CHAOTIC BEHAVIOR OF A COSMIC STRING IN THE GRAVITATIONAL FIELD OF A LORENTZIAN "WORMHOLE"

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The dynamics of a closed cosmic string in the space-time of a "wormhole" has been considered. Numerical solutions of the equations of motion of a string have been obtained, and the analysis of those solutions has been carried out for various initial data. It is shown that the scattering of the string by a "wormhole" has a chaotic character.

Cosmic strings are one-dimensional regions with concentrated energy density. They can emerge in a natural way owing to a spontaneous symmetry violation at phase transitions in the course of the universe evolution [1]. In the framework of different models of the Great Unification theory, strings manifest themselves as topological defects, being therefore stable formations. Cosmic strings attract the enhanced attention of scientists from the cosmological point of view, because they have such characteristics that allow them to be considered as nuclei of those density non-homogeneities at the early stages of the universe which govern the further formation of galaxies and galaxy clusters [1, 2].

Cosmic strings move, as a rule, with relativistic velocities and oscillate, with the strings of a dimension R possessing the characteristic oscillation frequency $\omega \propto R^{-1}$ (hereafter, the light velocity $c = 1$) [3]. In general, self-intersections of a string occur inevitably in the course of its oscillations, so that these intersections of string's elements may result in the isolation of a certain portion of the string and the formation of closed strings of smaller dimensions. The described gravitational decay of strings and their fragmentation result in that the minimal size of cosmic strings, which have arisen at the phase transition moment and survived until now, does not exceed the value $R_{\min} \approx 0.3$ Mps [1, 3].

A nonlinearity of the dynamic system (a cosmic string) makes it possible that essentially new dynamic modes, besides the well-studied regular solutions of the equations of motion, may exist under certain conditions. The motion of a cosmic string in those modes does not differ from a random one, although no external source of randomness is present [4]. Using the terms "chaotic" or "stochastic" instead of "random", one may assert that these concepts adequately reflect some fundamental intrinsic properties of both a cosmic string and any other nonlinear dynamic system, which constitutes an important and interesting subject for studying.

The mechanism that provides the existence of a chaotic mode in strictly deterministic systems is a local instability, resulting in that the initially close trajectories diverge exponentially in the phase space. Speaking more strictly, the "stochasticity" can be understood as the appearance of statistical properties in a system owing to its local instability.

In this work, the motion of a closed cosmic string in the gravitational field of a Lorentzian wormhole has been analyzed and this motion has been shown to have a chaotic behavior.

The string, during its evolution, sweeps up the world surface $x^\mu(\tau, \sigma)$ parameterized by the time-like parameter τ and the space-like parameter σ. The string equations of motion can be obtained making use of the well-known action functional

$$S = \frac{1}{2}\int d\tau d\sigma \sqrt{g} g^{ik} x^\mu_{,i} x^\nu_{,k} G_{\mu\nu}(x), \qquad (1)$$

where $x^\mu_{,i} = \left(x^\mu_{,\tau}, x^\mu_{,\sigma}\right) \equiv \left(\frac{\partial x^\mu}{\partial \tau}, \frac{\partial x^\mu}{\partial \sigma}\right)$, and $G_{\mu\nu}(x)$ is the metric tensor of the external space-time.

Varying Eq. (1) with respect to x^μ, the string equation of motion is

$$\frac{1}{\sqrt{g}}\left(\sqrt{g}g^{ik}x^\mu_{,k}\right)_{,i}+g^{ik}\Gamma^\mu_{\rho\nu}x^\rho_{,i}x^\nu_{,k}=0, \qquad (2)$$

where $\Gamma^\mu_{\rho\nu}$ are the Christoffel symbols determined by the metric tensor $G_{\mu\nu}(x)$. Varying Eq. (1) with respect to g^{ik} brings us to the equation

$$G_{\mu\nu}(x)\left[x^\mu_{,i}x^\nu_{,k}-\frac{1}{2}g_{ik}\ x^\mu_{,l}x^\nu_{,m}\ g^{lm}\right]=0, \qquad (3)$$

whence it follows immediately that the metric g_{ik} of the world sheet of the string coincides with that induced by the metric tensor of the external space-time. A reparametrizational invariance of action (1) allows the gauge to be fixed. It is convenient here to take advantage of an orthogonal gauge which is defined by the following conditions [1]:

$$G_{\mu\nu}(x)x^\mu_{,\tau}x^\nu_{,\sigma}=0, \qquad (4)$$

$$G_{\mu\nu}(x)(x^\mu_{,\tau}x^\nu_{,\tau}+x^\mu_{,\sigma}x^\nu_{,\sigma})=0. \qquad (5)$$

In this gauge, the string equations of motion become considerably simpler:

$$x^\mu_{,\tau\tau}-x^\mu_{,\sigma\sigma}+\Gamma^\mu_{\nu\rho}\left(x^\rho_{,\tau}x^\nu_{,\tau}-x^\rho_{,\sigma}x^\nu_{,\sigma}\right)=0. \qquad (6)$$

For a closed string, Eqs. (4)–(6) should be supplemented with periodical conditions $x^\mu(\tau,\sigma)=x^\mu(\tau,\sigma+2\pi)$.

Consider the dynamics of a cosmic string in the gravitational field of a Lorentzian wormhole, the metric of which is defined by the expression [5]

$$dS^2=-dt^2+dl^2+\left(b_0^2+l^2\right)\left(d\theta^2+\sin^2\theta\ d\phi^2\right), \qquad (7)$$

where b_0 if the radius of the "wormhole"'s mouth. In this case, the string equation of motion (6) and the kinematic constraint equations (4) and (5) become of the form

$$t_{,\tau\tau}-t_{,\sigma\sigma}=0, \qquad (8)$$

$$l_{,\tau\tau}-l_{,\sigma\sigma}-l\left(\theta^2_{,\tau}-\theta^2_{,\sigma}\right)-l\sin^2\theta\left(\phi^2_{,\tau}-\phi^2_{,\sigma}\right)=0, \qquad (9)$$

$$\theta_{,\tau\tau}-\theta_{,\sigma\sigma}+\frac{2l}{b_0^2+l^2}\left(\theta_{,\tau}l_{,\tau}-\theta_{,\sigma}l_{,\sigma}\right)-$$

$$-\sin\theta\cos\theta\left(\phi^2_{,\tau}-\phi^2_{,\sigma}\right)=0, \qquad (10)$$

$$\phi^2_{,\tau\tau}-\phi^2_{,\sigma\sigma}+\frac{2l}{b_0^2+l^2}\left(l_{,\tau}\phi_{,\tau}-l_{,\sigma}\phi_{,\sigma}\right)+$$

$$+2\operatorname{ctg}\theta\left(\phi_{,\tau}\theta_{,\tau}-\phi_{,\sigma}\theta_{,\sigma}\right)=0, \qquad (11)$$

$$-t^2_{,\tau}-t^2_{,\sigma}+l^2_{,\tau}+l^2_{,\sigma}+\left(b_0^2+l^2\right)\times$$

$$\times\left[\theta^2_{,\tau}+\theta^2_{,\sigma}+\sin^2\theta\left(\phi^2_{,\tau}+\phi^2_{,\sigma}\right)\right]=0, \qquad (12)$$

$$-t_{,\tau}t_{,\sigma}+l_{,\tau}l_{,\sigma}+\left(b_0^2+l^2\right)\left[\theta_{,\tau}\theta_{,\sigma}+\sin^2\theta\ \phi_{,\tau}\phi_{,\sigma}\right]=0. \qquad (13)$$

In order to solve the system of equations (8)—(13), we used a parametrization that describes a closed string:

$$t=t(\tau),\quad l=l(\tau),\quad \theta=\theta(\tau),\quad \phi=\sigma. \qquad (14)$$

Substitution of (14) into Eqs. (8)—(13) results in the following system of ordinary differential equations:

$$t_{,\tau}=E, \qquad (15)$$

$$l_{,\tau\tau}-l\theta^2_{,\tau}+l\sin^2\theta=0, \qquad (16)$$

$$\theta_{,\tau\tau}+\frac{2l}{b_0^2+l^2}l_{,\tau}\theta_{,\tau}+\sin\theta\cos\theta=0, \qquad (17)$$

$$l^2_{,\tau}+\left(b_0^2+l^2\right)\left(\theta^2_{,\tau}+\sin^2\theta\right)=E^2. \qquad (18)$$

The constant of integration E, which appears in Eqs. (15) and (18), is the total energy of the string and will play a role of the input parameter of the problem in what follows.

The description of the cosmic string dynamics in the gravitational field of a Lorentzian "wormhole" can be reduced to the study of the motion of a fictitious particle in a two-dimensional potential well. Really, it is easy to see that the equations that describe the string dynamics can be obtained from the Hamiltonian

$$H=\frac{P_l^2}{2}+\frac{P_\theta^2}{2\left(b_0^2+l^2\right)}+\frac{1}{2}\left(b_0^2+l^2\right)\sin^2\theta-\frac{E^2}{2}, \qquad (19)$$

taking into account the constraint $H=0$. The first term in the potential leads to the nonintegrability of the system and, as we will see below, to the emergence of a dynamic chaos.

The spatial part of metric (7) can always be inserted into the Euclidean space of a higher dimension [6], where the former can be interpreted as two asymptotically plane regions connected by a "bridge" [5]. Let the string be in either of those two asymptotically plane regions at the initial time, for example, at large positive l, and move towards the wormhole mouth, i.e. $\dot{l}<0$. In this case, numerical solutions of Eqs. (16)—(18) show that there exist two types of stable trajectories:

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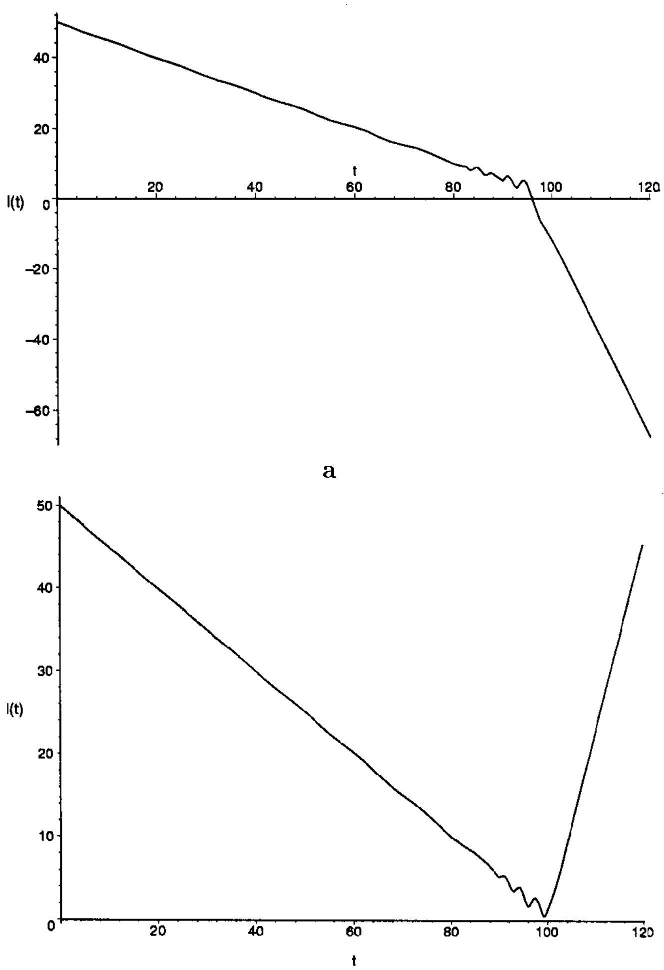

Fig. 1. Examples of the dependence of the proper distance $l(t)$ of a cosmic string to a wormhole mouth for cases 1 (a) and 2 (b)

1) the string traverses the "wormhole" mouth and goes away to the other asymptotically plane region (large by the absolute value but negative values of l);

2) the string is scattered by the "wormhole" and returns back to the initial asymptotically plane region (large positive values of l).

The corresponding types of the string motion are illustrated in Fig. 1.

In addition to the above-considered trajectories, there are infinitely many unstable trajectories, which are situated between those two. A strong dependence of the solution on the initial data is observed just in the intermediate range. Such a behavior of the solution serves as an indicator of the dynamic chaos in the system [4].

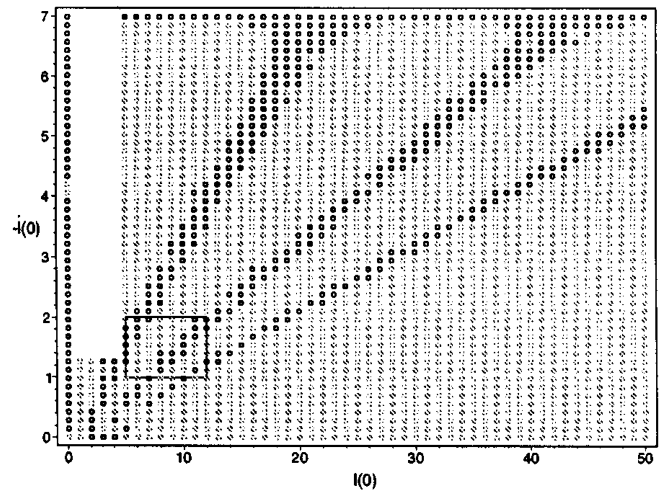

Fig. 2. Example of the phase-plane portrait for $E = 7$ and $\theta = 0$

For a more detailed research of the dynamics of the string, let us consider its phase-plane portraits, the latter being the surfaces of constant energy in the phase space of initial data. For the system of equations (16)—(18), the initial data are l, $\dot{l}$, θ, and $\dot{\theta}$, given at $\tau = 0$. The initial data comprise a four-dimensional phase space. So, to construct the phase-plane portrait, we consider a surface with fixed θ and $\dot{\theta}$:

$$\theta = \text{const}, \quad \dot{\theta}(0) = \sqrt{\frac{E^2 - \dot{l}^2(0)}{b_0^2 + l^2(0)} - \sin^2\theta}, \tag{20}$$

where the last equality is a consequence of constraint (18).

Fig. 2 illustrates an example of the phase-plane portrait, where the light areas correspond to the back scattering of the string and the dark ones to its capture by the "wormhole". The figure clearly shows the alternation of the areas and their boundaries. A more detailed investigation of the near-boundary layers reveals their fractal structure (Fig. 3), which is a typical indicator that dynamic chaos does present in the system.

Thus, we draw conclusion that the equations of motion for a closed cosmic string, which moves in the gravitational field of a Lorentzian "wormhole", have unstable solutions besides regular ones. The former come into action if the initial data for the system of equations (16)—(18) lie in the near-boundary regions of the phase-plane portrait. This instability manifests itself in that even a minor alteration of the initial data leads to qualitative variations of the solutions. Since the initial data are always known only with some accuracy, this

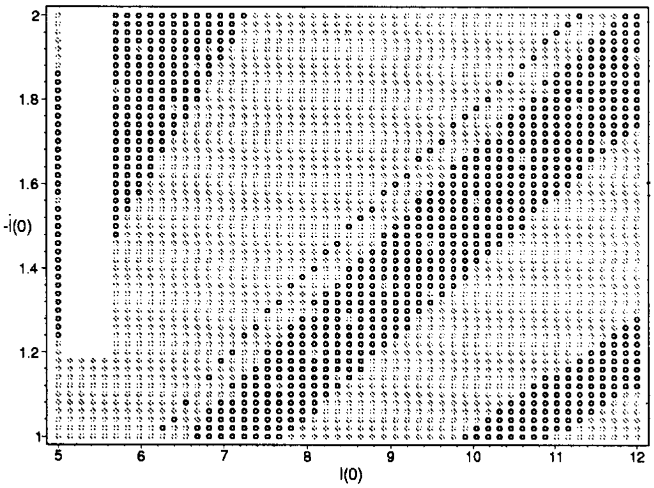

Fig. 3. A magnified part of the phase-plane portrait inside a rectangular in Fig. 2

instability makes it impossible to forecast the string's motion when the initial data are from the near-boundary regions.

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ХАОТИЧНА ПОВЕДІНКА КОСМІЧНОЇ СТРУНИ В ГРАВІТАЦІЙНОМУ ПОЛІ ЛОРЕНЦЕВОЇ “КРОТОВОЇ НОРИ”

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Р е з ю м е

Розглянуто динаміку замкненої космічної струни в просторі-часі “кротової нори”. Одержано числові розв'язки рівнянь руху струни і проведено аналіз цих розв'язків для різних початкових даних. Показано, що розсіяння струни “кротовою норою” має хаотичний характер.