MULTISPECIES ANYONS IN A MAGNETIC FIELD IN THE ANTISCREENING REGIME

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We study a system of multispecies anyons in the presence of an external magnetic field parallel to the anyonic Aharonov–Bohm fluxes (antiscreening regime). The lowest Landau level spectrum and the equation of state are found, and the connection with exclusion statistics is established. It is shown that distinguishable anyons with an infinitesimal statistical parameter in the antiscreening regime behave, in a certain sense, like fermions.

1. Introduction

The prominent features of statistics in two dimensions are:
(i) it can interpolate between the bosonic and fermionic limits [1, 2];
(ii) statistics of distinguishable particles is possible [3,4].

Of course, fundamental particles which live in three dimensions can only be bosons or fermions, but a nontrivial statistical phase can be emulated via the Aharonov–Bohm effect [5]. Specifically, consider "charge-flux composites," i.e. particles which are charges and point magnetic fluxes at the same time (two-dimensional cross sections of infinitely thin flux tubes). If one such particle with charge e_a and flux ϕ_a is taken around another one with charge e_b and flux ϕ_b, then, since the charges move in the topologically nontrivial vector potential of the flux tubes, the wave function of the system acquires a phase of $\exp[2i\pi\alpha_{ab}]$ with

$$\alpha_{ab} = \frac{e_a\phi_b + e_b\phi_a}{2\pi} \, . \qquad (1)$$

Furthermore, if the particles are identical, then an interchange of two of them will yield a phase of $\exp[i\pi\alpha_{aa}]$. Therefore,

(i) identical particles effectively exhibit intermediate or *fractional* statistics which continuously interpolates from bosonic to fermionic one as α_{aa} goes from 0 to 1 (if the "bare" particles are bosons; $\alpha_{aa} \mapsto \alpha_{aa} + 1$ if they are fermions);

(ii) distinguishable particles exhibit *mutual* fractional statistics as well, whenever α_{ab} is not an integer.

The coefficients α_{ab} form a matrix of *mutual statistics* parameters. Translation invariance dictates that it must be symmetric, as in Eq. (1).

In the so-called Chern–Simons field model [6] (which actually emerges as an effective model under certain conditions as a result of the dimensional reduction in gauge theories), it follows from the field equations that the magnetic field is proportional to the charge density. Therefore, a point charge is a point magnetic flux at the same time, with

$$\phi_a = \kappa e_a \, , \qquad (2)$$

the coefficient κ depending on the coefficient of the Chern–Simons term in the field Lagrangian. An accurate analysis [7] shows that, in this case, the mutual statistics parameter is, in fact, one half of the one in Eq. (1):

$$\alpha_{ab} = \frac{e_a\phi_b}{2\pi} \qquad (3)$$

[Eq. (2) guarantees that the matrix is symmetric].

Particles with fractional statistics are termed anyons; nonidentical particles with mutual fractional statistics are termed *multispecies anyons*. The quantum-mechanical problem for multispecies anyons is formulated as follows: the Schrödinger equation

$$H\psi = E\psi \tag{4}$$

plus the boundary conditions

$$P_{aj,ak}\psi = \exp[\mathrm{i}\pi\alpha_{aa}]\,\psi\ ; \tag{5}$$

$$P^2_{aj,bk}\psi = \exp[2\mathrm{i}\pi\alpha_{ab}]\,\psi\ , \tag{6}$$

where the operator $P_{aj,bk}$ corresponds to an anticlockwise interchange of the j-th particle of species a with the k-th particle of species b.

Just like the problem of N identical anyons [8], the generic multispecies many-anyon problem is not exactly solvable, since a wave function satisfying conditions (5), (6) cannot be represented as a linear combination of products of single-particle wave functions. There is, however, one particular case where a (partial) exact solution is available: anyons in a high external magnetic field (a physical example is elementary excitations in the fractional quantum Hall effect [9]). At temperatures much lower than the Landau gap (cyclotron frequency) $\omega_{\rm c}$, the degree of freedom corresponding to Landau excitations freezes out, and it is only the lowest Landau level (LLL) states that matter.

For identical anyons, the magnetic fluxes of particles can be either antiparallel or parallel to the external magnetic field; one talks, respectively, about the screening and antiscreening regimes (the fluxes screen the external magnetic field or add to it). For the screening regime, the higher Landau level states are always separated by a gap from the LLL. Starting from the exact quantum-mechanical spectrum, the equation of state (in the LLL approximation) was derived in [10]. The antiscreening regime was examined in [11]. It was shown that the equation of state follows from that in the screening regime upon invoking a periodicity in the single statistics parameter α, i.e., the invariance with respect to $\alpha \mapsto \alpha + 2$, as long as one neglects a number of states that join the LLL at the bosonic ($\alpha = 0$) point. Those states matter at small values of $\alpha \propto \omega_{\rm c}^{-1}$, and therefore can be neglected if the $\omega_{\rm c} \to \infty$ limit is taken first.

For multispecies anyons, the signs of particle fluxes may in general be different, leading to a combined screening-antiscreening regime. The pure screening regime (all fluxes antiparallel to the external field) has been analyzed [4, 12], and a generalization of the single-species anyon equation of state is obtained. In the present paper, we consider the pure antiscreening regime, extending the periodicity argument onto the multispecies case.

2. The Spectrum and Equation of State

We place the particles into a uniform magnetic field B and, in order to discretize the spectrum, add a harmonic potential, whose strength is to be nullified in the thermodynamic limit. By convention, we choose $e_aB < 0$ (in [10, 11], the opposite convention was used). The cyclotron frequencies

$$\omega_{{\rm c}a} = -\frac{e_aB}{2} \tag{7}$$

are species-dependent, as, in general, may be the harmonic frequencies ω_a. The many-body Hamiltonian is

$$H_N = \sum_{aj}\left[-\frac{2}{m_a}\partial_{aj}\bar\partial_{aj} - \omega_{{\rm c}a}(z_{aj}\partial_{aj} - \bar z_{aj}\bar\partial_{aj}) + \right.$$
$$\left. + \frac{m_a\omega_{{\rm t}a}^2}{2} z_{aj}\bar z_{aj}\right]\ , \tag{8}$$

where

$$\omega_{{\rm t}a} = \sqrt{\omega_{{\rm c}a}^2 + \omega_a^2}\ . \tag{9}$$

Separating out the long-distance behavior,

$$\psi_N = \exp\left(-\sum_{aj}\frac{m_a\omega_{{\rm t}a}}{2}z_{aj}\bar z_{aj}\right)\tilde\psi_N\ , \tag{10}$$

yields the Hamiltonian acting on $\tilde\psi_N$,

$$\tilde H_N = \sum_{aj}\left[-\frac{2}{m_a}\partial_{aj}\bar\partial_{aj} + (\omega_{{\rm t}a} - \omega_{{\rm c}a})z_{aj}\partial_{aj} + \right.$$
$$\left. + (\omega_{{\rm t}a} + \omega_{{\rm c}a})\bar z_{aj}\bar\partial_{aj} + \omega_{{\rm t}a}\right]\ . \tag{11}$$

We are interested in the high magnetic field limit where $\omega_{{\rm c}a} \gg \omega_a$. As is evident from Eq. (11) with account for (9), the energy of any eigenstate whose wave function depends on $\bar z_{aj}$ will be proportional to $\omega_{{\rm c}a}$; such states do not belong to the lowest Landau level. Therefore, the single-particle LLL basis is given by the set of functions $\{z^\ell,\ \ell = 0,1,2,\ldots\}$ (ℓ is the angular momentum). For $\alpha_{ab} = 0$, when the system is one of multispecies bosons, the complete set of many-body LLL eigenfunctions is given by

$$\tilde{\psi}^{(\mathrm{Bose})}_{N\{\ell_{aj}\}} = \prod_a \left\{ \prod_j z_{aj}^{\ell_{aj}} \right\}_{\mathrm{sym}} , \qquad (12)$$

where the symmetrization is performed over the coordinates of particles of the same species only. For nonzero α_{ab}, the key observation is that a multispecies generalization of the Laughlin factor, $\prod_{(aj)<(bk)}(z_{aj} - z_{bk})^{\alpha_{ab}}$, satisfies the interchange conditions (5) and (6). A candidate multibody eigenfunction, then, is

$$\tilde{\psi}_{N\{\ell_{aj}\}} = \prod_{(aj)<(bk)} (z_{aj} - z_{bk})^{\alpha_{ab}} \prod_a \left\{ \prod_j z_{aj}^{\ell_{aj}} \right\}_{\mathrm{sym}} . \qquad (13)$$

Here, $(aj) < (bk)$ is understood as "$a < b$ or ($a = b$ and $j < k$)", so that each pair of particles is counted only once. It is easy to see that this will be an eigenfunction of Hamiltonian (11) if the quantity

$$\varpi \equiv \omega_{\mathrm{t}a} - \omega_{\mathrm{c}a} \qquad (14)$$

is species-independent. From this requirement, one obtains ω_α, through Eq. (9); ω_a^2 will be positive (and tend to zero, rendering the system free, when $\varpi \to 0$) only when all $\omega_{\mathrm{c}a}$ are positive.

The energy of a state (13) is

$$E_{\{\ell_{aj}\}} = \sum_a N_a\omega_{\mathrm{t}a} + \left[\sum_{aj} \ell_{aj} + \sum_a \frac{N_a(N_a-1)}{2}\alpha_{aa} + \right.$$

$$\left. + \frac{1}{2}\sum_{ab} N_a N_b \alpha_{ab} \right] \varpi , \qquad (15)$$

and the LLL spectrum is obtained by letting ℓ_{aj}'s run from 0 to ∞ with the restriction $\ell_{aj} \le \ell_{a,j+1}$. This makes it possible to calculate the many-body partition function, and ultimately the equation of state in the thermodynamic limit, which is [12]

$$\beta P = \sum_a \rho_{\mathrm{L}a} \ln\left(1 + \frac{\nu_a}{1 - \sum_b \alpha_{ab}\nu_{ba}} \right) . \qquad (16)$$

Here,

$$\rho_{\mathrm{L}a} = \frac{m_a\omega_{\mathrm{c}a}}{\pi} \qquad (17)$$

is the (species-dependent) multiplicity per unit area of the single-particle Landau level (which becomes degenerate in the thermodynamic limit $\varpi \to 0$),

$$\nu_a = \rho_a/\rho_{\mathrm{L}a} \qquad (18)$$

is the filling factor of the Landau level, and $\nu_{ba} = \rho_b/\rho_{\mathrm{L}a}$.

The wave functions given by Eq. (13) are, in general, valid only in the pure screening regime, $\alpha_{ab} \ge 0$; otherwise, they may be singular. For $\alpha_{ab} < 0$, the nonsingular two-body factor with the correct interchange behavior is $(\bar{z}_{aj} - \bar{z}_{bk})^{-\alpha_{ab}}$. However, as noted above, the wave functions depending on $\bar{z}_{aj}$ do not belong to the lowest Landau level. To obtain those that do, one infers a periodicity in the statistical parameters. Clearly, redefining $\alpha_{aa} \mapsto \alpha_{aa} + 2$ and $\alpha_{ab} \mapsto \alpha_{ab} + 1$ ($a \ne b$) leaves conditions (5) and (6) invariant. Therefore, a set of wave functions

$$\tilde{\psi}_{N\{\ell_{aj}\}} = \prod_{a,\, j<k} (z_{aj} - z_{ak})^{\alpha_{aa}+2} \times$$

$$\times \prod_{a<b,\, jk} (z_{aj} - z_{bk})^{\alpha_{ab}+1} \prod_a \left\{ \prod_j z_{aj}^{\ell_{aj}} \right\}_{\mathrm{sym}} \qquad (19)$$

with ℓ_{aj} defined as above forms the LLL basis of the eigenfunctions of (11) as long as $\alpha_{aa} \ge -2$ and $\alpha_{ab} \ge -1$. The corresponding eigenvalues of energy are

$$E_{\{\ell_{aj}\}} = \sum_a N_a\omega_{\mathrm{t}a} + \left[\sum_{aj} \ell_{aj} + \right.$$

$$\left. + \sum_a \frac{N_a(N_a-1)}{2}(\alpha_{aa}+2) + \frac{1}{2}\sum_{ab} N_a N_b(\alpha_{ab}+1) \right] \varpi ; \qquad (20)$$

proceeding in the same manner as in the screening regime yields the equation of state

$$\beta P = \sum_a \rho_{\mathrm{L}a} \times$$

$$\times \ln\left(1 + \frac{\nu_a}{1 - (\alpha_{aa}+2)\nu_a - \sum_{b\ne a}(\alpha_{ab}+1)\nu_{ba}} \right) . \qquad (21)$$

3. Analysis

Just like for a single species, one notes a striking contrast between the two regimes. It is known [12] that Eq. (16) describes the so-called Haldane exclusion statistics [13]. The latter is defined in terms of multiparticle Hilbert space dimension: It is postulated that adding a particle of species b to the system reduces the dimension of the Hilbert space (the number of states available) for particles of species a by g_{ab} (multispecies bosons and multispecies fermions corresponds to $g_{ab} = 0$ and $g_{ab} = \delta_{ab}$, respectively). It is possible to see both from the exact thermodynamic equations and a simple mean-field argument (replacing all the particle fluxes with an average magnetic field) that $g_{ab} = \alpha_{ab}$ in the screening regime. In particular, in the limit $\alpha_{ab} \to 0^+$, all correlations between different species vanish in Eq. (16) . With Eq. (21), however, the expressions for the exclusion statistics parameters become

$$g_{aa} = \alpha_{aa} + 2 \; ; \tag{22}$$

$$g_{ab} = \alpha_{ab} + 1 \qquad (a \neq b) \; . \tag{23}$$

In the limit $\alpha_{ab} \to 0^-$, adding a particle of a given species still excludes *two* states for all particles of the same species and *one* state for all particles of other species. That is to say, distinguishable particles start behaving like *fermions* for infinitesimally small (negative) α_{ab}.

The explanation of this apparent strangeness is the following. Consider what happens when an α_{ab} passes through zero. Any state of the form (19), which was valid when α_{ab} was negative, remains valid when it turns positive [it can be expressed as a linear combination of several states (13)]. The reverse is not true, however: It is only those states (and/or linear combinations thereof) of the form (13) for which the symmetrized factor vanishes at $z_{aj} \to z_{bk}$ that "survive" the change from $\alpha_{ab} \geq 0$ to $\alpha_{ab} < 0$ without changing their analytic form (the potential singularity getting cancelled). All the others start to depend on $\bar{z}_{aj}$ in addition to, or instead of, z_{aj} (see [11] for a complete analysis of the two-body case). Their energies will then contain terms proportional to $\alpha_{ab}B$. Since the energies of all the states, in fact, depend on α_{ab} continuously, so will do the exact equation of state at any finite B. However, if the $B \to \infty$ limit is taken first, then the above-mentioned states (detaching from the LLL) are to be thrown away. The discontinuity in the number of states belonging to the LLL implies a discontinuity in the thermodynamic properties.

The exact equation of state for an arbitrary magnetic field is apparently out of reach, as the general multibody problem cannot be solved. It is possible, however, to corroborate the above-described qualitative picture at the level of the second virial coefficient. The virial expansion for a multispecies equation of state is defined as

$$\beta P = \sum_{n_1 \ldots n_s} a_{n_1 \ldots n_s} \lambda^{2(n_1 + \ldots + n_s) - 1} \rho_1^{n_1} \cdots \rho_s^{n_s} \; , \tag{24}$$

where $\lambda = \sqrt{2\pi\beta/m}$ is the thermal wavelength. For two identical anyons in a magnetic field, the second virial coefficient is [14] ($\alpha \equiv \alpha_{11}$)

$$a_2(\alpha) = \frac{\lambda^2}{x} \left[-\frac{1}{4} \frac{1 - \mathrm{e}^{-2x}}{1 + \mathrm{e}^{-2x}} + \right.$$

$$\left. + \frac{\alpha}{2} + \frac{1 - \mathrm{e}^{2\alpha x}}{(1 + \mathrm{e}^{-2x})(1 - \mathrm{e}^{-2x})} + (\mathrm{e}^{x(|\alpha| + \alpha)} - 1) \right] \tag{25}$$

with $x = \beta\omega_c$. The mixed virial coefficient is given by [4]

$$a_{11}(\alpha_{12}) = a_2(\alpha_{12}) + a_2(\alpha_{12} + 1) \; , \tag{26}$$

so that any features of a_2 for $\alpha \in [0, 2]$ directly map onto the corresponding features of a_{11} for $\alpha \in [0, 1]$. For $x \gg 1$, one has

$$a_2(\alpha) = \frac{1}{2\rho_L}(-1 + 2\alpha) \qquad \text{for} \quad \alpha \in [0, 1] \; , \tag{27}$$

and

$$a_2(\alpha) = \frac{1}{2\rho_L} \left[-1 + 2\alpha + 4(1 - \mathrm{e}^{2\alpha x}) \right]$$

$$\text{for} \quad \alpha \in [-1, 0] \; , \tag{28}$$

which is continuous in α for any finite x. However, if the $x \to \infty$ limit is taken first, Eq. (28) becomes

$$a_2(\alpha) = \frac{1}{2\rho_L} (-1 + 2\alpha + 4) \qquad \text{for} \quad \alpha \in [-1, 0[\; , \tag{29}$$

which is the same as (27) with the replacement $\alpha \to \alpha + 2$. Now the continuity of $a_2(\alpha)$ across the point $\alpha = 0$ is broken.

A simple semiclassical argument supports this picture and explains relations (22), (23). For a single particle, the normal modes in the magnetic field in the presence of a harmonic potential are circular orbits with frequencies $\omega_t - \omega_c = \varpi$ and $-\omega_t - \omega_c$. The excitations of these modes correspond to the splitting of a Landau level and to different Landau levels, respectively [12]. The different signs correspond to different directions of rotation; consequently, it is only states with a positive angular momentum (that is, states whose wave functions depend on z and not $\bar{z}$) that belong to the lowest Landau level.

The single-particle LLL spectrum is $l = 0, 1, \dots$. The exclusion principle for fermions dictates that if one particle is in the $l = 0$ ground state, the next one must occupy the $l = 1$ (or higher) state. Now, the allowed values of relative angular momentum of two identical anyons are $\alpha + 2n$ with n integer according to Eq. (5). Semiclassically, therefore, if alpha is positive and the first particle sits in the ground state, the second one may go into the $l = \alpha$ state. In view of the aforesaid, this corresponds to "excluding α states" for the subsequent particle, so that the exclusion statistics parameter is $g = \alpha$. For two distinguishable anyons, the allowed values are $\alpha + n$, which leads to the same result. The situation, however, is drastically different for negative α. The $l = \alpha$ state now does not belong to the LLL. For identical anyons, the lowest allowed (within the LLL) value is $l = \alpha + 2$, which explains Eq. (22). For distinguishable anyons, it is $l = \alpha + 1$, explaining Eq. (23).

4. Conclusion

We have found the lowest Landau level spectrum and the equation of state for multispecies anyons in the antiscreening regime, when all anyon magnetic fluxes are parallel to the external magnetic field. The presence of the latter can be expected, of course, to break the mirror symmetry, but the effect turns out to be crucial even for infinitesimal fluxes. In the $B \to \infty$ limit, for a small α, identical and distinguishable anyons alike are little different from bosons in the screening regime; but in the antiscreening regime, distinguishable anyons start behaving like fermions, while identical anyons behave like "superfermions", with one particle excluding two states for the subsequent ones.

A question remaining open is what happens in the "mixed" regime, when some fluxes are parallel and some ones are antiparallel to the external field. Numerical simulations [15] seem to indicate that the exclusion statistics parameter matrix is not symmetric in this case, with one of the off-diagonal parameters actually being negative. The exact quantum-mechanical spectrum is not available in this case. Even the two-body problem appears to be solvable only numerically (since the center-of-mass motion does not separate); it should be possible, however, to at least use that numerical solution to make contact with the exclusion-statistics description.

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ЕНІОНИ БАГАТЬОХ СОРТІВ У МАГНІТНОМУ ПОЛІ

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Р е з ю м е

Вивчається система еніонів багатьох сортів в присутності зовнішнього магнітного поля, паралельного еніонним ааронов-бомівським магнітним потокам (режим антиекранування). Знайдено спектр розщепленого найнижчого рівня Ландау та рівняння стану; встановлено зв'язок із ексклюзійною статистикою. Показано, що еніони, що розрізняються, з інфінітезимальним статистичним параметром в режимі антиекранування поводять себе в певному сенсі подібно до ферміонів.