HYDRODYNAMIC VORTICES IN OPEN SYSTEMS WITH A MATTER SINK

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The evolution of hydrodynamic vortices (acceleration, stationary behavior, and diffusion) in multicomponent multiphase open systems with a matter sink in some region is considered. The mechanism of vortex development and dissipation influence on its stabilization are analyzed.

1. Introduction

The non-linear character of hydrodynamic equations allows one to get analytical solutions only for several specific models. At small velocities, these equations may be linearized, and their solutions can be obtained by standard methods [1, 2]. Moreover, at small Reynolds numbers, one can neglect viscosity and solve the Euler equations for the ideal liquid. In the general case, the motion of a viscous liquid is described by numerical methods.

At the end of the 19th century, an exact solution (velocity profile) of the Navier—Stokes equations, which nullify the term with the viscosity coefficient in the equation for azimuthal velocity, was proposed: $v_\varphi \sim r$ $(r < R)$ and $v_\varphi \sim 1/r$ $(r > R)$, where R is the radius of a cylindrical region of "rigid-body" rotation. In this case, a radial velocity equals to zero. This velocity profile is known as a Rankine vortex [3]. Let us note that the region of "rigid-body" rotation of liquid is arbitrary and determines by the initial conditions.

In [4], a new mechanism of origin and development of instable hydrodynamic vortices was proposed, and the velocity profiles, which nullify terms with viscosity in the Navier—Stokes equations with a nonzero radial velocity connected with a volumetric sink inside a certain region of liquid, were considered. In the case of the presence of initial vorticity, the azimuthal velocity appears to be instable and increases exponentially with time. Such initial vorticity may appear on the account of natural conditions (e.g., the Earth rotation and contradirectional air flows for atmospheric vortices). This mechanism was used for the description of power atmospheric vortices such as tornados and typhoons.

In the present paper, we consider the mechanisms of stabilization and decay (diffusion) of such instable cylindrically symmetric vortices. The full evolution of an instable vortex can be divided into three stages: exponential (in the time) acceleration of the vortex core, its stabilization, and decay. The acceleration of the vortex lasts till the moment, when a turbulent dissipation become strong enough to compensate the inflow of kinetic energy to the region of the vortex core (the sink region). The turbulent dissipation leads to the stabilization of the vortex motion, which may go on a quite long period of time, while the volumetric sink exists. After that, the vortex spreads in space (diffusion or decay of a vortex), loses its energy, and disappears.

2. Instability of Rotational Motion in the Case of a Cylindrically Symmetric Sink Region

One of the key aspects of the instability mechanism we consider is the presence of a volumetric sink of matter q in a certain region of space (we call it as “internal” region) and a possibility of the infinite matter inflow from the “external” region (open system). This can be realized in a multicomponent system, in which some components may convert into others on account of chemical or phase transitions and drop out of the collective motion. We will write hydrodynamic equations for the collective motion of the components as the equations for one component with a volumetric sink. In this case, the continuity equation and the Navier—Stokes equations have a form

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\rho\mathbf{v}=q, \tag{2.1}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}\right]=-\nabla p+\eta\triangle\mathbf{v}+(\zeta+\eta/3)\nabla\mathrm{div}\mathbf{v}, \tag{2.2}$$

where ρ is the matter density, $p(\mathbf{r},t)$ is the pressure, η, ζ are the first and second viscosity respectively, and q is a volumetric sink (or source) of mass. Here, we neglect the gravitation force. We will consider a model of incompressible liquid (gas) with $\rho=\mathrm{const}$, for validity of which it is enough to satisfy the conditions $v\ll c_s$ and $l/\tau\ll c_s$, where τ,l are quantities of the order of intervals of time and space, on which the velocity of a liquid changes noticeably, c_s — speed of sound in the liquid. In this case, the volumetric sink q may depend on time, but not on coordinates. For example, if the volumetric sink has a characteristic time of existence T (the saturation of a reservoir of the sink), we may model this situation with the equation

$$q(t)=q_0e^{-t/T}. \tag{2.3}$$

For an incompressible liquid, the term with the second viscosity becomes zero, and the hydrodynamic equations (2.1), (2.2) read

$$\mathrm{div}\mathbf{v}=q(t)/\rho, \tag{2.4}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\frac{\nabla p}{\rho}+\nu\triangle\mathbf{v}, \tag{2.5}$$

where $\nu=\eta/\rho$ is the kinematic viscosity.

Further, we restrict ourselves by the case where the matter sink exists only in a certain cylindrical region with radius R. Then

$$\mathrm{div}\mathbf{v}=\frac{q(t)}{\rho}\equiv\begin{cases}-1/\tau(t), & r<R,\\ 0, & r>R,\end{cases} \tag{2.6}$$

where $\tau(t)$ is a given function of time.

Consider a plane motion of liquid, assuming it to be homogeneous along the axis z ($\frac{\partial}{\partial z}=0,\ v_z=0$). Since we will be interested in vortex solutions of the hydrodynamic equations, it is convenient to use the polar coordinate system, where these equations are written in the form

$$\begin{cases}\dfrac{\partial v_r}{\partial t}+(\mathbf{v}\nabla)v_r-\dfrac{v_\varphi^2}{r}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial r}+\\ +\nu\left[\triangle v_r-\dfrac{v_r}{r^2}-\dfrac{2}{r^2}\dfrac{\partial v_\varphi}{\partial\varphi}\right],\\ \dfrac{\partial v_\varphi}{\partial t}+(\mathbf{v}\nabla)v_\varphi+\dfrac{v_rv_\varphi}{r}=-\dfrac{1}{\rho r}\dfrac{\partial p}{\partial\varphi}+\\ +\nu\left[\triangle v_\varphi-\dfrac{v_\varphi}{r^2}+\dfrac{2}{r^2}\dfrac{\partial v_r}{\partial\varphi}\right],\end{cases} \tag{2.7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi}=q/\rho, \tag{2.8}$$

where

$$(\mathbf{v}\nabla)=v_r\frac{\partial}{\partial r}+\frac{v_\varphi}{r}\frac{\partial}{\partial\varphi}, \tag{2.9}$$

$$\triangle=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}. \tag{2.10}$$

Further, we will analyze several velocity profiles of liquid motion in vortices with the purpose of uncovering the principal mechanisms of their development. From all possible motions, the survivors will be those, in which dissipation plays the least role. That is, we will find those solutions of the hydrodynamic equations, in which the terms with the kinematic viscosity are nullified, namely, the square brackets in Eqs. (2.7):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2v_r}{\partial\varphi^2}-\frac{v_r}{r^2}=\frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}, \tag{2.11}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2v_\varphi}{\partial\varphi^2}-\frac{v_\varphi}{r^2}=-\frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}. \tag{2.12}$$

Consider the case where the radial velocity $v_r=0$, and only the azimuthal velocity $v_\varphi(r)$, which depend only on r, is non-zero.

At such a restriction, we get from Eq. (2.12) that

$$v_\varphi=\omega r+\Omega/r. \tag{2.13}$$

Then, Eq. (2.7) allows us to determine the pressure from the relation (cyclostrophic regime)

$$\frac{\partial p}{\partial r}=\rho\frac{v_\varphi^2}{r}. \tag{2.14}$$

The continuity equation (2.8) is satisfied identically and does not introduce any additional relations. Dividing the motion region into the external ($r > R$) and internal ($r < R$) ones and making the velocity and the pressure continuous on the boundary ($r = R$), we get the known velocity profile for a Rankine vortex:

$$v_\varphi(r) = \begin{cases} \omega r, & r \leqslant R, \\ \omega R^2/r, & r > R. \end{cases} \tag{2.15}$$

Here, ω is the constant angular velocity of a "rigid-body" rotation. Notice that the quantity R separates the region of "rigid-body" rotation from the region of "irrotary" rotation, where $v_\varphi \sim 1/r$, and is determined by initial values. The distribution of the pressure is given by the expressions

$$p(r) = \begin{cases} p_0 + \rho\omega^2 r^2/2, & r \leqslant R, \\ p_\infty - \rho\omega^2 R^4/2r^2, & r > R, \end{cases} \tag{2.16}$$

where p_0 is the pressure at the center of a vortex $r = 0$, and p_∞ is the pressure at $r \to \infty$.

From the condition of continuity of the pressure at $r = R$, we get

$$p_0 = p_\infty - \rho\omega^2 R^2. \tag{2.17}$$

Further, we will consider the peculiarities of vortex motion at the presence of a volumetric sink of matter $q \neq 0$. We look for only axially symmetric solutions ($\frac{\partial}{\partial\varphi} = 0$) with the velocity profiles which satisfy conditions (2.11) and (2.12). In this case, the corresponding solutions have a form

$$v_r(r,t) = \begin{cases} -\beta(t) r, & r \leqslant R, \\ -B(t) R^2/r, & r > R, \end{cases}$$

$$v_\varphi(r,t) = \begin{cases} \omega(t) r, & r \leqslant R, \\ \Omega(t) R^2/r, & r > R. \end{cases} \tag{2.18}$$

One can substitute these solutions in the continuity equation (2.6) and get the relation

$$2\beta(t) = \frac{1}{\tau(t)}. \tag{2.19}$$

In this case, the Navier-Stokes equation for a velocity component $v_\varphi(r,t)$ gives

$$\begin{cases} \dot\omega - 2\beta\omega = 0, & r \leqslant R, \\ \dot\Omega = 0, & r > R. \end{cases} \tag{2.20}$$

Wherefrom, with regard for (2.19), we get

$$\omega(t) = \omega_0 \exp\left\{ \int\limits_0^t \frac{dt'}{\tau(t')} \right\}, \quad \Omega = \Omega_0 = \text{const.} \tag{2.21}$$

In the simplest case where the sink is time-constant $\tau(t) = \tau$, the azimuthal velocity is given by the expression

$$v_\varphi(r,t) = \begin{cases} \omega_0 r\, e^{t/\tau}, & r \leqslant R, \\ \Omega_0 \frac{R^2}{r}, & r > R. \end{cases} \tag{2.22}$$

We see that the exponential acceleration of a vortex occurs in the internal region, but, in the external region, the liquid rotates with a constant initial "angular velocity". Therefore, a tangential step of velocity exists on the boundary (except the initial moment of time, if we assume $\Omega_0 = \omega_0$), which grows with time. A tangential step of velocity is unstable and leads to a small-scale turbulence on the boundary, which destroys the solutions that nullify terms with viscosity in the Navier—Stokes equations. Moreover, the acceleration of the velocity is limited by the value of the order of the speed of sound, when the incompressible liquid approximation becomes invalid and the second viscosity begins to play a role. Due to the large dissipation of energy, these factors lead to the stabilization of a vortex, and its angular velocity stops to increase. Furthermore, the change (decrease) of the sink intensity with time $q(t)$ in the internal region (saturation of a sink or external source depletion) can additionally stabilize the vortex. We analyze these mechanisms of vortex stabilization in detail in the next sections.

At last, we will use the Navier—Stokes equation for the radial velocity $v_r(r,t)$:

$$\begin{cases} (\beta^2 - \omega^2) r = -\frac{1}{\rho}\frac{\partial p}{\partial r}, & r \leqslant R, \\ (\beta^2 + \omega_0^2)\frac{R^4}{r^3} = \frac{1}{\rho}\frac{\partial p}{\partial r}, & r > R. \end{cases} \tag{2.23}$$

This yields the pressure distribution as

$$p(r,t) = \begin{cases} p_0(t) + \frac{\rho r^2}{2}(\omega^2(t) - \beta^2), & r \leqslant R, \\ p_\infty - \frac{\rho R^4}{2r^2}(\omega_0^2 + \beta^2), & r > R, \end{cases} \tag{2.24}$$

where

$$p_0(t) = p_\infty - \frac{\rho R^2}{2}(\omega^2(t) + \omega_0^2). \tag{2.25}$$

Continuity of the radial velocity requires the condition $B = \beta$. At the same time, the azimuthal velocity v_φ (2.22) has a step which grows with time. This can be eliminated by taking onto the account the viscosity in the near-surface layer $r = R$.

3. Accounting the Viscosity on the Boundary of a Sink

On the boundary of a sink ($r = R$) a tangential velocity step occurs, in the region of which friction (with the turbulent viscosity coefficient ν^* [4]) arises, drags the liquid in a small layer (with a characteristic thickness l) near $r = R$, and eliminate this tangential step. We describe this by letting the functions ω, Ω depend on radius r. We extract apart a stationary solution with the initial velocity profile from the azimuthal velocity, i.e. we look for the velocity profiles in the form

$$v_\varphi(r,t) = \begin{cases} \omega_0 r + \omega(r,t) r, & r < R, \\ \omega_0 \frac{R^2}{r} + \Omega(r,t) \frac{R^2}{r}, & r > R. \end{cases} \tag{3.26}$$

The unknown functions $\omega(r,t)$ and $\Omega(r,t)$ must satisfy the initial conditions

$$\omega(r,0) = 0, \ \Omega(r,0) = 0, \tag{3.27}$$

and be continuous on the boundary:

$$\omega(R,t) = \Omega(R,t).$$

One can substitute the velocity profile (3.26) in the Navier—Stokes equation

$$\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} = \nu^* \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\varphi}{\partial r} \right) - \frac{v_\varphi}{r^2} \right] \tag{3.28}$$

and get the equations

$$\frac{\partial \omega}{\partial t} = \nu^* \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} (3\nu^* + \beta r^2) \frac{\partial \omega}{\partial r} + 2\beta\omega, \tag{3.29}$$

$$\frac{\partial \Omega}{\partial t} = \nu^* \frac{\partial^2 \Omega}{\partial r^2} - \frac{1}{r} (\nu^* - \beta R^2) \frac{\partial \Omega}{\partial r}. \tag{3.30}$$

We find the general solution of these equations by the variable separation method (β is assumed to be constant)

$$\omega(r,t) = \int d\lambda \, C(\lambda) \, e^{\lambda t} \, \frac{e^{-\beta r^2/4\nu^*}}{r^2} M \left(\frac{\lambda}{2\beta}, \frac{1}{2}, -\frac{\beta r^2}{2\nu^*} \right), \tag{3.31}$$

$$\Omega(r,t) = \int d\lambda \, \tilde{C}(\lambda) \, e^{\lambda t} \, \frac{1}{r^\sigma} \, K_\sigma \left(\sqrt{\frac{\lambda}{\nu^*}} \, r \right), \tag{3.32}$$

where $\sigma = \frac{\beta R^2}{2\nu^*} - 1$, $M(\mu, \nu, x)$ is the Whittaker function which determines a solution limited at zero of the equation $M''(\mu,\nu,x) + \left(-\frac{1}{4} + \frac{\mu}{x} + \frac{1/4 - \nu^2}{x^2} \right) M(\mu,\nu,x) = 0$, and $K_\sigma(x)$ is the McDonald function of order σ (modified Bessel function).

In order that the solution behavior will be qualitatively clear, we use an approximation, in which the effect of dissipation is small and the acceleration of angular velocity in time remains the same, i.e. $\omega(t) \propto e^{t/\tau}$. Then, the final solution which satisfies all the initial and boundary conditions reads

$$v_\varphi(r,t) =$$
$$= \begin{cases} \omega_0 \, r + \omega_0 \, r \, (e^{t/\tau} - 1), & r < R, \\ \omega_0 \, \frac{R^2}{r} + \omega_0 \, R (e^{t/\tau} - 1) \left(\frac{R}{r} \right)^{\sigma+1} \frac{K_\sigma\left(\frac{r}{l}\right)}{K_\sigma\left(\frac{R}{l}\right)}, & r > R, \end{cases} \tag{3.33}$$

where

$$\sigma = \frac{R^2}{(2l)^2} - 1, \quad l = \sqrt{\nu^* \tau}. \tag{3.34}$$

Note that, at $R \gg l$, we have an exponentially decreasing velocity profile of the liquid dragged by friction in the external region

$$K_\sigma(r/l) \sim \sqrt{\frac{l}{r}} \, e^{-r/l}.$$

Here, l plays the role of the characteristic thickness of a layer, where a turbulence motion occurs.

At all this, the radial part of velocity (and the pressure as a consequence) remains without any changes:

$$v_r(r,t) = \begin{cases} -\beta r, & r < R, \\ -\beta \frac{R^2}{r}, & r > R, \end{cases} \tag{3.35}$$

$$p(r,t) = \begin{cases} p_0(t) + \frac{\rho r^2}{2} (\omega_0^2 \, e^{2t/\tau} - \frac{1}{4\tau^2}), & r < R, \\ p_\infty - \frac{\rho R^4}{2r^2} (\omega_0^2 + \frac{1}{4\tau^2}), & r > R. \end{cases} \tag{3.36}$$

4. Stabilization Mechanisms of a Vortex

Some of the causes which stabilize the rotation of a vortex are the saturation of the sink reservoir of matter or the depletion of an external source, at the expense of what its intensity decreases with time. For clarity, let us consider a model with the exponential sink behavior

$$q(t) = q_0 \, e^{-t/t_q}, \tag{4.1}$$

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where t_q is the characteristic time of a decrease of the sink.

In this case, we have

$$\tau(t) = \tau_0 \, e^{t/t_q}, \tag{4.2}$$

wherefrom, with the use of expression (2.21), we get the dependence of the angular velocity on time in the internal region

$$\omega(t) = \omega_0 \exp\left\{ \frac{t_q}{\tau_0} \left(1 - e^{-t/t_q}\right) \right\}. \tag{4.3}$$

In the limiting cases of large and small times (in comparison with t_q) we have

$$\omega(t) = \begin{cases} \omega_0 \, e^{t/\tau_0}, & t \ll t_q, \\ \omega_0 \, e^{t_q/\tau_0}, & t \gg t_q, \end{cases} \tag{4.4}$$

i.e., at small times, we have exponential acceleration and, at large times, a tendency to saturation and stabilization.

Another power stabilization mechanism is the turbulent energy dissipation in the region of a tangential velocity step, i.e. on the sink boundary $r = R$. The kinetic energy of a small-scale ripple transforms to heat, and the average amount of the energy, which is dissipated in the unit time in the unit volume, equals [1]

$$\varepsilon \sim \frac{(\triangle u)^3}{l^*}, \tag{4.5}$$

where $\triangle u$ is a variation of the average velocity of a turbulence motion on a characteristic length l^*, which determines sizes of the turbulence motion region. In this case, the turbulent viscosity is of the order of $\nu^* \sim \triangle u \cdot l^* \sim \nu \cdot \mathrm{Re}$, where Re is the Reynolds number. In our case, for estimations, we can take

$$l^* \sim l = \sqrt{\nu^* \tau}, \quad \triangle u \sim v_\varphi(\mathrm{Re}). \tag{4.6}$$

Hence, the energy dissipation in the unit time on the unit length (of the axis z) of the surface $r = R$ is

$$\frac{dE_{\mathrm{dis}}}{dt} \sim 2\pi\rho R \, (\triangle u)^3 \sim 2\pi\rho \, \omega_0^3 \, R^4 e^{3t/\tau}. \tag{4.7}$$

On the other hand, the speed of change of the kinetic energy in the internal region (neglecting the dissipation) on the unit length (of the z axis) equals

$$\frac{dE_{\mathrm{kin}}}{dt} = \frac{\rho}{2} \frac{d}{dt} \int_0^R 2\pi r dr \, v^2(r,t) = \frac{1}{2} \pi\rho \frac{\omega_0^2}{\tau} R^4 \, e^{2t/\tau}. \tag{4.8}$$

In the rough approximation, comparing expressions (4.7) and (4.8), we can estimate the stabilization time of a vortex as

$$t_d \sim \tau \, \ln \frac{1}{4\omega_0 \tau}. \tag{4.9}$$

In this case, the maximum velocity on the sink boundary is

$$v_{\max} = \omega_0 R \, e^{t_d/\tau} \sim \frac{R}{4\tau}. \tag{4.10}$$

The considered theory of the unstable behavior of vortices has a certain simplification, namely, it demands the incompressible liquid (gas), which takes part in the acceleration of a vortex. The condition of incompressibility holds well for motions with velocities less than the velocity of sound in a liquid,

$$v < c_s.$$

Since the velocity of motion rapidly increases in time in our problem, we reach formally (neglecting the energy dissipation) the velocity of sound c_s after the time interval

$$t_s \sim \tau \ln \frac{c_s}{\omega_0 R}. \tag{4.11}$$

The order of such a time interval gives us the limit of validity of the formulas with exponential acceleration (2.22). After this, a stage of vortex stabilization due to liquid compressibility and the second viscosity originating with it begins. Of course, this mechanism acts along with a dissipation.

Three mechanisms considered above are actually in concurrence among themselves (and, of course, among the other secondary mechanisms). The fact, which mechanism is realized in a given certain case, depends on the liquid under consideration and on initial conditions, especially on the initial angular velocity ω_0, radius of a sink region R, and time-behavior of a volumetric sink $\tau(t)$. In other words, the evolution is determined by the hierarchy of characteristic times t_q, t_d, and t_s. For example, if $t_d < t_s$, which is the same as

$$\frac{R}{4\tau} < c_s,$$

then the dissipation comes earlier and the velocity of sound cannot be reached. Inasmuch as at reaching the sound velocity (if it is possible at all), the role of a stabilizer is played, in fact, also by the dissipation (via turbulence and compressibility), then the time of the beginning of stabilization is

$$t_D \sim \min\{t_d, t_s\}. \tag{4.12}$$

In addition, since our system is an open one, we can assume that t_q is large enough (actually, we need $t_q > t_D$), i.e. the sink exists longer than the interval of vortex acceleration.

Under the above-presented conditions, we have three stages of vortex evolution:
1. *Acceleration* ($0 < t < t_D$). The angular velocity of a "rigid-body" rotation increases with time: at first as $\propto e^{t/\tau}$, then more slowly, and finally reaches a saturation.
2. *Stationary stage* ($t_D < t < t_q$). The mechanisms of acceleration (on account of sink) and dissipation compensate each other, and a constant angular velocity is maintained.
3. *Decay* ($t_q < t < t_q + t_r$). After the depletion of the source of kinetic energy (termination of the action of a volumetric sink), a vortex decays according to the standard scenario (see the next section; there, a characteristic time t_r is also estimated).

5. Diffusion of a Vortex

Let us take the vortex decay beginning t_q as a time reference point. Since already there is no matter sink, the radial velocity equals zero, $v_r = 0$, and it remains only the azimuthal velocity $v_\varphi(r,t)$. It is known that if the initial azimuthal velocity profile is given as $v_\varphi(r,0) = v_0(r)$, then its profile at any point of time is given by the expression [5]

$$v_{\varphi(r,t)} = \frac{1}{2\nu r t}\int_0^\infty r''\,dr''\,\Omega_0(r'')\,e^{-r''^2/(4\nu t)}\times$$

$$\times\int_0^r r'\,dr'\,e^{-\frac{r'^2}{4\nu t}}\,I_0\left(\frac{r'r''}{2\nu t}\right), \tag{5.1}$$

where

$$\Omega_0(r) = (\mathrm{rot}\,\mathbf{v_0})_z = \frac{1}{r}\frac{d(rv_0)}{dr}$$

and $I_0(x)$ is a modified Bessel function.

We neglect the azimuthal velocity in the external region, since the exponentially decreasing turbulent tail there is thin enough. So, we can approximately write

$$v_0(r) = \begin{cases} \omega_{\max} r, & r < R, \\ 0, & r > R. \end{cases} \tag{5.2}$$

Then, in our case, we get

$$v_\varphi(r,t) = \frac{1}{r}\int_0^r r'\,dr'\,\Omega(r',t), \tag{5.3}$$

where the vorticity $\Omega = (\mathrm{rot}\,\mathbf{v})_z$ equals [5]

$$\Omega(r,t) = \frac{\omega_{\max}}{\nu^* t}\,e^{-r^2/4\nu^* t}\int_0^R r''\,dr''\,e^{-r''^2/4\nu^* t}\,I_0\left(\frac{rr''}{2\nu^* t}\right). \tag{5.4}$$

The analysis of expression (5.4) shows that a vortex “spreads” in space. The maximum of its azimuthal velocity moves away and diminishes with time, and the total kinetic energy of a vortex decreases. Particularly, the vorticity at the center of a vortex ($r = 0$) equals

$$\Omega(0,t) = \frac{\omega_{\max}}{\nu^* t}\int_0^R r''\,dr''\,e^{-\frac{r''^2}{4\nu^* t}}\,I_0\,(0) =$$

$$= 2\omega_{\max}\left(1 - e^{-\frac{R^2}{4\nu^* t}}\right),$$

wherefrom, it is seen that a characteristic time of its decrease is of the order of $\frac{R^2}{4\nu^*}$.

Thus, the characteristic time of decay of the entire vortex is of the order of

$$t_r \sim \frac{R^2}{4\nu^*}. \tag{5.5}$$

The turbulent viscosity ν^* appears to be much larger than the ordinary kinematic viscosity ν. In particular, estimations in [4] show that ν^* is by one order larger than ν. At the same time, it is known [5] that this quantity in powerful atmospheric vortices may be by 5—6 orders higher than ν. Hence, the characteristic time of the vortex decay may vary in a large enough range.

6. Conclusions

We have investigated the time-instable vortex solutions of the hydrodynamic equations for incompressible liquid which describe multicomponent multiphase open systems with chemical or phase transitions. The existence of such solutions is guaranteed by a volumetric sink of one of the components in a certain finite volume and by its inflow from the surrounding environment. For the first time, such solutions were investigated in [4]. Let us emphasize that the corresponding velocity profiles of these vortices are exact solutions of the non-linear hydrodynamic equations, which nullify terms with viscosity, i.e. satisfy the condition $\Delta\mathbf{v} = 0$. At a non-zero initial vorticity, these solutions must be dominant.

The time-space evolution of the above-mentioned instable vortices is considered. A characteristic time of vortex acceleration is determined by the intensity of a volumetric sink. Usually, such an acceleration mechanism is in a concurrence with the dissipation which increases with the acceleration and, after a certain time, leads to the vortex stabilization. The stabilized vortex may exist long enough, in fact, as long as the volumetric sink exists. After that, the vortex decays in space via the diffusion mechanism [5].

The described mechanism of a hydrodynamic instability may be realized in nature as, for example, a mechanism of the origin of such powerful atmospheric vortices as tornados and typhoons [4] or origination of vortices in a supersaturated ^{3}He—^{4}He solution which accelerate the process of its decomposition (phase separation) [6].

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ГІДРОДИНАМІЧНІ ВИХОРИ У ВІДКРИТИХ СИСТЕМАХ ЗІ СТОКАМИ РЕЧОВИНИ

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Р е з ю м е

Розглятуто еволюцію гідродинамічних вихорів (розвиток, стаціонарну поведінку та розпад) у багатокомпонентних багатофазних відкритих системах за наявності об'ємного стоку у деякій області. Проаналізовано механізм розкрутки вихору та вплив дисипації на його стабілізацію.