# PARTICLE DIFFUSION IN EXTERNAL FIELD OF RANDOM LANGMUIR WAVES: SHORT AND LONG TIMES

**V.I. ZASENKO, A.G. ZAGORODNY, J. WEILAND**[1]

**Bogolyubov Institute for Theoretical Physics, Nat. Acad. Sci. of Ukraine**
*(14b, Metrolohichna Str., Kyiv 03143, Ukraine),*

[1]**Department of Electromagnetics,**
**Chalmers University of Technology and Euroatom-VR Association**
*(41296 Göteborg, Sweden)*

UDC 539


Temporal behavior of the velocity dispersion of particles undergoing an external field of random Langmuir waves is considered. It is shown that the early stage of evolution of the dispersion may affect its power law asymptotics. The results of simulation are recovered with solutions of the generalized Fokker—Planck equation that was obtained from the microscopic description without averaging over the small field correlation time scale.

## 1. Introduction

Turbulent and transport phenomena are usually characterized by different time and space scales. To find the behavior of a physical system on a large scale, the averaging over a small one is frequently used. For a number of problems, this step is not trivial. The transition from the kinetic description to the hydrodynamic one with regard for large-scale fluctuations was developed in [1]. In [2], the averaging for systems which may be treated as multifractals is proposed. In the present paper, we consider a stochastic acceleration of particles. It is shown that the behavior of the distribution function at long times depends on its variation on the field correlation time scale. For this reason, the straightforward averaging in this and similar problems may be inappropriate.

For the classical diffusion of a gas of neutral particles, the collisional time is small as compared to the interval between collisions. In plasmas, there exists the additional *anomalous* diffusion of particles due to their interaction with collective excitations – electromagnetic waves. When plasma is in the equilibrium state the anomalous diffusion may be neglected. However, the equilibrium state for plasmas is not usual. In most cases, plasma is inhomogeneous and, apart of this, may be penetrated by particle and electromagnetic beams. A non-equilibrium state causes a growth of the intensity of collective motions and, therefore, an increase of their influence on transport processes. It is commonly supposed that the confinement time in controlled thermonuclear fusion devices is determined by the anomalous transport which remains, for a long time, to be one of the fundamental and unsolved problems in the plasma theory.

More simple is the problem of stochastic acceleration concerning particle diffusion in an external random field. However, the quantitative description is developed, in this case, only for weak fields. The difficulties appear with a growth of the field intensity, and their analysis may help to solve the more general problem of anomalous transport.

The particle interaction with a field of random waves is continuous in contrast with pair collisions. Physical fields, even if they are considered as random, have regular features at small time and space scales. The measure of a temporal interval of their regularity is the correlation time. Another parameter such as the particle bounce time may be associated with the field intensity. In the case of small intensities, when the bounce time much exceeds the field correlation time, a particle is subjected to a time-varying field. The particle does not feel the regularity of a spatial field profile, and its interaction with the field may be treated as a sequence of instantaneous collisions. The diffusion in a weak field is similar to Brownian motion, and it is governed by the Fokker—Planck equation with the quasilinear diffusion coefficient. If the bounce time decreases, being comparable with the correlation time, the particle distribution function obeys the generalized Fokker—Planck equation with the time-dependent diffusion coefficient [3]. This equation takes into account that a particle feels a regular field profile and for a while may be trapped in a potential well. The time dependence of the diffusion coefficient is substantial


*ISSN 0503-1265. Ukr. J. Phys. 2005. V. 50, N 8*

in a temporal interval which is less than the correlation time. After that, the asymptotic quasilinear value of the diffusion coefficient is attained, and further the evolution proceeds according to the quasilinear equation. The broadening of the distribution function occurs intensively at the very beginning of the evolution. Much more time is needed to reach such a state through a quasilinear evolution. This is a reason for different behaviors of the distribution functions governed by the quasilinear and generalized Fokker—Planck equations. The quasilinear diffusion coefficient is obtained from the time-dependent one by the integration over the correlation time. Thus, the time averaging brings into a different behavior of the distribution function not only at the beginning but at long times as well, though the long-time behavior is governed in both cases by the same quasilinear equation.

The problem of stochastic acceleration is simpler than the self-consistent treatment of the distribution function and the field. Despite this, even for a low field intensity, when the quasilinear diffusion equation is valid, there is a discrepancy in the estimation of a long-time behavior of the velocity dispersion [4–6]. It is more convenient to follow the evolution not of the distribution function but its moments. The most representative is the dispersion of particles that corresponds to the second moment of the distribution function. In [6], it was pointed out that the deviation from a linear growth [4,5] is caused by the dependence of the diffusion coefficient on the velocity, and a unified power law was proposed. However, the simulation [3] has shown that the fractional power is not common for different spectra; here, it is considered in more details. Our conclusions would be based on the solutions of the Fokker—Planck equation for long times. Previously, we have established a consistency between the simulations and the solutions of the Fokker—Planck equation. As far as the consistency was found for different spectra in the whole temporal interval of a simulation starting from the scale less than the correlation time, there is a good reason to assume that the Fokker—Planck equation gives a correct description for the distribution function at very long times.

## 2. Simulation and Generalized Diffusion Equation

We made simulation of the particle motion in the external electric field of random Langmuir waves. The correlation function of the potential is of the form

$$\langle\varphi^2\rangle_{xt} = \varphi_0^2 \exp\left(-\frac{(\Delta k\, x)^2}{4}\right)\cos(\omega t - k_0 x). \qquad (1)$$

The parameters of the random field are the dimensionless amplitude of the potential and the spectrum width

$$\sigma = \frac{e}{m}\varphi_0\frac{k_0^2}{\omega^2} \qquad \text{and} \qquad d = \frac{\Delta k}{k_0}.$$

Another important parameter is the Kubo number which is the ratio of the field correlation time to the bounce period of a particle in a potential well; it is given in terms of these parameters as

$$Q = \sqrt{\sigma}/d. \qquad (2)$$

The Kubo number may be considered as the measure of the field intensity or the dimensionless field correlation time. In the case of small Kubo numbers, the particle diffusion is similar to Brownian motion. The distribution function is governed by the Fokker—Planck equation with the quasilinear diffusion coefficient obtained by the integration over the field correlation time. When the Kubo number is not small, we should take into account the particle motion on a time scale of the order of the correlation time.

The results of the simulation are compared with those obtained from the microscopic description in terms of the distribution function. The evolution of the particle distribution function $F$ may be given as

$$F(x,v,t) = \int dx'dv'W(x,v,t;x',v',0)F(x',v',0), \qquad (3)$$

where $F(x,v,0)$ is its initial value, and $W(x,v,t;x',v',t')$ is the transition probability.

The equation for the transition probability of a particle in the external random field was derived in [7]. It reads

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)W(x,v,t;x',v',t') =$$

$$= \left(\frac{e}{m}\right)^2\frac{\partial}{\partial v}\int\limits_{t'}^{t} d\tau\int dy du W(x,v,t;y,u,\tau)\times \qquad (4)$$

$$\times\langle E(x,t)E(y,\tau)\rangle\frac{\partial}{\partial u}W(y,u,\tau;x',v',t'),$$

$$W(x,v,t';x',v',t') = \delta(x-x)\delta(v-v').$$

Such a nonlinear integro-differential equation cannot be solved directly. To simplify this equation, the following consideration are taken into account.

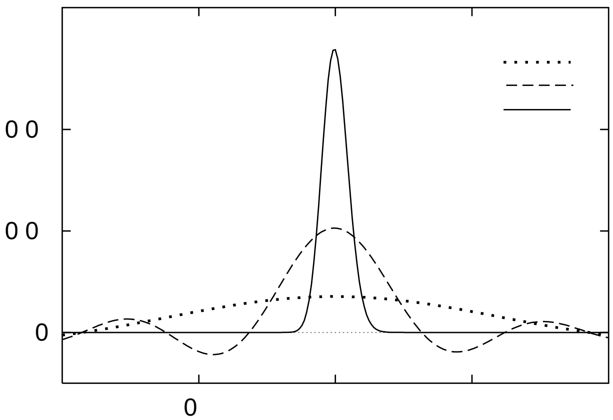


Fig. 1. Profile of the diffusion coefficient (6) for different instants. In all figures, time is normalized by wave period $2\pi/\omega$, and length by $2\pi/k_0$

1. $E(x,t)$ is a uniform and stationary process, so $W(x,v,t;y,u,\tau) = W(x-y,v,u,t-\tau)$. It follows from Eq. (1) that the Lagrangian correlation function $\langle E(x,t)E(x-v(t-t'),t')\rangle = \langle E^2\rangle_{v(t-t'),t-t'}$ strongly decays for $|t-t'| > \tau_{\rm cor}$, where the correlation time is determined by the spectrum width $\tau_{\rm cor} \simeq 2\pi/(\Delta k\, v)$.

2. Strong decay of the correlation function means that the main contribution to the integral over time comes from a short interval $t-\tau < \tau_{\rm cor}$. If the particle spread in the velocity space at $\tau_{\rm cor}$ is small, $\langle\Delta v\rangle^2_{\tau_{\rm cor}}/v^2 \ll 1$, then the exact $W(x,v,t;y,u,\tau)$ may be substituted by its zeroth-order approximation $\delta(x-y-v(t-\tau))\,\delta(v-u)$. The simulation shows that the spread of velocities is small for Kubo numbers $Q \ll 1$, and such a case is treated by the quasilinear theory. For moderate $Q \sim 1$, the initial spread substantially exceeds what is observed for $Q \ll 1$, but if the spectrum is narrow, $d \ll 1$, it remains small enough, nevertheless, (cf. Fig. 2) to justify the free-particle approximation.

3. The simulation and *posteriori* numerical solution show that $W(x,v,t;x',v',0)$ evolves slowly on a scale $\tau_{\rm cor}$, except on a small fraction of the interval $[0,\tau_{\rm cor}]$. This means that $\partial W(y,u,\tau;x',v',t')/\partial u$ in Eq. (4) can be replaced for $\partial W(x,v,t;x',v',t')/\partial v$.

Taking this into account, Eq. (4) may be simplified:

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)W(x-x',v,v',t-t') =$$

$$=\left(\frac{e}{m}\right)^2\frac{\partial}{\partial v}\left(\int\limits_{t'}^{t} d\tau\langle E^2\rangle_{v\tau,\tau}\right)\frac{\partial}{\partial v}W(x-x',v,v',t-t').$$

After the integration over $x$, we obtain the generalized Fokker—Planck equation for the transition probability in the velocity space $v$, which can be considered also as the distribution function of velocities with the initial condition in a form of a $\delta$-function:

$$\frac{\partial w(v,v_0,t)}{\partial t} = \frac{\partial}{\partial v}D(v,t)\frac{\partial}{\partial v}w(v,v_0,t). \qquad (5)$$

Here,

$$D(v,t) = \left(\frac{e}{m}\right)^2\int\limits_0^t\langle E^2\rangle_{v\tau,\tau}d\tau \qquad (6)$$

is the time-dependent diffusion coefficient. For a small Kubo number, we may put $\tau_{\rm cor}$ to zero and obtain the quasilinear diffusion coefficient

$$D_{\rm ql}(v) = D(v,t\to\infty) =$$

$$= \left(\frac{e}{m}\varphi_0\right)^2\sqrt{\pi}\frac{\omega^2}{\Delta k|v|^3}\exp-\left(\frac{\omega-k_0v}{\Delta k\, v}\right)^2. \qquad (7)$$

The Fokker—Planck equation with the time-dependent diffusion coefficient (6) makes it possible to describe the early evolution of the distribution function at $t < \tau_{\rm cor}$, that is important for a moderate Kubo number $Q = 1 \div 3$. Note that, to obtain Eq. (6), we do not average over a small but finite time interval. Thus, the transition probability is determined at a small time scale $t < \tau_{\rm cor}$ i.e. in the prekinetic stage. The solutions of Eq. (5) with the diffusion coefficients (6), (7) were found numerically.

Along with this, an analytical WKB approximation for solutions is proposed as

$$w(v,v_0,t) = C(t)\exp[-\frac{1}{4}Z^2(v,v_0,t)], \qquad (8)$$

where

$$Z(v,v_0,t) = \int\limits_{v_0}^{v}\frac{du}{\sqrt{\int\limits_0^t D(u,\tau)d\tau}},$$

and $C(t)$ can be found from the condition of normalization

$$C^{-1} = \int dv\exp[-\frac{1}{4}Z^2(v,v_0,t)].$$

 ISSN 0503-1265. Ukr. J. Phys. 2005. V. 50, N 8

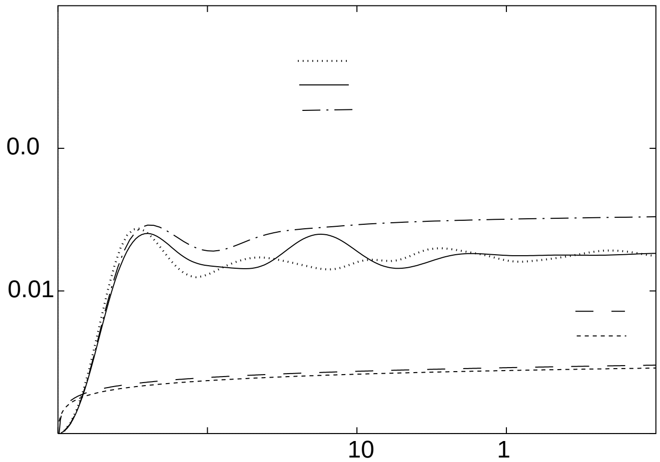


Fig. 2. Dispersion vs time. Initial stage: $d = 0.04$, $\sigma = 0.01$, $v_0 = 1.0$, $Q = 2.5$

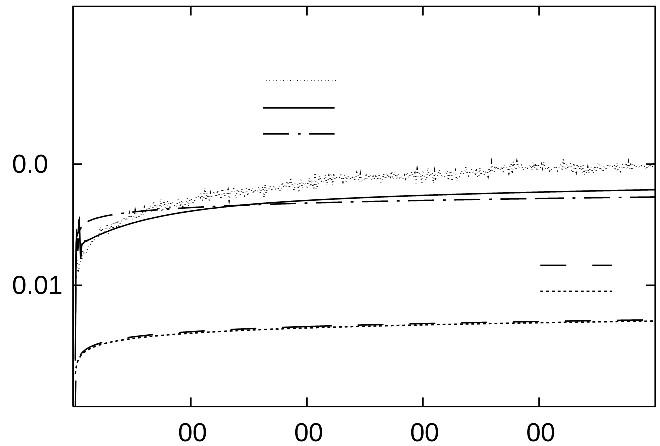


Fig. 3. The same as in Fig. 2 at longer times

## 3. Dispersion over Small Time Scale

The transition probability at a small time scale is determined by a temporal variation of the diffusion coefficient (6). Its profile for different instants is shown in Fig. 1. It evolves from a very broad distribution through oscillations to the asymptotic quasilinear value attained at $t \simeq \tau_{\rm cor}$.

For moderate Kubo numbers, the temporal variation of the diffusion coefficient substantially influences the initial stage of evolution of the distribution function. This is seen from the observation of the particle dispersion

$$\langle \Delta v^2 \rangle_t = \int dv (v - \bar{v}(t))^2 w(v, v_0, t),$$

i.e. the mean square deviation from the average velocity

$$\bar{v}(t) = \int dv\, v\, w(v, v_0, t).$$

In Figs. 2, 3, the velocity dispersion obtained in the simulation is compared with those calculated from the generalized Fokker—Planck equation (numerical and WKB solutions). The solutions of the Fokker—Planck equation with the quasilinear diffusion coefficient are given for comparison as well. The latter does not reproduce the fast initial growth of the dispersion and gives underestimated values on the whole. Though the evolution of the distribution functions governed by both these equations proceeds at long times in accordance with the quasilinear description, the curves corresponding to $D(v, t)$ is not obtained by a shift of those with $D(v)$. They are characterized by different power laws. The dispersion at very long times is considered in the following section.

## 4. Dispersion at Long Times

In this section, we discuss the power law for the velocity dispersion which may be attributed, at first glance, to the rough scaling invariance of the diffusion equation. The scaling-based consideration is as follows. As was already mentioned, after the initial stage, the time-dependent coefficient attains the asymptotic (quasilinear) value. The quasilinear diffusion coefficient (7) drops sharply to zero at small $v$ and slowly decays at large $v$. So, asymptotically, the diffusion is determined by the particle spreading in the region of large $v$. The rough estimation of the diffusion coefficient for large $v$ gives

$$\lim_{v \to \infty} D_{ql}(v) \sim \frac{1}{|v|^3}.$$

In such an approximation, the solution of the Fokker—Planck equation for the diffusion coefficient is a function of velocity and time taken as $v^5/t$:

$$f(v, t) = C(t)\, f_a \left( \frac{v^5}{t} \right),$$

where $C(t) \sim t^{-1/5}$ provides the normalization. Consequently, the asymptotic behavior of the dispersion obeys the power law $\langle \Delta v^2 \rangle = \int dv v^2 f(v, t) \sim t^{2/5}$.

The simulation shows a power-law dispersion, but the exponent is different from that was obtained from

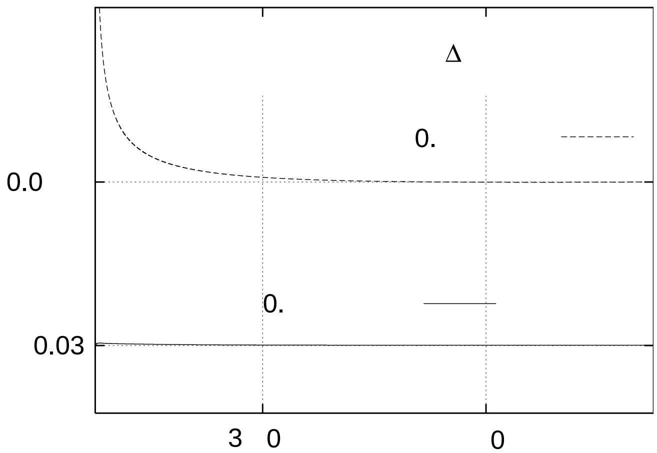


Fig. 4. $Y = \langle \Delta v^2 \rangle / t^p$ vs $t$. Power law for the broad spectrum. $d = 0.4$, $\sigma = 0.01$, $v_0 = 1$, $Q = 0.25$. WKB solutions of the Fokker—Planck equation with $D(v)$. Dispersion normalized by $t^p$ over a long time for the narrow initial distribution (bottom curve, $p = 0.295$) and some broad initial distribution (upper curve, $p = 0.242$)

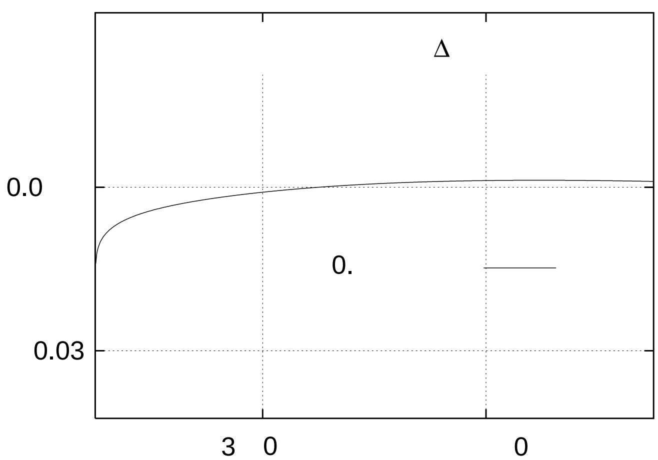


Fig. 5. Over very long times, the fractional power $p$ for a narrow initial distribution is changed from 0.295 to 0.242

such scaling consideration and depends on the intensity and the spectrum width. The numerical and WKB solutions of the Fokker—Planck equation give the same power law as the simulation for a longer time interval (the near asymptotics). However, the WKB solution being extended to very long times (the far asymptotics) shows a variation of the power exponent. That is, the power law may be assigned to the dispersion on relatively long but finite intervals. If such temporal intervals are spaced a long distance apart, a particular exponent may be assigned with a certain accuracy to

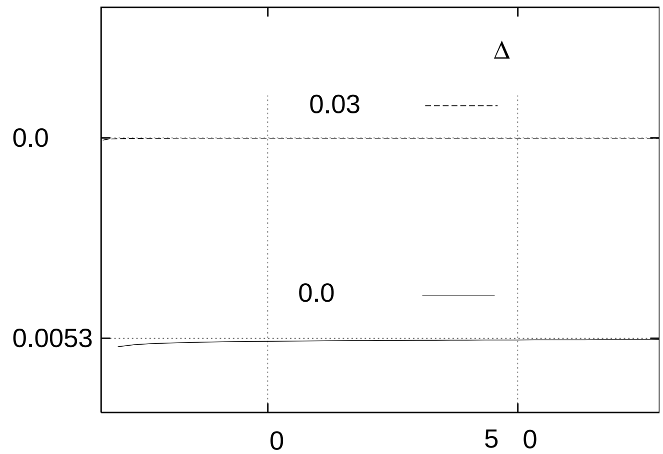


Fig. 6. $Y = \langle \Delta v^2 \rangle / t^p$ vs $t$. Power law for the narrow spectrum. $d = 0.04$, $\sigma = 0.01$, $v_0 = 1$, $Q = 2.5$. WKB solutions of the Fokker—Planck equation with $D(v)$ and $D(v, t)$

each of them. The illustration is the bottom curve in Fig. 4 and the curve in Fig. 5; they were obtained for a narrow initial distribution of particles at different time scales.

If the initial particle distribution in the velocity space is broad, then its near asymptotics is similar to the far asymptotics for the narrow initial distribution (the upper curve in Fig. 4 and the curve in Fig. 5). It will be easily understood, as far as it needs some time for a narrow distribution being transformed to a broad one.

This consideration may explain the difference in the dispersion behavior for moderate Kubo numbers given by the Fokker—Planck equation with the time-dependent and quasilinear diffusion coefficients not only at short (Fig. 2) but long times as well. The fast initial broadening of the distribution function shifts it for a far stage of the evolution comparatively with that governed by the quasilinear diffusion equation from the very beginning (Fig. 6).

## 5. Conclusions

The numerical simulation of the particle diffusion in the external field of random Langmuir waves has shown the fast initial growth of the velocity dispersion for moderate Kubo numbers. In this case, the evolution of the distribution function at short and long times is described by the Fokker—Planck equation with the time-dependent diffusion coefficient. The quasilinear diffusion coefficient, which is obtained by the integration over the field correlation time, may be put in correspondence to the time-dependent diffusion coefficient as its asymptotic value. The quasilinear diffusion equation does not recover the early stage of evolution of the distribution function and consequently distorts its behavior at long times.

One of the typical problems of non-equilibrium plasma is to find its stationary state. It is established by the balance between an increment of instability and a decrement due to dissipation processes. In particular, it could be a diffusion similar to that considered in this paper. One of the time scales in this problem is given by the inverted increment. The fact that a stationary state may be reached at much longer times does not yet give grounds for the averaging over a small scale of the field correlation time. Such an averaging may distort a description at a time scale of the inverted increment and leads to the wrong determination of a stationary state.


1. *Klimontovich Yu.L.* Kinetic Theory of Electromagnetic Processes. — Berlin: Springer, 1982.
2. *Bar'yakhtar V.G., Gonchar V.Yu., Shcertzer D., Yanovsky V.V.* // Ukr. Fiz. Zh. — 1999. — **44**. — P. 1169—1173.
3. *Zasenko V., Zagorodny A., Weiland J.* // Phys. Plasmas. — 2005. — **12**. — P. 062311.
4. *Doveil F., Gresillon D.*// Phys. Fluids. — 1982. — **25**. — P. 1396.
5. *Hirose A., Edler K., Bolser E., Ishihara O.*// IEEE Transaction on Plasma Science. — 2003. — **31**. — P. 3—6.
6. *Eijnden E. Vanden* // Phys. Plasmas. — 1997. — **4**. — P. 1486—1488.
7. *Orszag S.A., Kraichnan R.H.*// Phys. Fluids. — 1967. — **10**. — P. 1720—1736.


## ДИФУЗІЯ ЧАСТИНОК У ЗОВНІШНЬОМУ ПОЛІ ВИПАДКОВИХ ЛЕНГМЮРІВСЬКИХ ХВИЛЬ: МАЛІ ТА ВЕЛИКІ ЧАСИ

*В.І. Засенко, А.Г. Загородній, Я. Вейланд*


Р е з ю м е

Розглянуто часову поведінку дисперсії швидкості частинок, що рухаються у зовнішньому полі випадкових ленгмюрівських хвиль. Показано, що початковий етап еволюції дисперсії може впливати на степеневий закон її поведінки на великих часах. Результати числового моделювання відтворюються на основі узагальненого рівняння Фоккера—Планка, отриманого з мікроскопічного опису без усереднення за малими часовими масштабами.


## ДИФФУЗИЯ ЧАСТИЦ ВО ВНЕШНЕМ ПОЛЕ СЛУЧАЙНЫХ ЛЕНГМЮРОВСКИХ ВОЛН: МАЛЫЕ И БОЛЬШИЕ ВРЕМЕНА

*В.И. Засенко, А.Г. Загородний, Я. Вейланд*


Р е з ю м е

Рассмотрено временное поведение дисперсии скорости частиц, движущихся во внешнем поле случайных ленгмюровских волн. Показано, что начальный этап эволюции дисперсии может влиять на степенной закон ее поведения на больших временах. Результаты числового моделирования воспроизводятся на основе обобщенного уравнения Фоккера—Планка, полученного из микроскопического описания без усреднения по малому временному масштабу корреляции поля.


*To be continued in the 8D issue*