A PLATONIC MONOPOLE, A RELATED RIEMANN SURFACE, AND AN IDENTITY OF RAMANUJAN

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We develop the Ercolani—Sinha construction of SU(2) monopoles and make it effective for (a five-parameter family of centered) charge 3 monopoles. In particular, we show how to solve the transcendental constraints arising on the spectral curve. For a class of symmetric curves, the transcendental constraints become a number-theoretic problem, and a recently proven identity of Ramanujan provides a solution. The Ercolani—Sinha construction provides a gauge-transform of the Nahm data.

1. Introduction

Various disciplines of modern science including geometry, physics, chemistry, and biology have discovered objects possessing the symmetries of Platonic solids [1].[1]

Here, we will discuss the appearance of the Platonic solids in high-energy physics: namely, static magnetic monopoles of higher charge. These may arise as extended objects when coupling the Yang—Mills and Higgs fields. In a particular limit, the usual second-order field equations may be reduced to first-order equations, the Bogomolny equations. Even in this limit, the resulting equations are extremely complicated, and the only analytic solutions which have been found over the last decades are those possessing the symmetry of certain Platonic solids. In this note we describe some of our recent results [4] aimed at constructing the solutions without these restrictions of symmetry.

The construction we shall describe reveals an intriguing interplay between algebraic geometry and number theory, between ideas of Riemann and Ramanujan. Our starting point is the general θ-functional solution of Ercolani and Sinha [8] for the Nahm equation associated to an su(2) monopole of charge n. In particular, we will restrict our consideration to the case of monopoles of charge 3 and show that the Hitchin constraints [11] for the monopole curve can be solved. The case of a 3-monopole with tetrahedral symmetry appears in our development as a particular case. A recently proved identity of Ramanujan for hypergeometric functions plays a principal role in singling out the case of tetrahedral symmetry.

2. What is Monopole?

Although magnetic monopoles (magnetic charges) are absent in the classical Maxwell theory of electromagnetism, Dirac reconsidered the problem [7] in 1931.[2] He proved that the quantum mechanics of an electrically charged particle can be formulated in the presence of magnetic charges provided certain quantization conditions are satisfied. Dirac's theory leads however to a mathematical difficulty: the vector potential corresponding to a magnetic charge cannot be globally defined and has singularities. Because the gauge group of electromagnetism is U(1), the Dirac theory is called "Abelian".

In 1974, 't Hooft [22] and Polyakov [20] discovered that "non-Abelian" gauge theories (for example, theories

[1]The authors of [1] write in their introduction: " ... Plato was so captivated by the perfect forms of the five regular solids that in his dialogue *Timaeus* he associates them with what, at that time, were believed to be basic elements of the world, namely, earth, fire, water and ether. Kepler also attributed cosmic significance to the Platonic solids. In his book *Mysterium Cosmographicum* he presents a version of the solar system as nested Platonic solids, the radii of the intervening concentric spheres being related to the orbits of the planet. This model had the compelling feature that the existence of only five Platonic solids explained why there were only the six planets known at that time. The models of both Plato and Kepler are, of course, entirely false, but more promising applications have since come to light."

[2]The recent monograph [17] presents an excellent introduction to the subject as well a complete set of references.

with gauge group SU(k), $k > 1$) can admit magnetic monopole solutions without singularities. The gauge theory in question is modeled by the Landau—Ginsburg equation of superconductivity [10], where now the scalar fields are taken to be matrix functions with values in SU(k). The Lorentz-invariant Lagrangian density is given as

$$\mathcal{L} = \frac{1}{8}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}\mathrm{Tr}(D_\mu\Phi D^\mu\Phi) - \frac{\lambda}{4}(1-|\Phi|^2)^2, \tag{1}$$

where $|\Phi|^2 = -\frac{1}{2}\mathrm{Tr}\,\Phi^2$ is the (non-negative) squared norm of the Higgs field. The gauge potential and covariant derivative are

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j], \qquad D_i\Phi = \partial_i\Phi + [A_i, \Phi].$$

Here, both A_i and Φ are $k \times k$ skew-Hermitian zero-trace matrix functions of the spatial coordinates x_1, x_2, x_3.

When $k = 2$, 't Hooft and Plyakov independently found the static solution to this theory which solves the field equations

$$D_\mu D_\mu\Phi = -\lambda(1-|\Phi|^2)\Phi,$$

$$D_\mu F_{\mu\nu} = -[D_\nu\Phi, \Phi].$$

Assuming the spherical and reflection symmetries,

$$\Phi = \imath h(r)\frac{x^a}{r}\sigma^a,$$

$$A_i = -\imath\frac{1}{2}(1-k(r))\varepsilon_{ija}\frac{x^j}{r^2}\sigma^a,$$

where σ^a ($a = 1, 2, 3$) are the Pauli matrices, and $h(r)$ and $k(r)$ are functions just of the distance r from the origin. The functions $h(r)$ and $k(r)$ obey the ODEs,

$$\frac{\mathrm{d}^2 h}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{2}{r^2}k^2 h - \lambda(1-h^2)h,$$

$$\frac{\mathrm{d}^2 k}{\mathrm{d}r^2} = \frac{1}{r^2}(k^2-1)k + 4h^2k^2.$$

For $\lambda \neq 0$, these ODEs can only be solved numerically, but, at $\lambda = 0$, Prasad and Sommerfeld found the analytic solution [21]

$$h(r) = \coth 2r - \frac{1}{2r}, \quad k(r) = \frac{2r}{\sinh 2r}. \tag{2}$$

The energy for this special solution is 2π.

The understanding of the limit $\lambda = 0$ was achieved by Bogomolny. His argument is the following. The energy of a static field

$$E = -\frac{1}{4}\int \mathrm{Tr}(B_iB_i) + \mathrm{Tr}(D_i\Phi D_i\Phi)\mathrm{d}^3x,$$

where $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$ may be rewritten as

$$E = -\frac{1}{4}\int_{\mathbb{R}^3}\mathrm{Tr}(B_i + D_i\Phi)(B_i + D_i\Phi)\mathrm{d}^3x +$$

$$+\frac{1}{2}\int_{\mathbb{R}^3}\partial_i(\mathrm{Tr}(B_i\Phi))\mathrm{d}^3x.$$

This expression is obtained by using the identity

$$\partial_i(\mathrm{Tr}(B_i\Phi)) = \mathrm{Tr}((D_iB_i)\Phi) + \mathrm{Tr}(B_iD_i\Phi) = \mathrm{Tr}(B_iD_i\Phi),$$

where the Bianchi identity $D_iB_i = 0$ has been used. By the Stokes theorem, the second integral in the expression for E reduces to the integration over the sphere at infinity. Taking into account the quantization of the magnetic flux, one finds that this term equals $2\pi n$, where n is the monopole charge. For $n > 0$, there is a non-trivial bound achieved when $B_i = -D_i\Phi$. This last equation is known as the "Bogomolny equation" and the bound as the "Bogomolny bound". This equation is the starting point for the theory of static monopoles. Bogomolny showed that the Prasad—Sommerfeld monopole solution has monopole charge 1.

We observe that, in the scalar case $k = 1$, the equations are the standard equations of magnetostatics

$$\mathrm{grad}\,\Phi = \mathrm{curl}\,\mathrm{A}.$$

We are now in a position to define an su(k) charge n-monopole: this is a solution of the Bogomolny equation

$$\frac{1}{2}\varepsilon_{ijk}F_{jk} = D_i\Phi, \tag{3}$$

with asymptotic behaviour

$$|\Phi| = 1 - \frac{n}{r} + O\left(\frac{1}{r}\right), \quad r \to \infty, \tag{4}$$

where $|\Phi|^2 = -\frac{1}{2}\mathrm{Tr}\Phi^2$ and the positive integer n is the magnetic charge or monopole number. We shall focus throughout on the case $k = 2$, where the gauge group is SU(2).

3. Nahm Equations and Hitchin Constraints

Finding the closed-form solutions to the Bogomolny equations for monopole charge $n > 1$ meets many mathematical difficulties. At present, the only known way to write down analytic solutions to the Bogomolny equation in terms of standard functions is based on the Nahm transformation [18].

3.1. Nahm Data

The Nahm transformation reduces the Bogomolny equation with magnetic charge n to the finding of $n \times n$ matrices T_1, T_2, T_3 $\in$ SU(2) depending on a real parameter $s \in [0, 2]$ subject to the following conditions

— T_i satisfy Nahm's equations

$$\frac{\mathrm{d}T_i}{\mathrm{d}s} = \frac{1}{2}\sum_{j,k=1} \varepsilon_{ijk}[T_j, T_k], \tag{5}$$

where ε is the completely antisymmetric tensor of the third rank;

— $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0$ and $s = 2$, the residues of which form the irreducible n-dimensional representation of SU(2);

— $T_i(s) = -T_i^\dagger(s)$ and $T_i(s) = T_i^\dagger(2 - s)$.

Nahm's equations admits the Lax formulation [11]. Upon setting

$$A_{-1} = T_1 + iT_2, \quad A_0 = -2iT_3, \quad A_1 = T_1 - iT_2,$$

$$A = A_{-1}\zeta^{-1} + A_0 + A_1\zeta, \quad M = \frac{1}{2}A_0 + A_1\zeta,$$

then

$$\frac{dA}{ds} = [A, M], \quad \text{orequivalently} \quad [\frac{d}{ds} + M, A] = 0. \tag{6}$$

Nahm's equations (5) describe a linear flow on a complex torus, which is the Jacobian of an algebraic curve. This algebraic curve is, in fact, the monopole spectral curve $\mathcal{C}$ which may be explicitly read off from the Lax equation

$$P(\eta, \zeta) = \det\big(\eta + (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2\big) = 0. \tag{7}$$

If solutions of Nahm's equations are known, then solutions of the Bogomolny equation may be reconstructed as follows. Introduce the operator

$$\Delta = \left(\imath\frac{\mathrm{d}}{\mathrm{d}z} + x_0\right) + e_j(x_j + \imath T_j), \tag{8}$$

where e_j ($j = 1, 2, 3$) are the imaginary quaternions and $z = s - 1$. Let $\boldsymbol{v}$, an n-component vector of quaternions, be the unique solution to

$$\Delta^+\boldsymbol{v} = 0$$

and normalized by the condition

$$\int_{-1}^{1} \boldsymbol{v}^+\boldsymbol{v}\mathrm{d}z = 1.$$

Here, the superscript + means both the transposed and quaternionic conjugate. Note that $\boldsymbol{v}$ has the form

$$\boldsymbol{v}(z, x_0, x_1, x_2, x_3) = \exp\{\imath x_0 z\}\boldsymbol{N}(z, x_1, x_2, x_3).$$

Then a solution of the Bogomolny equation (3) with asymptotic (4) is given by

$$\begin{aligned} \Phi(x_1, x_2, x_3) &= \int_{-1}^{1} z\boldsymbol{N}^+\boldsymbol{N}\mathrm{d}z, \\ A_i(x_1, x_2, x_3) &= \int_{-1}^{1} \boldsymbol{N}^+\partial_i\boldsymbol{N}\mathrm{d}z. \end{aligned} \tag{9}$$

We remark that the reconstruction procedure has thus far only be carried out numerically.

3.2. Ercolani—Sinha construction

Ercolani and Sinha showed how one may utilize the Lax representation of Nahm's equations to reduce its integration to solving a spectral problem associated with the curve $\mathcal{C}$. Denote $z = s - 1$, then $z \in [-1, 1]$ for $s \in [0, 2]$, and we have that[3]

$$A_0(z) = A_0^\dagger(z), \quad A_1(z) = -A_{-1}^\dagger(z),$$

[3]The matrices of Ercolani—Sinha and Nahm are related by $T_i^{\mathrm{ES}}(z) = T_i^{\mathrm{Nahm}}(z + 1)$, whence $T_i^{\mathrm{ES}}(z) = T_i^{\mathrm{Nahm}}(z + 1) = T_i^{\mathrm{Nahm}\,t}(2 - [z + 1]) = T_i^{\mathrm{Nahm}\,t}(1 - z) = T_i^{\mathrm{ES}\,t}(-z)$.

$$A_\alpha(z) = A_\alpha^t(-z), \quad \alpha = 1,2,3. \tag{10}$$

By considering the differential operator

$$\frac{d}{dz} + M(z) = \frac{d}{dz} + \frac{1}{2}A_0(z) + A_1(z)\zeta,$$

related to the Lax equation (6), Ercolani and Sinha showed how the spectral theory of this equation enables the integration of the Lax equation. The z-dependence of the term $A_1(z)$ means that

$$\left(\frac{d}{dz} + \frac{1}{2}A_0(z)\right)\varphi = -\zeta A_1(z)\varphi$$

is not of standard eigenvalue form. By considering the gauge transformation

$$Q_\alpha(z) = C^{-1}(z)A_\alpha(z)C(z), \quad \varphi = C(z)\Psi,$$

they obtain the standard eigenvalue equation with the "potential" $\mathcal{U}(z) = Q_0(z)$

$$\left(\frac{d}{dz} + \mathcal{U}(z)\right)\Psi = -\zeta Q_1(0)\Psi, \tag{11}$$

if and only if $C(z)$ satisfies

$$C(z)^{-1}C'(z) = \frac{1}{2}\mathcal{U}(z),$$

equivalently $\left(\frac{d}{dz} + \frac{1}{2}\mathcal{U}(z)\right)C^{-1} = 0. \quad (12)$

The gauge transform was chosen so that $Q_1(z) = A_1(0) = Q_1(0)$. From (10), we see that it is a symmetric matrix, and (by an overall constant gauge transformation) we may assume that it is a diagonal one:

$$A_1(0) = Q_1(0) = \mathrm{diag}(\rho_1, \dots, \rho_n). \tag{13}$$

We see from (11) that ρ_j (which may be assumed distinct) correspond to the roots of $P(\eta,\zeta)/\zeta^{2n}$ near $\zeta = \infty$,

$$\frac{P(\eta,\zeta)}{\zeta^{2n}} \sim \prod_{j=1}^{n}\left(\frac{\eta}{\zeta^2} - \rho_j\right). \tag{14}$$

As a consequence, we find that, at ∞_j,

$$\frac{\eta}{\zeta} \sim \rho_j\,\zeta, \qquad \mathrm{d}\left(\frac{\eta}{\zeta}\right) \sim \rho_j\,\mathrm{d}\zeta = \left(-\frac{\rho_j}{t^2} + O(1)\right)\mathrm{d}t, \tag{15}$$

where $t = 1/\zeta$ is a local coordinate.

The integration of (11) is now carried out using the idea of a Baker—Akhiezer function [2] pioneered by Krichever in the context of integrable systems [15]. In the present monopole setting, Ercolani and Sinha [8] found θ-functional expressions for the wave function $\boldsymbol{\Psi}(z,P) = (\Psi_j(z,P), \dots, \Psi_j(z,P))^T$ and potential $\mathcal{U} = (\mathcal{U}_{jl})_{j,l=1,\dots,n}$ as follows.

Let $\mathcal{C}$ be an algebraic curve of genus g equipped with canonical homology basis $(\mathfrak{a}_1,\dots,\mathfrak{a}_g,\mathfrak{b}_1,\dots,\mathfrak{b}_g)$ in $H_1(\mathcal{C},\mathbb{Z})$. Let $\mathrm{d}\boldsymbol{v}$ be the vector of holomorphic differentials normalized by

$$\oint_{\mathfrak{a}_i} \mathrm{d}v_j = \delta_{ij}, \quad i,j = 1,\dots,g.$$

The Riemann period matrix is then

$$\tau = (\tau_{ij})_{i,j=1,\dots,g}, \quad \tau_{ij} = \oint_{\mathfrak{b}_i} \mathrm{d}v_j.$$

Ercolani and Sinha established the following formulae

$$\Psi_j(z,P) = g_j(P)\exp\left\{z\int_{P_0}^{P}\Omega_\infty - \nu_j z\right\}\Theta(z,P),$$

$$\mathcal{U}_{jl}(z) = -(\rho_j - \rho_l)c_{jl}\exp\{z(\nu_i - \nu_j)\}\,\Theta(z,P_l),$$

where

$$\Theta(z,P) = \frac{\theta\left(\int_{P_0}^{P}\mathrm{d}\boldsymbol{v} - \boldsymbol{U}z - \int_{gP_0}^{\mathcal{D}_j}\mathrm{d}\boldsymbol{v} - \boldsymbol{K}\right)}{\theta\left(\int_{P_0}^{P}\mathrm{d}\boldsymbol{v} - \int_{gP_0}^{\mathcal{D}_j}\mathrm{d}\boldsymbol{v} - \boldsymbol{K}\right)} \times$$

$$\times\frac{\theta\left(\int_{P_0}^{\infty_j}\mathrm{d}\boldsymbol{v} - \int_{gP_0}^{\mathcal{D}_j}\mathrm{d}\boldsymbol{v} - \boldsymbol{K}\right)}{\theta\left(\int_{P_0}^{\infty_j}\mathrm{d}\boldsymbol{v} - \int_{gP_0}^{\mathcal{D}_j}\mathrm{d}\boldsymbol{v} - \boldsymbol{U}z - \boldsymbol{K}\right)}.$$

In these formulae, $\theta(\boldsymbol{z})$ is the canonical Riemann θ-function,

$$\theta(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^g}\exp\left\{\imath\pi\boldsymbol{k}^T\tau\boldsymbol{k} + 2\imath\pi\boldsymbol{k}^T\boldsymbol{z}\right\},$$

with a Riemann period matrix τ built from the curve $\mathcal{C}$, $\boldsymbol{K}$ is the vector of Riemann constants, and $\boldsymbol{U} = (U_1,\dots,U_g)$ is the "winding vector", that is, the vector of $\mathfrak{b}$-periods of the normalized differential $\Omega_\infty(P)$ of the second kind with unique second-order poles on all sheets at infinity,

$$U_j = \oint_{\mathfrak{b}_j} \Omega_\infty(P), \quad \oint_{\mathfrak{a}_j} \Omega_\infty(P) = 0, \quad j = 1, \dots, g,$$

$$\Omega_\infty(P) = -\left(\frac{\rho_i}{t^2} + O(1)\right) \mathrm{d}t, \quad i = 1 \dots, n.$$

Finally, $G_j(P)$ are meromorphic functions normalized as $G_j(\infty_j) = 1$ $(j = 1, \dots, k)$ and with divisor

$$(G_j) = \mathcal{D}_j + \infty_1 + \dots \widehat{\infty_j} + \dots + \infty_k - \sum_{l=1}^{g+k-1} Q_l.$$

The divisor of poles $\sum_{l=1}^{g+k-1} Q_l$ is chosen in such a way that

$$c_{i,j} = -c_{j,i}, \quad i, j = 1, \dots, k,$$

$$c_{i,j} = \lim_{P \to \infty_j} \zeta G_j(P), \quad P = (\zeta, \eta) \in \mathcal{C}.$$

Although this θ-functional integration of (11) may be carried out, there still remains the major problem of satisfying the boundary conditions given in the definition of n-monopole. This leads to strong constraints on the underlying algebraic curve and it is not clear if such smooth curves indeed exist.

3.3. Hitchin data

The algebro-geometric formulation of these constraints was given by Hitchin [11,12].

The monopole curve

$$\eta^n + \eta^{n-1} a_1(\zeta) + \dots + \dots \eta^r a_{n-r}(\zeta) + \dots +$$

$$+\eta a_{n-1}(\zeta) + a_n(\zeta) = 0., \tag{16}$$

where $a_r(\zeta)$ are polynomials of maximum degree $2r$, should satisfy:

— H1. Polynomials $a_r(\zeta)$ are of the form

$$a_r(\zeta) = \chi_r \left[\prod_{l=1}^r \left(\frac{\overline{\alpha}_l}{\alpha_l}\right)^{1/2}\right] \prod_{k=1}^r (\zeta - \alpha_r)\left(\zeta - \frac{1}{\overline{\alpha}_r}\right),$$

$\alpha_r \in \mathbb{C}, \ \chi \in \mathbb{R}.$

The genus of the curve (16) is $(n-1)^2$.

— H2. The "winding vector" appearing in the above θ-formulae is a half-period

$$\boldsymbol{U} = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\tau \boldsymbol{m}. \tag{17}$$

We call two integer vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ appearing in formula (17) as the Ercolani—Sinha vectors or, for short, ES-vectors. This is an equivalent form of Hitchin's constraint on the triviality of a certain line-bundle (see [8] and [4]).

— H3. The function $F(z) = \theta(\boldsymbol{U}z + \boldsymbol{K})$ has zeros only at $z = 0$ and $z = 2$ when $0 \le z \le 2$, where θ is the θ-function of the curve, $\boldsymbol{K}$ is the vector of Riemann constants, and $\boldsymbol{U}$ is the winding vector defined in H2.

4. Symmetric 3-Monopoles

In what follows, we restrict our consideration to monopoles of charge 3 for which the algebraic curve (of genus four) takes the form

$$\eta^3 + \chi(\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3) \times$$

$$\times \left(\zeta + \frac{1}{\overline{\lambda}_1}\right)\left(\zeta + \frac{1}{\overline{\lambda}_2}\right)\left(\zeta + \frac{1}{\overline{\lambda}_3}\right) = 0. \tag{18}$$

We formulate first various necessary results for the more general curve of genus 4 possessing the $\mathbb{Z}_3$ symmetry

$$\mathcal{C}: \quad w^3 = (z - \lambda_1)\cdots(z - \lambda_6), \tag{19}$$

where the six points $\lambda_i \in \mathbb{C}$ are assumed distinct and ordered according to the rule $\arg(\lambda_1) < \arg(\lambda_2) < \dots \arg(\lambda_6)$. Let $\mathcal{R}$ be the automorphism of $\mathcal{C}$ defined by

$$\mathcal{R}: \ (z, w) \to (z, \rho w), \qquad \rho = \exp\{2\imath\pi/3\}.$$

Introduce the cut structure and homology basis $(\mathfrak{a}_1, \dots, \mathfrak{a}_4; \mathfrak{b}_1, \dots, \mathfrak{b}_4)$ of $H_1(\mathcal{C}, \mathbb{Z})$ as shown in Fig. 1.[4] This has pairing $\mathfrak{a}_i \circ \mathfrak{a}_j = \mathfrak{b}_i \circ \mathfrak{b}_j = 0$ and $\mathfrak{a}_i \circ \mathfrak{b}_j = -\mathfrak{b}_i \circ \mathfrak{a}_j = \delta_{ij}$. In the homology basis introduced, we have

$$\mathcal{R}(\mathfrak{b}_i) = \mathfrak{a}_i, \quad i = 1, 2, 3, \quad \mathcal{R}(\mathfrak{b}_4) = -\mathfrak{a}_4. \tag{20}$$

[4] We recall here a paragraph from [9]: "According to Betti, Riemann said he got the idea of *cuts* from conversation with Gauss (1777—1855) [[19], p.90]. Letters of Klein and Schering attest to Gauss' influence on Riemann's theory of hypergeometric series. This influence came partly from Gauss papers. Still, it is striking to consider the over 70 year old Gauss sketching plans of such an etherial construction to very young Riemann."

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The curve $\mathcal{C}$ admits four independent holomorphic differentials,

$$\mathrm{d}u_1 = \frac{\mathrm{d}z}{w},\ \mathrm{d}u_2 = \frac{\mathrm{d}z}{w^2},\ \mathrm{d}u_3 = \frac{z\mathrm{d}z}{w^2},\ \mathrm{d}u_4 = \frac{z^2\mathrm{d}z}{w^2},$$

but the first of these is the most essential when describing the period matrix of the curve $\mathcal{C}$. Denote

$$x_l = \oint_{\mathfrak{a}_l} \mathrm{d}u_1, \quad l = 1,\dots 4, \qquad \boldsymbol{x} = (x_1,\dots,x_4)^T.$$

The following statement was proved in [16,23].

Proposition 4.1. [Wellstein, 1899; Matsumoto, 2000]. *Let $\mathcal{C}$ be the triple covering of $\mathbb{P}^1$ with six distinct points $\lambda_1,\dots,\lambda_6$,*

$$w^3 = \prod_{i=1}^{6}(z-\lambda_i). \tag{21}$$

Then the Riemann period matrix is of the form

$$\tau = \rho\left(H - (1-\rho)\frac{\boldsymbol{x}\boldsymbol{x}^T}{\boldsymbol{x}^T H \boldsymbol{x}}\right), \tag{22}$$

where $H = \mathrm{diag}(1,1,1,-1)$ and $\rho = \exp\{2\pi\imath/3\}$ is a root of unity. Then τ is positive definite if and only if

$$\bar{\boldsymbol{x}}^T H \boldsymbol{x} < 0. \tag{23}$$

The branch points can be expressed via θ-constants. Following Matsumoto [16] (see also [6]), we introduce the set of characteristics

$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}, \qquad \boldsymbol{b} = -\boldsymbol{a}H, \quad a_i \in \left\{\frac{1}{6},\frac{3}{6},\frac{5}{6}\right\} \tag{24}$$

and denote $\theta\begin{bmatrix} \boldsymbol{a} \\ -H\boldsymbol{b} \end{bmatrix}(\tau) = \theta\{6\boldsymbol{a}\}$. Here, $\theta\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}(\tau)$ denotes θ-constant

$$\theta\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}(\tau) =$$

$$= \sum_{\boldsymbol{k}\in\mathbb{Z}^g} \exp\left\{\imath\pi(\boldsymbol{k}+\boldsymbol{a})^T\tau(\boldsymbol{k}+\boldsymbol{a}) + 2\imath\pi(\boldsymbol{k}+\boldsymbol{a})^T\boldsymbol{b}\right\}.$$

Proposition 4.2. [Diez 1991, Matsumoto 2000]. *Consider the curve (19) and transform it by a Möbius transformation to the form*

$$\mathcal{C}(\lambda):\ w^3 = z(z-1)(z-\Lambda_1)(z-\Lambda_2)(z-\Lambda_2). \tag{25}$$

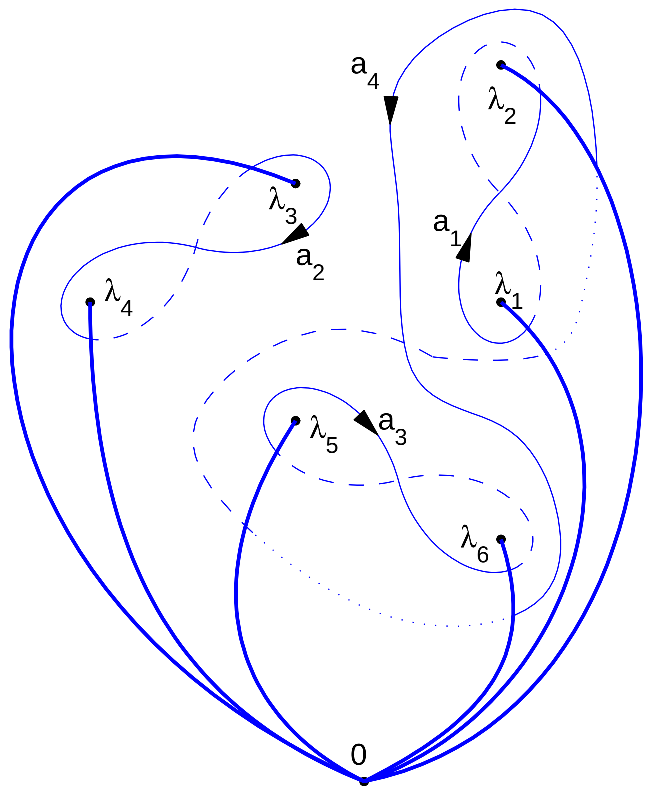

Fig. 1. Cut structure and $\mathfrak{a}$-cycles. Because of the action of the automorphism $\mathcal{R}$ given in (20) the $\mathfrak{b}$-cycles are constructed from the same arcs as the $\mathfrak{a}$-cycles but are placed on other sheets

Let τ be the period matrix of (19) given in Proposition 4.1. Then

$$\Lambda_1 = \left(\frac{\theta\{3,3,3,5\}}{\theta\{1,1,3,3\}}\right)^3, \quad \Lambda_2 = -\left(\frac{\theta\{1,5,3,3\}}{\theta\{1,1,5,5\}}\right)^3,$$

$$\Lambda_3 = -\left(\frac{\theta\{1,1,3,3\}}{\theta\{5,1,1,1\}}\right)^3. \tag{26}$$

These formulae enable us to define a curve up to a Möbius transformation given a period matrix.

Now return to curve (18), whose branch points are supposed to be such that $\lambda_1 \in \mathbb{R}$ and $\overline{\lambda_2} = \lambda_3$. The following theorem is proved in [4].

Theorem 4.3. [Braden and Enolskii, 2005]. *Conditions H1 and H2 are simultaneously satisfied for the curve $\mathcal{C}$ defined by ES-vectors of the form*

$$\begin{pmatrix} \boldsymbol{n}^T \\ \boldsymbol{m}^T \end{pmatrix} = \begin{pmatrix} n_1 & m_1 - n_1 & -m_1 & n_4 \\ m_1 & -n_1 & n_1 - m_1 & m_4 \end{pmatrix}, \tag{27}$$

where n_4, m_4 are integer solutions to the equation

$$3n_1m_1 - 3m_1^2 - 3n_1^2 - n_4m_4 - 2m_4n_1 +$$

$$+2n_4n_1 - 4n_4m_1 - n_4^2 - 2m_1m_4 - m_4^2 = 0, \qquad (28)$$

satisfying to the inequality (lying inside hyperboloid)

$$3(n_1^2 - n_1m_1 + m_1^2) - n_4^2 - n_4m_4 - m_4^2 < 0. \qquad (29)$$

The period matrix τ is defined in terms of the fractions

$$\frac{x_i}{x_1} = \frac{n_i + \rho^2 m_i}{n_1 + \rho^2 m_1},\ i = 2,3, \quad \frac{x_4}{x_1} = \frac{-n_4 + \rho^2 m_4}{n_1 + \rho^2 m_1}. \qquad (30)$$

This theorem shows that the whole construction of a monopole solution of the Bogomolny equation associated with curve (18) is governed by the ES-vectors. In this case, the set of parameters, on which the solution depends, is a set of integers. Each ES-vector forms a half-integer θ-characteristic we denote as $\begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{m}^T \end{bmatrix}$.

Let us analyze some solutions of (28). One can check that

$$n_4 = 2n_1 - m_1, \qquad m_4 = -3n_1 \qquad (31)$$

is a solution (amongst others) of (28). For solution (31), we have

Proposition 4.4. [Braden and Enolskii, 2005]. *The ES-vectors corresponding to a monopole curve of the form*

$$\eta^3 + \chi(\xi^6 + b\xi^3 - 1) = 0 \qquad (32)$$

are given by

$$\begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{pmatrix} = \begin{pmatrix} n_1 & m_1 - n_1 & -m_1 & n_1 - m_1 \\ m_1 & -n_1 & n_1 - m_1 & -3n_1 \end{pmatrix}. \qquad (33)$$

Moreover, the equation (with respect to t) involving the standard hypergeometric function $F(a, b; c; t)$,

$$\frac{2n_1 - m_1}{m_1 + n_1} = \frac{F\left(\frac{1}{3}, \frac{2}{3}; 1; t\right)}{F\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t\right)}, \quad n_1, m_1 \in \mathbb{N}, \qquad (34)$$

has algebraic solution

$$t = \frac{-b + \sqrt{b^2 + 4}}{2\sqrt{b^2 + 4}}.$$

Here, χ is obtained by

$$\chi^{\frac{1}{3}} = -(n_1 + m_1)\, \frac{2\pi}{3\sqrt{3}}\, \frac{\alpha}{(1 + \alpha^6)^{\frac{1}{3}}}\, F\left(\frac{1}{3}, \frac{2}{3}; 1, t\right) \qquad (35)$$

with $\alpha^6 = t/(1-t)$.

This theorem contains an amazing fact: the transcendental equation (34) is a disguised form of an algebraic equation. Moreover, the comparison with the known results [13] for tetrahedral monopoles leads one to the conclusion that Eq. (34) should be satisfied at $b = 5\sqrt{2}$ with ES-vectors given by $n_1 = m_1 = 1$. For this value of the parameter b, the polynomial $\zeta^6 - b\zeta^3 - 1$ appearing here is that given by Klein as the polynomial invariant under the action of the tetrahedral group [14]. The resulting monopole curve with the evaluated constant χ is reduced after a rotation to the form

$$\eta^3 + \imath \frac{\Gamma\left(\frac{1}{3}\right)^3 \Gamma\left(\frac{1}{6}\right)^3}{48\sqrt{3}\pi^{3/2}} \zeta(\zeta^4 - 1) = 0. \qquad (36)$$

The proof of this connection between (34) and the tetrahedral monopole and, more generally, the solution of (34) are given using a recently proven result of Ramanujan which we now describe.

5. Ramanujan Identity

Let n be a natural number. A modular equation of degree n and signature r ($r = 2, 3, 4, 6$) is a relation between α and β of the form

$$n\frac{F(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha)}{F(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha)} = \frac{F(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta)}{F(\frac{1}{r}, \frac{r-1}{r}; 1; \beta)}. \qquad (37)$$

When $r = 2$, we have the complete elliptic integral $K(k) = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}; 1; k^2)$, and (37) yields the usual modular relations. By interchanging $\alpha \leftrightarrow \beta$ we may interchange $n \leftrightarrow 1/n$. This, together with the iteration of these modular equations, means we may obtain relations with n being an arbitrary rational number. Our equation (34) is precisely of this form for the signature $r = 3$ and with α to be at least $1/2$.

In his second notebook, Ramanujan presented the results pertaining to these generalized modular equations and various theta function identities.[5] For example, if

[5] In the abstract to [3], the authors write: “In his famous paper on modular equations and approximations to π Ramanujan offers several series representations for $1/\pi$, which he claims are derived from “corresponding theories” in which the classical base q is replaced by three other bases. The formulas for $1/\pi$ were only recently proved by J.M. and P.B.Borwein in 1987, but these “corresponding theories” have never been heretofore developed. However, on six pages of his notebooks, Ramanujan gives approximately 50 results without proofs in these theories. The purpose of this paper is to prove all of these claims ... ”

$n = 2$ in the signature $r = 3$, then α and β are related by

$$(\alpha\beta)^{\frac{1}{3}} + ((1-\alpha)(1-\beta))^{\frac{1}{3}} = 1. \quad (38)$$

He also states that (for $0 \leq p < 1$)

$$(1+p+p^2)\, F\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{p^3(2+p)}{1+2p}\right) =$$

$$= \sqrt{1+2p}\, F\left(\frac{1}{3}, \frac{2}{3}; 1, \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right). \quad (39)$$

Ramanujan's results were derived in [3]. An account of the history and the associated theory of these equations may be found in the last volume dedicated to Ramanujan's notebooks [5].

One finds that Eq. (37) is solved for $n = 2$ for $t = \frac{1}{2} + \frac{5\sqrt{3}}{18}$. With $t = \frac{1}{2} - \frac{5\sqrt{3}}{18}$ (for which $b = 5\sqrt{2}$), we get $n = 1/2$, the case relevant for the tetrahedral monopole. Expressions analogous to (38) are known for $n = 3$, 5, 7 and 11 [3, 7.13, 7.17, 7.24, 2.28, respectively]. Thus, we may solve (34) by iteration for rational numbers, whose numerator and denominator have these as their only factors. A theory exists then for solving (37), and this has been worked out for various low primes.

6. Conclusion

We have considered the curves of genus 4 possessing the $\mathbb{Z}_3$-symmetry and shown that its parameters can be fixed in such a way as to satisfy the Hitchin conditions $H1$ and $H2$. As for the third condition, we are only able (at present) to check it numerically. In Fig. 2, the imaginary and real parts of the function $F(z)$,

$$F(z) = \theta(\boldsymbol{U}z + \boldsymbol{K}), \qquad z \in [0, 2],$$

are given for the ES-vectors corresponding to a tetrahedral monopole.

We have proven here that the existence of a tetrahedral monopole can be deduced from the θ-functional formula written for the more general curve by successively restricting the moduli of the curve to satisfy conditions $H1-H3$. We have not addressed here the question of whether there exist values of the ES-vectors different from the tetrahedral case for which a 3-monopole exists, but we believe that the method developed allows us to answer this question. That will be done in the forthcomming publication [4].

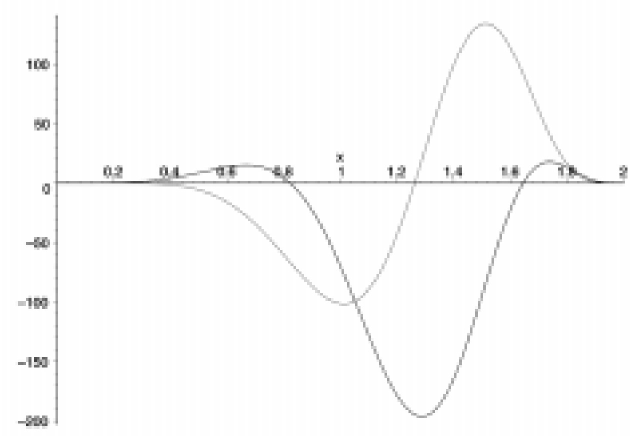

Fig. 2. Real and imaginary parts of the function $F(z)$ at $z \in [0, 2]$. That's seen that Re$F(z)$ and Im$F(z)$ vanish simultaneously only at the points $z = 0$ and $z = 2$

We also believe that it is possible to extend our analysis by the methods developed to further cases such as the 4-monopole with octahedral symmetry, whose existence is proved in [13] and related curve is of the form

$$\eta^4 + \frac{3\Gamma\left(\frac{1}{4}\right)^8}{64\pi^2}(\zeta^8 + 14\zeta^4 + 1) = 0 \quad (40)$$

and others.

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ПЛАТОНІВ МОНОПОЛЬ, ЗВ'ЯЗАНА З НИМ ПОВЕРХНЯ РІМАНА ТА ТОТОЖНІСТЬ РАМАНУДЖАНА

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Р е з ю м е

Розроблено конструкцію Ерколані—Сінха для SU(2)-монополів і застосовано її до п'ятипараметричної сім'ї монополів із зарядом три. Зокрема показано, як розв'язати трансцендентні зв'язки на спектральній кривій. Для класу симетричних кривих трансцендентні зв'язки постають як теоретико-числова проблема, розв'язок якої дає нещодавно доведена тотожність Рамануджана. Конструкція Ерколані—Сінха приводить до калібрувального перетворення даних Нама.

ПЛАТОНОВ МОНОПОЛЬ, СВЯЗАННАЯ С НИМ ПОВЕРХНОСТЬ РИМАНА И ТОЖДЕСТВО РАМАНУДЖАНА

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Р е з ю м е

Разработана конструкция Эрколани—Синха для SU(2)-монополей, которая применена к пятипараметрическому семейству монополей с зарядом три. В частности показано, как разрешить трансцендентные связи на спектральной кривой. Для класса симметричных кривых трансцендентные связи сводятся к теоретико-числовой проблеме, решение которой дает недавно доказанное тождество Рамануджана. Конструкция Эрколани—Синха приводит к калибровочному преобразованию данных Нама.