SOLITON-LIKE STATES OF THE ORDER PARAMETERS NEAR THE LIFSHITS POINTS

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This paper is dedicated to Victor G. Bar'yakhtar on the occasion of his 75th birthday

The soliton-like states of order parameters have been considered in the framework of a model which describes phase transitions for a single-component non-uniform order parameter and takes into account its higher-order derivatives. Exact solutions of the variational equation have been constructed in the framework of the φ^6-model. The scale invariance of the equations in models with arbitrary nonlinearity has been studied. The criteria of integrability have been found for scale-invariant equations.

1. Introduction

As a rule, the inhomogeneities of the order parameter (OP) of the system emerging in the vicinity of phase transformation points, if being taking into account, enlarge the system energy. However, although these inhomogeneities arise and evolve in the critical range as a result of the phase instability, there is always a lot of reasons that allow one to consider the critical inhomogeneities of the OP as the equilibrium extremals of thermodynamic potentials. In particular, such a consideration is valid for the modulated structures of the OP which are energetically beneficial even in an unconfined medium [1]. The importance of the model with a single-component OP stems from the fact that many phase transformations, which are described by a multicomponent OP, can be effectively described, making use of a single-component one, in the vicinity of the critical point [1].

In order to describe phase transformations in the systems with a single-component non-uniform OP, the following expression for the thermodynamic potential is used [2,3]:

$$\Phi = \int_{-\infty}^{+\infty} \Big[(\bar\varphi'')^2 - g\bar\varphi^2(\bar\varphi')^2 - -\gamma(\bar\varphi')^2 + r\bar\varphi^2 + \frac{s}{2}\bar\varphi^4 + \frac{1}{3}\bar\varphi^6\Big] d\bar x, \quad (1)$$

where $\bar\varphi(\bar x)$ is the OP; g, γ, r, and s are the material parameters ($g=\gamma=r=s=0$ is the Lifshits point); and the length of the specimen is considered infinite. In the homogeneous case, model (1) describes the phase transition of the second kind from the phase with $\bar\varphi=0$ into the phase with $\bar\varphi=\sqrt{-\frac{r}{s}}$, which takes place when r changes its sign. In this case, the system possesses only one degree of freedom for $\bar\varphi$. Making allowance for the spatial derivatives of the OP in the thermodynamic potential extends the number of the degrees of freedom of the system, which interact with one another through short-range forces, to a continuum and makes it possible to take into account both the correlations and the interaction of the $\bar\varphi(\bar x)$-fluctuations.

The Euler–Poisson equation for functional (1) looks like

$$\bar\varphi^{IV} + g\left(\bar\varphi^2\bar\varphi'' + 2\bar\varphi\,(\bar\varphi')^2\right) + \gamma\bar\varphi'' + r\bar\varphi + s\bar\varphi^3 + \bar\varphi^5 = 0. \quad (2)$$

Below, we consider the case $s=0$. Carrying out the transformations

$$\bar\varphi(x) = \sqrt{|\gamma|}\varphi(\bar x\sqrt{|\gamma|}), \quad r = q\gamma^2, \quad \bar x\sqrt{|\gamma|} = x, \quad (3)$$

Eq. (2) can be rewritten in the form which includes only 2 material parameters, g and q:

$$\varphi^{IV} + g\left(\varphi^2\varphi'' + 2\varphi\,(\varphi')^2\right) + \sigma\varphi'' + q\varphi + \varphi^5 = 0, \quad (4)$$

where $\sigma = \operatorname{sign}\gamma$.

2. Soliton-like States of the OP

The soliton-like states are searched for in the form

$$\varphi(x) = \frac{1}{a\,\mathrm{ch}(b\cdot x)}, \quad (5)$$

where the parameters a and b, after relation (5) having been substituted into Eq. (2), are expressed in terms of the material parameters.

In the case $\gamma > 0$, substituting relation (5) into Eq. (2) brings about the system of equations for a and b:

$$\begin{cases} b^4 + b^2 + q = 0, \\ g - a^2 - 10(ab)^2 = 0, \\ 24(ab)^4 - 3g(ab)^2 + 1 = 0. \end{cases} \quad (6)$$

Examining the compatibility conditions of the system of equations (6) in the phase plane (q, g^2), we may draw a curve corresponding to valid combinations of the parameters a and b in relation (5). This curve represents the plot of the function

$$g^2(q) = \frac{(10\beta + 1)^2}{3\beta(2\beta + 1)}, \quad (7)$$

where

$$\beta \equiv b^2 = \frac{\sqrt{1 - 4q} - 1}{2}, \quad q < 0. \quad (8)$$

Function (7) possesses the asymptote $g^2 = \frac{50}{3}$ at $q \to -\infty$ and the minimum $g_0^2 = \frac{32}{3}$ at $q = -\frac{7}{16}$.

If $\gamma < 0$, the system of equations for a and b reads

$$\begin{cases} b^4 + b^2 - q = 0, \\ g + a^2 - 10(ab)^2 = 0, \\ 24(ab)^4 - 3g(ab)^2 + 1 = 0. \end{cases} \quad (9)$$

The compatibility condition for the system of equations (9) looks like

$$g^2 = \frac{(10\beta - 1)^2}{3\beta(2\beta - 1)}, \quad (10)$$

where

$$\beta = \frac{\sqrt{1 - 4q} + 1}{2}, \quad q < 0. \quad (11)$$

Function (10) also has the asymptote $g^2 = \frac{50}{3}$ at $q \to -\infty$ and grows monotonously up to $+\infty$ at $q \to -\frac{1}{4}$.

3. Model with a Generalized Nonlinearity

Consider a model characterized by the thermodynamic potential with a nonlinearity of a more general kind than that in Eq. (1). In so doing, we admit $\gamma = r = s = 0$, so that

$$\Phi = \int \left[(\varphi'')^2 - g\varphi^{2n} (\varphi')^{2m} + \frac{\lambda}{2k}\varphi^{2k} \right] dx, \quad (12)$$

where the numbers n, m, and k are not necessarily integers. There may be several terms of the form $\varphi^{2n} (\varphi')^{2m}$ with different n and m. The variational equation for functional (12) is as follows:

$$\varphi^{\mathrm{IV}} + g(2m - 1)(n\varphi^{2n-1}(\varphi')^{2m} + $$
$$+ m\varphi^{2n}(\varphi')^{2m-2}\varphi'') + \frac{\lambda}{2}\varphi^{2k-1} = 0. \quad (13)$$

Integrating Eq. (13), we find its first integral:

$$2\varphi'\varphi''' - (\varphi'')^2 + g\,(2m - 1)\,(\varphi')^{2m}\,\varphi^{2n} + \frac{\lambda}{2k}\varphi^{2k} = D, \quad (14)$$

where D is an integration constant. Since we are interested in solutions that tend, together with their derivatives, to 0 at infinity, we admit $D = 0$ hereafter.

Confining ourselves to the subset of solutions with $D = 0$ (which include all the soliton-like ones), we can preserve the initial variational symmetry of Eq. (13); below, we are going to determine the relevant conditions for that.

Let us introduce a new function

$$U(x) = \sqrt{\varphi'}. \quad (15)$$

Then, Eq. (14) reads

$$4U^3U'' + g\,(2m - 1)\,U^{4m}\varphi^{2n} + \frac{\lambda}{2k}\varphi^{2k} = 0. \quad (16)$$

By introducing the function $V(\varphi) = U^3$, Eq. (17) can be rewritten in the following form:

$$V'' + \frac{3g(2m - 1)}{4}\varphi^{2n}V^{\frac{4m-5}{3}} + \frac{\lambda}{2k}\varphi^{2k}V^{-\frac{5}{3}} = 0. \quad (17)$$

Equation (17) is an equation for the $V(\varphi)$ function. This equation is invariant in respect of the following scale transformation:

$$\begin{cases} \varphi* = \mu\varphi, \\ V*(\varphi*) = \mu^{\Delta}V(\varphi). \end{cases} \quad (18)$$

Therefore, it can be converted to the form, which does not include the explicit dependence on the argument. For this purpose, we pass to new variables:

$$\eta(\xi) = \varphi^{-\Delta}V(\varphi), \xi = \ln\varphi. \quad (19)$$

Then, Eq. (18) reads

$$\eta'' + (2\Delta - 1)\eta' + (\Delta - 1)\eta +$$
$$+ \frac{3g(2m - 1)}{4}\eta^{\frac{4m-5}{3}} + \frac{\lambda}{2k}\eta^{-\frac{5}{3}} = 0. \quad (20)$$

At arbitrary m and q, Eq. (20) can be integrated analytically only in the case $\Delta = \frac{1}{2}$. If $m = \frac{1}{2}$, it can integrated analytically also at $\Delta = \frac{5}{18} \pm \frac{1}{9}\sqrt{22}$.

Therefore, if the differential equation of a higher order for the order parameter is characterized by a variational scale symmetry, all its soliton-like solutions can be found exactly, at least with 3 integration constants, notwithstanding the fact that the initial symmetry disappears already in the first integral of this differential equation.

1. *Klepikov V.F.* Phase Transitions (Incommensurate Structures). — Kharkiv: NSC KIPT Publishing House, 1996 (in Russian).

2. *Toledano J.C., Toledano P.* The Landau Theory of Phase Transitions. — Singapore: World Scientific, 1987.

3. *Klepikov V.F.* // J. de Phys. C. — 1988. — **49**, N 12. — P. 1805 — 1853.

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СОЛІТОНОПОДІБНІ СТАНИ ПАРАМЕТРІВ ПОРЯДКУ ПОБЛИЗУ ТОЧОК ЛІФШИЦЯ

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Р е з ю м е

Досліджено солітоноподібні стани параметрів порядку в моделі з вищими похідними, яка описує фазові перетворення для однокомпонентного неоднорідного параметра порядку. Побудовано точні розв'язки варіаційного рівняння в моделі φ^6. Досліджено масштабну інваріантність рівнянь в моделях з довільною нелінійністю. Для масштабно інваріантних рівнянь знайдено критерії інтегровності.