i}}{(1+2\mathrm{i})^2}\right). \quad (30)$$

Thus, *an analog* $\mathcal{F} : f(x) \to \tilde{f}(p)$ *of the Fourier transformation for our oscillator is given by the formula*

$$\mathcal{F} f(x) = \tilde{f}(p) = \frac{2^{2l}}{2\pi} \int_{-\infty}^{\infty} f(x) F(x,p) |\Gamma(l+\mathrm{i}x)|^2 dx,$$

where the kernel $F(x,p)$ *is given by formula* (30). The transformation $\mathcal{F}$ is linear and isometric, that is, it conserves the scalar product. Therefore, the inverse transformation $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1} \tilde{f}(p) = f(x) = \frac{2^{2l}}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(p) \overline{F(x,p)} |\Gamma(l+\mathrm{i}p)|^2 dp,$$

where $\overline{F(x,p)}$ means a complex conjugate of $F(x,p)$. The Plancherel formula

$$\int_{-\infty}^{\infty} |f(x)|^2 |\Gamma(l+\mathrm{i}x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(p)|^2 |\Gamma(l+\mathrm{i}p)|^2 dp$$

holds.

10. A Limit to the Quantum Harmonic Oscillator

We have constructed an infinite number of models of the quantum oscillator. These models are characterized by the number l. The model corresponding to l will be denoted by osc_l. We state that

$$\lim_{l\to\infty} l^{-1/2}\, \mathrm{osc}_l = \mathrm{osc}, \quad (31)$$

where osc denotes the standard quantum harmonic oscillator. Formula (31) means that

$$\lim_{l\to\infty} l^{-1/2}\, Q_l = \mathcal{Q}, \quad \lim_{l\to\infty} l^{-1/2}\, P_l = \mathcal{P}, \quad (32)$$

where $Q \equiv Q_l$, $P \equiv P_l$ are the position and momentum operators for osc_l, and $\mathcal{Q}$, $\mathcal{P}$ are the position and momentum operators for osc. A validity of relations (32) follows from the fact that, under this limit, relations (11) turn into

$$[H, \mathcal{Q}] = -\mathrm{i}\mathcal{P}, \quad [H, \mathcal{P}] = \mathrm{i}\mathcal{Q}, \quad [\mathcal{Q}, \mathcal{P}] = \mathrm{i}.$$

It follows from the formula for the action of the operators Q_l and P_l upon the basis e_n^l, $n = 0, 1, 2, \cdots$, that, in the limit $l \to \infty$, one gets the formulas

$$\mathcal{P}\tilde{e}_n = \frac{\mathrm{i}}{\sqrt{2}} (\sqrt{n+1}\, \tilde{e}_{n+1} - \sqrt{n}\, \tilde{e}_{n-1}),$$

$$\mathcal{Q}\tilde{e}_n = \frac{1}{\sqrt{2}} (\sqrt{n+1}\, \tilde{e}_{n+1} + \sqrt{n}\, \tilde{e}_{n-1}),$$

where $\tilde{e}_n = \mathrm{i}^{-1} e_n^l$. They are the standard formulas for $\mathcal{Q}$ and $\mathcal{P}$.

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su(1, 1)-МОДЕЛІ КВАНТОВОГО ОСЦИЛЯТОРА

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Р е з ю м е

На базі представлень дискретної серії алгебри Лі su(1, 1) побудовано моделі квантового осцилятора. Оператори положення та імпульсу в цих моделях збігаються відповідно з операторами J_2 і J_1 цих представлень. Як і в стандартному квантовому гармонічному осциляторі, оператори положення та імпульсу в цих моделях мають неперервні прості спектри, що покривають всю дійсну вісь. Власні функції цих операторів знайдено в явному вигляді. Показано, що звичайний квантовий гармонічний осцилятор є границею осциляторів, побудованих у цій роботі, тобто останні можуть розглядатися як деформації квантового гармонічного осцилятора.