SYMMETRIES OF AN 8-COMPONENT EQUATION OF THE DIRAC—KÄHLER TYPE

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A complex analog of the Dirac—Kähler equation (CDK) as a system of 8 (but not 16) equations for 8 independent complex components with nonzero mass $m = \sqrt{\kappa_1 \kappa_2}$ is proposed. This equation is written in three Bose (2.11), (2.13), (2.14) and two Fermi (2.19), (2.24) forms. It is shown (Theorem 3) that, irrespective of $m \neq 0$, the CDK equation is invariant relative to the algebra $\tilde{A}_8$ of purely matrix transformations, whose 8×8-matrices are constructed from 4×4-matrices of the Pauli—Gürsey invariance algebra A_8 for the massless Dirac equation $\gamma\,\partial\psi = 0$. Six generators of the algebra $\tilde{A}_8$ generate the internal symmetry group for the CDK equation which can be identified with the isospin group SU(2) of the compound-field $\Psi = (\psi_1, \psi_2)$. It is shown (Theorems 4, 5) that the CDK equation (in any form) is invariant relative to two *nonequivalent representations* $\mathcal{P}^{\mathrm{S}}$ and $\mathcal{P}^{\mathrm{TSV}}$ of the Poincare group $\mathcal{P} \supset \mathcal{L}$ which are generated by the spinor $2\mathcal{L}^{\mathrm{S}}$ (3.20) and, respectively, tensor-scalar-vector $\mathcal{L}^{\mathrm{TSV}}$ (3.29) matrix representations of the Lorentz group $\mathcal{L}$. The operator connecting the Bose and Fermi forms of the CDK equation is found: by the action of this operator, the Fermi compound-field $\Psi = (\psi_1, \psi_2)$ is expressed through the system $\mathcal{F} = (\mathcal{B}^{\mu\nu}, \phi, V^\mu)$ of three $\mathcal{P}$-irreducible Bose fields. An equation of the CDK type is given (without any discussion) in the 5-dimensional Minkowski space.

1. Introduction

For recent years, a great interest is paid to the equation Dirac—Kähler (DK) (see review [1] and references therein) which has a long history and is called the equation Ivanenko—Landau—Kähler by certain authors [2, 3]. In the language of differential forms, the DK equation looks as

$$(d - \delta + m)\Phi = 0,$$

$$\Phi = \sum_0^4 \varphi_{\mu_1 \ldots \mu_p} dx^{\mu_1} \Lambda \cdots \Lambda dx^{\mu_p}, \qquad (1.1)$$

where the components $\varphi_{\mu_1 \ldots \mu_p}$ are skew-symmetric tensors of rank p. As shown in many works (e.g., [3—5]), it splits into four Dirac equations

$$(i\gamma^\mu \partial_\mu - m)\psi_{(b)} = 0; \;\; \psi_{(b)} = (\psi^a_{(b)}), \;\; a, b = \overline{1,4}, \qquad (1.2)$$

with the irreducible Dirac matrices γ^μ (i.e., it can be written as

$$(i\Gamma^\mu_{(16)} \partial_\mu - m)\Psi = 0,$$

$$\Psi = \mathrm{column}(\psi^a_{(1)}, \ldots, \psi^a_{(4)})^4_{a=1}, \qquad (1.3)$$

where $\Gamma^\mu_{(16)}$ are the corresponding reducible Dirac 16×16-matrices).

The interaction of the multicomponent Dirac—Kähler field with the electromagnetic field of potentials A^μ is introduced in the standard way ($\partial_\mu \to \partial_\mu - ieA_\mu$), like the transition to quasirelativistic approximations. This stimulated the application of the DK equation to specific problems in certain field and quantum-mechanical models (e.g., [6—7]). In particular, work [6] indicates the advantages of the use of the DK equation in the problems, to which the Duffin — Kemmer — Petiau equation or Proca equation was earlier applied. The DK equation was widely used (e.g., [8—9]) for the construction of the theory of fermions on lattices. In particular, the representations of the corresponding symmetry groups of Fermi fields on lattices were constructed on the basis of representations of the symmetry groups of the DK equation [8] and were used in quantum chromodynamics.

In addition, we mention various generalizations of the DK equation with the purpose of a significant increase in the number of components of the field $\Phi \sim \Psi$ [10—11] [i.e. in the dimension of matrices Γ^{μ} in (1.2)] and the consideration of both the DK equations in spaces of higher dimensionalities [3] and their relation to the DK equation in the form (1.1)=(1.3). Some works (e.g., [2,14]) consider the problems of the general relativity theory with the use of equations of the DK type. While performing the group-based and other analyses of the DK equation and its modifications, the various general theoretical and conceptual questions were discussed, including a distinctive revision [13] of the conception of the spin of a Dirac fermion.

In the present work, we give attention to the expediency to consider an equation „more fundamental" in a certain sense than the DK equation (1.1)=(1.3). By taking the simplest version of the DK equation where the tensors $\varphi_{\mu_1 \dots \mu_p}$ in Eq. (B.1) are real (real-valued) functions in the 4-dimensional Minkowski space M(1,3) (i.e., when the DK equation (B.1) is a system of 16 differential first-order equations for 16 real functions in M(1,3)), we will construct, with the proper justification, a simpler 8-component equation of the DK type for 8 complex functions (see the CDK equation (2.11) below and its different forms). This equation is self-sufficient in the sense that its complete analysis (in particular, the analysis of the symmetries of this equation and their consequences, its solutions in various quantum-mechanical bases of a certain physical content, the types of quantization, and other interrelated aspects) does not require the embedding of the CDK equation into more general schemes (e.g., into equations with the greater number of components or into the models of equations with higher spatial dimensionalities). We do not see any obstacles in the practical use of this equation for many problems, in which the DK equation (more complicated and "less fundamental") was used. Therefore, it is actual to carry out the complete analysis of the CDK equation (2.11) on the modern axiomatic level and with the same detailing, as this made for the ordinary Dirac equation, for example in [14] (but such an analysis cannot be executed in the scope of the present paper).

The main purpose of this paper is to analyze, in the first turn, the general theoretical conceptual questions. To avoid the indefiniteness and ambiguities, we will carry out, in Section 3, the detailed, transparent, and clear group analysis (in fact, on the level of the axiomatic approach to the theory of fields and particles) of the CDK equation by the Bargman—Wigner method which, in particular, adequately identifies the mass and spin of the field (and of its components) satisfying some equation (a system of equations), being invariant relative to *a certain representation* of the Poincare group. In Section 2, we present various forms of the CDK equation which illustrate the propositions of the proved theorems on the groups of symmetries of the CDK equation. It is most interesting that, despite $m \neq 0$, the operators of essentially different Poincare-symmetries of the CDK equation (Bose and Fermi ones), as well as those of its internal symmetry (not connected with transformations in the Minkowski space), are expressed in terms of the corresponding elements of the symmetries of the *massless* Dirac equation (a short systemized description of its symmetries is given in Section 2.2 together with a new result on the connection of the standard spin matrices with the matrices of the internal symmetry).

2. Definition of an 8-component Equation and Its Different Forms

2.1. Main definitions and clarifying considerations

We will use the real Cartesian (contravariant) coordinates $x^{\mu}, \mu = 0,1,2,3 \equiv \overline{0,3}, x^0 = ct$, for 4-vectors $x \equiv (x^{\mu}) \in$ M(1,3); the metric tensor $g^{\mu\nu} = g_{\mu\nu}, g = \mathrm{diag}(1,-1,-1,-1)$; the Levi—Civita tensors $\varepsilon^{\mu\nu\rho\sigma}$ and ε^{jkl} with norms $\varepsilon^{0123} = \varepsilon^{123} = +1$; the summation rule over repeated Greek (upper and lower ones) and Latin indices; the d'Alembert operator $\Box \equiv \partial^{\mu}\partial_{\mu} = \partial_0^2 - \Delta$.

For real skew-symmetric tensors $\varphi_{\mu_1 \dots \mu_4}$ of a proper rank and a variance, it is convenient to write Eq. (1.1) in the explicitly covariant form as the system of equations

$$\begin{cases} \partial_{\nu} B^{\mu\nu} + \partial^{\mu}\varphi + m^2 A^{\mu} = 0, \quad \partial_{\mu} A^{\mu} = \varphi, \\ \partial_{\nu} \varepsilon B^{\mu\nu} + \partial^{\mu}\tilde{\varphi} + m^2 \tilde{A}^{\mu} = 0, \quad \partial_{\mu}\tilde{A}^{\mu} = \tilde{\varphi}, \\ B_{\mu\nu} + \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - \varepsilon_{\mu\nu\rho\sigma}\partial^{\rho}\tilde{A}^{\sigma} = 0 \end{cases} \tag{2.1}$$

for a scalar φ, pseudoscalar $\tilde{\varphi}$, vector A^{μ}, pseudovector $\tilde{A}^{\mu}$, antisymmetric tensor

$$B^{\mu\nu} = -B^{\nu\mu}; \quad \varepsilon B^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} B_{\rho\sigma},$$

$$\varepsilon B^{01} = B^{23}, \quad \varepsilon B^{12} = -B^{01}, \quad \text{etc.,} \tag{2.2}$$

i.e., for the polycovariant (compound-field)

$$F = (\varphi, \tilde{\varphi}, A^{\mu}, \tilde{A}^{\mu}, B^{\mu\nu}) \tag{2.3}$$

as a collection of the indicated independent tensors and pseudotensors. That is, we have the system of 16 equations (which are not repeated) for 16 independent components of the tensors in collection (2.3). This system is invariant relative to a representation of the universal covering $\mathcal{P}$ of the proper orthochronous Poincare group $\mathrm{P}_+^\uparrow = \mathrm{T}(4)\times)\mathrm{L}_+^\uparrow$ generated by the representation

$$(0,0)\oplus(0,0)\oplus(0,1)\oplus(1,0)\oplus(\tfrac{1}{2},\tfrac{1}{2})\oplus(\tfrac{1}{2},\tfrac{1}{2}) \qquad (2.4)$$

of the universal covering $\mathcal{L} = \mathrm{SL}(2,\mathrm{C})$ of the proper orthochronous Lorentz group $\mathrm{L}_+^\uparrow \subset \mathrm{O}(1,3)$.

It becomes clear from (2.4) (see also [1]) that the set of independent (basic) solutions of the system of equations (2.1) for the compound-field $F(2.3)$ contains the spin states which are repeated twice for bosons described by the fields indicated in (2.3). Here, we are faced with an analogy with the Maxwell equations

$$\partial_\nu B^{\mu\nu} = j^\mu, \quad \partial_\nu \varepsilon B^{\mu\nu} = 0; \quad \partial_\nu \equiv \partial/\partial x^\nu, \qquad (2.5)$$

for the tensor of intensities

$$B \equiv (B^{\mu\nu}): \quad B^{oj} = -B^{jo} = E^j, \ \ B^{jl} = \varepsilon^{jln} H^n, \qquad (2.6)$$

In terms of the vectors of electric $\vec{E} = (E^j)$ and magnetic $\vec{H} = (H^j)$ intensities (in the Gauss corrected system of units), these equations take the form

$$\left\{ \begin{array}{ll} \partial_0 \vec{E} = \mathrm{rot}\vec{H} - \vec{j}, & \mathrm{div}\vec{E} = \rho, \\ \partial_0 \vec{H} = -\mathrm{rot}\vec{E}; & (\vec{j},\rho) \equiv (j^\mu). \end{array} \right. \qquad (2.5\mathrm{a})$$

Indeed, Eq. (2.5)—(2.5a) is invariant [at $j^\mu(x) \equiv 0$] relative to the $\mathcal{P}$-representation generated by the reducible representation $(0,1)\oplus(1,0)$ of the group $\mathcal{L}$. However, in terms of the complex tensor

$$\mathcal{B} = B - i\varepsilon B \equiv (\mathcal{B}^{\mu\nu}): \quad \mathcal{B}^{oj} = \mathcal{E}^j \equiv E^j - iH^j, \qquad (2.7)$$

which is self-dual in the sense that

$$\varepsilon\, \mathcal{B}^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{B}_{\rho\sigma} = i\mathcal{B}^{\mu\nu} \Leftrightarrow$$

$$\Leftrightarrow \quad \varepsilon\, \vec{\mathcal{E}} \equiv \varepsilon\vec{E} - i\varepsilon \vec{\mathrm{H}} = i\vec{\mathcal{E}}, \qquad (2.8)$$

the Maxwell equations (2.5) have also an explicitly covariant form

$$\partial_\nu \mathcal{B}^{\mu\nu} = j^\mu, \qquad (2.9)$$

In terms of a 3-component complex function $\vec{\mathcal{E}} \equiv (\mathcal{E}^j)$, they look as

$$\partial_0 \vec{\mathcal{E}} = i\mathrm{rot}\vec{\mathcal{E}} + \vec{j}, \quad \mathrm{div}\vec{\mathcal{E}} = \rho. \qquad (2.9\mathrm{a})$$

Moreover, the complex field $\mathcal{B} \equiv (\mathcal{B}^{\mu\nu})$ or $\vec{\mathcal{E}} = \vec{E} - i\vec{H}$ (which was used, in fact, by Oppenheimer) is an irreducible $\mathcal{P}$-covariant: it is transformed according to an irreducible $\mathcal{P}$-representation generated by the irreducible (0,1)-representation of the group $\mathcal{L}$. (This $\mathcal{P}$-representation [at $j = (\rho,\vec{j}) \equiv 0$] corresponds to the invariance group of Eqs. (2.9)=(2.9a)). Therefore, just the *complex field* $\mathcal{B}$ or $\mathcal{E}$, rather than B or $(\vec{E},\vec{H})$, should be considered as a *photon field in terms of the intensities.* In this sense, Eq. (2.9)=(2.9a) for the complex electromagnetic field (2.7) is more fundamental, than the historically primary Maxwell equations (2.5a)=(2.5).

2.2 Bose forms of the CDK equation

In view of the mentioned analogy, we define the components of a complex compound-field in terms of the components of compound-field $F(2.3)$ as

$$\mathcal{F} \equiv (\phi, V^\mu, \mathcal{B}^{\mu\nu}): \quad \phi = \varphi - i\tilde{\varphi},$$

$$V^\mu = A^\mu - i\tilde{A}^\mu, \ \ \mathcal{B}^{\mu\nu} = B^{\mu\nu} - i\varepsilon B^{\mu\nu}. \qquad (2.10)$$

The system of equations (2.1) yields the following system of equations in the explicitly covariant form for the complex components of compound-field $\mathcal{F}$ (2.10):

$$\left\{ \begin{array}{l} \partial_\nu \mathcal{B}^{\mu\nu} + \partial^\mu\phi + m^2 V^\mu = 0, \quad \partial_\mu V^\mu = \phi, \\ \mathcal{B}^{\mu\nu} + \partial^\mu V^\nu - \partial^\nu V^\mu - i\varepsilon^{\mu\nu o\sigma}\partial_\rho V_\sigma = 0. \end{array} \right. \qquad (2.11)$$

This system of equations can be written without repeated equations, i.e., in the form of the system of 8 equations for 8 independent complex components of the irreducible $\mathcal{P}$-covariants

$$\mathcal{F} \equiv (\vec{\mathcal{E}} = (\mathcal{E}^j), \phi, V = (\vec{V}, V^0)), \qquad (2.12)$$

namely:

$$\left\{ \begin{array}{l} \partial_0\vec{\mathcal{E}} - i\mathrm{rot}\vec{\mathcal{E}} + \mathrm{grad}\phi - m^2\vec{V} = 0, \\ \partial_0\vec{V} + i\mathrm{rot}\vec{V} + \mathrm{grad}V^{\ddot{o}} + \vec{\mathcal{E}} = 0, \\ \partial_0\phi + \mathrm{div}\vec{\mathcal{E}} + m^2 V^0 = 0, \\ \partial_0 V^0 + \mathrm{div}\vec{V} + \phi = 0. \end{array} \right. \qquad (2.13)$$

It is useful to write the last system of equations in a matrix form for the column composed from the corresponding components of the compound-field $\mathcal{F}$ (2.12) as

$$\left. \begin{array}{l} \alpha\partial\mathcal{E} - m^2 V = 0 \\ \overset{\times}{\alpha}\partial V + \mathcal{E} = 0 \end{array} \right\} \Leftrightarrow \qquad (2.14)$$

$$\begin{bmatrix} \alpha\partial & -m^2 \\ \mathrm{I}_4 & \overset{\times}{\alpha}\partial \end{bmatrix} \mathcal{F} = 0, \quad \mathcal{F} \equiv \begin{bmatrix} \mathcal{E} \\ V \end{bmatrix}, \qquad (2.14a)$$

where I_4 is the unit 4×4-matrix, $\overset{\times}{\alpha} \equiv C\alpha C$, C is the operator of complex conjugation, $C\mathcal{F} \equiv \mathcal{F}^*$,

$$\alpha\partial \equiv \alpha^\mu\partial_\mu = \begin{bmatrix} \partial_0 & i\partial_3 & -i\partial_2 & \partial_1 \\ -i\partial_3 & \partial_0 & i\partial_1 & \partial_2 \\ i\partial_2 & -i\partial_1 & \partial_0 & \partial_3 \\ -\partial_1 & -\partial_2 & -\partial_3 & -\partial_0 \end{bmatrix}, \qquad (2.15)$$

$$\mathcal{E} = \begin{bmatrix} \vec{\mathcal{E}} \\ \phi \end{bmatrix} \equiv \begin{bmatrix} \mathcal{E}^1 \\ \mathcal{E}^2 \\ \mathcal{E}^3 \\ \phi \end{bmatrix}, \quad V = (V^\mu) \equiv \begin{bmatrix} V^1 \\ V^2 \\ V^3 \\ V^0 \end{bmatrix} \qquad (2.16)$$

(the explicit form of the matrices α^μ and $\overset{\times}{\alpha}{}^\mu$ becomes clear in view of the form (2.15) of the operator $\alpha^\mu\partial_\mu$ and the definition $\overset{\times}{\alpha} \equiv C\alpha C$). The 8-component equation (2.14)=(2.14a) or Eq. (2.11)=(2.13) in the componentwise form will be called further the *complex equation* Dirac—Kähler (CDK) in the Bose form.

2.3. Fermi form of the CDK equation

In terms of the compound-field $\mathcal{F}$ (2.12), we define two 4-component fields $\chi_r = (\chi_r^\alpha)_{\alpha=1}^4$ $(r = 1, 2)$ of the same dimensionality as

$$\chi_1^j = \mathcal{E}^j, \quad \chi_1^4 = \phi;$$

$$\chi_2^j = \kappa_2 V^j, \quad \chi_2^4 = \kappa_2 V^0, \qquad (2.17)$$

where $\kappa_2 \neq 0$ is a real positive constant with the dimension of mass (in units $\hbar = c = 1$). By using the matrices α^μ given by equalities (2.15), we define five matrices $\tilde{\gamma}^{\tilde{\mu}}$, $\tilde{\mu} = \overline{0,4}$, as

$$\tilde{\gamma}^\mu = C\alpha^\mu, \quad \tilde{\gamma}^4 = \alpha^0\alpha^1\overset{\times}{\alpha}{}^2\alpha^3. \qquad (2.18)$$

In these notations, Eqs. (2.14)=(2.14a) (after the multiplication of the first equation in (2.14) by the operator C) take the form

$$\left.\begin{aligned} \tilde{\gamma}\partial\chi_1 - \kappa_1\chi_2 = 0, \\ \tilde{\gamma}\partial\chi_2 + \kappa_2\chi_1 = 0, \end{aligned}\right\} \Leftrightarrow (\tilde{\Gamma}\partial - M)\tilde{\chi} = 0, \qquad (2.19)$$

where

$$\kappa_1 \equiv m^2/\kappa_2, \quad \tilde{\gamma}\partial \equiv \tilde{\gamma}^\mu\partial_\mu, \;\; \tilde{\Gamma}\partial \equiv \tilde{\Gamma}^\mu\partial_\mu,$$

$$\tilde{\chi} \equiv \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}, \;\; M \equiv \begin{bmatrix} 0 & \kappa_1 \\ -\kappa_2 & 0 \end{bmatrix}, \;\; \tilde{\Gamma}^\mu \equiv \begin{bmatrix} \tilde{\gamma}^\mu & 0 \\ 0 & \tilde{\gamma}^\mu \end{bmatrix}.$$

Using (2.15) and (2.18), we get the matrices $\tilde{\gamma}^{\tilde{\mu}}$ in the explicit form as

$$\tilde{\gamma}^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} C, \quad \tilde{\gamma}^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} C,$$

$$\tilde{\gamma}^2 = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} C, \quad \tilde{\gamma}^3 = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} C,$$

$$\tilde{\gamma}^4 \equiv \tilde{\gamma}^{0123} = i. \qquad (2.20)$$

These matrices were introduced in [15,16] and used upon the study of the interrelations between the massless Dirac equation and the system of equations for coupled electromagnetic and scalar fields. They satisfy the standard Clifford—Dirac (CD) commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \qquad (2.21)$$

Under the action of the nonsingular operator [15]

$$W = \begin{vmatrix} 0 & 0 & C_+ & C_- \\ C_+ & iC_+ & 0 & 0 \\ 0 & 0 & C_- & C_+ \\ C_- & iC_- & 0 & 0 \end{vmatrix}, \quad C_\pm \equiv \tfrac{1}{2}(C \pm 1), \qquad (2.22)$$

matrices $\tilde{\gamma}^\mu$ (2.20) are transformed into the Dirac matrices $\gamma^\mu = W\tilde{\gamma}^\mu W^{-1}$ in the standard Pauli—Dirac (PD) representation, and the transformation

$$\mathcal{E} \to \psi_1 = W\mathcal{E}, \quad V \to \psi_2 = \kappa_2 WV \qquad (2.23)$$

transforms Eq. (2.14)=(2.14a) or (2.19) into the equation

$$\left.\begin{aligned} \gamma\partial\psi_1 - \kappa_1\psi_2 = 0 \\ \gamma\partial\psi_2 + \kappa_2\psi_1 = 0 \end{aligned}\right\} \Leftrightarrow (\Gamma\partial - M)\Psi = 0, \qquad (2.24)$$

where

$$\Psi \equiv \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \Gamma\partial \equiv \Gamma^\mu\partial_\mu, \quad \Gamma^\mu \equiv \begin{vmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{vmatrix}, \qquad (2.24a)$$

and γ^μ are the Dirac matrices in the PD-representation. Based on the clear motives, we call Eq. (2.24), as distinct from Eq. (2.14a) [and its forms (2.14), (2.13), and (2.11)], as the CDK equation in the (standard) *Fermi form* and call matrices $\tilde{\gamma}^\mu$ (2.20) as the Dirac matrices in the *Bose representation* (briefly, the B-*representation*). The CDK equations in any form yield the equalities

$$(\Box + m^2)\mathcal{F} = 0, \quad (\Box + m^2)\psi_{1,2} = 0; \tag{2.25}$$

$$\Box \equiv \partial^\mu \partial_\mu, \quad m^2 \equiv \kappa_1 \kappa_2 > 0.$$

That is, each from the $\mathcal{P}$-covariants which belongs to the compound-field $\mathcal{F}$ or Ψ is a field with mass

$$m = \sqrt{\kappa_1 \kappa_2} > 0.$$

for any values of the parameters κ_1 and κ_2.

R e m a r k 1. Upon the derivation of equalities (2.25), the *different signs* of κ_1, κ_2 in Eqs. (2.24) *play the decisive role:* at the same signs of κ_1, κ_2 in Eqs. (2.24), the fields ψ_1, ψ_2 would satisfy the equations $(\Box - m^2)\psi_{1,2} = 0$, i.e. would be fields with imaginary mass.

T h e o r e m 1. The *nonunitary nonsingular operator*

$$V \equiv \frac{1}{\sqrt{m}} \begin{vmatrix} \sqrt{\kappa_2} & 0 \\ 0 & \sqrt{\kappa_1} \end{vmatrix} \tag{2.26}$$

equalizes the mass parameters κ_1, κ_2 in Eq. (2.24), and the nonunitary nonsingular operator

$$\widehat{V} = UV; \quad U \equiv \frac{1}{\sqrt{2}} \begin{vmatrix} i & 1 \\ -i & 1 \end{vmatrix} \tag{2.27}$$

splits Eq. (2.24) into a subsystem of two independent Dirac equations

$$(i\gamma\,\partial \mp m)\psi_\mp = 0. \tag{2.28}$$

P r o o f of this theorem is carried out by the direct calculation of the corresponding transformed quantities, **qed.**

The CDK equation under study differs from the ordinary DK equation (2.2) (or from its Fermi form) at several essential points. First, as was noted in Introduction, the ordinary DK equation in the Fermi form is split [4,5] into four Dirac equations (1.2), whereas the CDK equation (2.2) is split (moreover, by the transformation nonunitary at $\kappa_1 \neq \kappa_2$) only into two Dirac equations (2.28). This means that Eq. (2.24) is not equivalent to the system of two independent equations (2.28) at $\kappa_1 \neq \kappa_2$. In this case, the different mass parameters κ_1, κ_2 define the „relative share" of the 4-component quantities ψ_1, ψ_2 in the 8-component column Ψ, which can be significant in the presence of an interaction of the field Ψ with other fields.

There is some analogy with the "large" and "small" components inthe ordinary stationary Dirac equation in the presence of the interaction with an external field. There is also a certain analogy of the factors κ_1, κ_2 with the dielectric permittivity and magnetic permeability for the electromagnetic field $(\vec{E}, \vec{H})$ in media, at the expense of which the magnetic component $\mu\vec{H}$ is small as compared to the electric one $\varepsilon\vec{E}$ of the electromagnetic field in the medium.

Finally, the CDK equation contains 8, rather than 16 components. At this point, the 8-component CDK equation (2.11)$\sim$(2.24) is more fundamental, than the 16-component DK equation (2.1) $\sim$(1.3), as the 4-component Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ with the irreducible matrices γ^μ is more fundamental, than the equation derived from the last by the substitution $\psi = \mathrm{Re}\psi + i\mathrm{Im}\psi$ with the following transition to a 8-component equation for the system of fields $(\mathrm{Re}\psi, \mathrm{Im}\psi)$. The symmetry properties of the CDK equation with arbitrary parameters κ_1, κ_2 and $m = \sqrt{\kappa_1\kappa_2}$ are expressed through the constructive elements of the symmetries of the Dirac massless equation which are given below in Section 3.2.

3. Invariance Groups of the CDK Equation

3.1.

First, we present the necessary information about the infinitesimal representations of the groups $\mathcal{P} \supset \mathcal{L}$ and $\mathcal{L}$.

The local $\mathcal{P}$-transformations of any N-component covariant $\mathcal{A} \equiv (\mathcal{A}^n), n = \overline{1, N}$, of the group $\mathcal{P}$ have the form

$$\mathcal{A}(x) \to \mathcal{A}'(x) = F(\omega)\mathcal{A}(\Delta^{-1}(x - a)) \overset{\mathrm{i}}{=}$$

$$\overset{\mathrm{i}}{=} (\mathrm{I}_N - a^\rho \partial_\rho - \tfrac{1}{2}\omega^{\rho\sigma} j^{(\mathcal{A})}_{\rho\sigma})\mathcal{A}(x), \tag{3.1}$$

where $a \equiv (a^\rho$ and $\omega^{\rho\sigma} = -\omega^{\sigma\rho}$ are real parameters of the group $\mathcal{P}$ with the commonly known physical content, the symbol „ i " above the sign of equality means „infinitesimally" (i.e., in a vicinity of the unity of the group $\mathcal{P}$),

$$F(\omega) \overset{\mathrm{i}}{=} \mathrm{I}_N - \tfrac{1}{2}\omega^{\rho\sigma} s^{(\mathcal{A})}_{\rho\sigma} \tag{3.2}$$

is the N-dimensional matrix representation of the group $\mathcal{L}$, I_N is the unit $N\times N$-matrix,

$$\Lambda \stackrel{\mathrm{i}}{=} \mathrm{I}_4 - \tfrac{1}{2}\omega^{\rho\sigma} s^{(\mathrm{V})}_{\rho\sigma}; \tag{3.3}$$

and the matrices $\mathrm{s}^{(\mathrm{V})}_{\rho\sigma}$ and $\mathrm{s}^{(\mathcal{A})}_{\rho\sigma}$ satisfy the commutation relations

$$[s_{\mu\nu}, s_{\rho\sigma}] = -g_{(\mu\rho} s_{\nu\sigma)} \equiv$$

$$\equiv -g_{\mu\rho}s_{\nu\sigma} - g_{\rho\nu}s_{\sigma\mu} - g_{\nu\sigma}s_{\mu\rho} - g_{\sigma\mu}s_{\rho\nu}. \tag{3.4}$$

Moreover, the matrices $\mathrm{s}^{(\mathrm{V})}_{\rho\nu}$ generate a vector $(\frac{1}{2},\frac{1}{2})$-representation of the group $\mathcal{L}$, and the operators

$$\partial_\rho = \partial/\partial x^\rho,\ j^{(\mathcal{A})}_{\rho\nu} = m_{\rho\sigma} + s^{(\mathcal{A})}_{\rho\sigma}$$

$$m_{\rho\sigma} \equiv x_\rho\partial_\sigma - x_\sigma\partial_\rho \tag{3.5}$$

satisfy the commutation relations for the $\mathcal{P}$-generators in the covariant form

$$[\partial_\rho, \partial_\sigma] = 0,$$

$$[\partial_\rho, j_{\mu\nu}] = g_{\rho\mu}\partial_\nu - g_{\rho\nu}\partial_\mu, \tag{3.6a}$$

$$[j_{\mu\nu}, j_{\rho\sigma}] = -g_{\mu\rho}j_{\nu\sigma} - g_{\rho\nu}j_{\sigma\mu} -$$

$$-g_{\nu\sigma}j_{\mu\rho} - g_{\sigma\mu}j_{\rho\nu}. \tag{3.6b}$$

The Casimir operator $\hat{w}$, being the square of the Pauli—Lubanski vector w^μ, is defined here as

$$\hat{w} \equiv w_\mu w^\mu; \quad w^\mu \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} j^{(\mathcal{A})}_{\nu\rho}\partial_\sigma. \tag{3.7}$$

For the sake of definiteness, we start from form (2.24) of the CDK equation (i.e., in the PD-representation of the matrices γ^μ) and write the formulas for all the necessary matrices, namely the Lorentz spins $s^{(\mathcal{A})}_{\rho\sigma}$, with the necessary comments. First, we recall that the relativistic group of invariance of the Dirac massless equation (like the equation with $m \neq 0$) is defined by the spinor representation $\mathcal{P}^{\mathrm{S}}$ of the group $\mathcal{P}$ according to formulas (3.1), (3.2), and (3.5), in which the matrices $s^{(\mathcal{A})}_{\rho\sigma}$ with $\mathcal{A} = \mathrm{S}$ look as

$$s^{(\mathrm{S})}_{\rho\sigma} = \tfrac{1}{2}[\gamma_\rho, \gamma_\sigma] \tag{3.8}$$

and generate the reducilble spinor representation

$$\mathcal{L}^{\mathrm{S}} = \left(0, \tfrac{1}{2}\right) \oplus \left(\tfrac{1}{2}, 0\right) \tag{3.9}$$

of the group $\mathcal{L}$ by formula (3.2).

3.2. On the additional symmetry of the dirac massless equation

The massless equation

$$\gamma^\mu\partial_\mu\psi = 0 \tag{3.10}$$

has, besides three Poincare-symmetries (see [16]), an additional symmetry, being invariant (see [17]) relative to the Pauli — Gürsey algebra A_8, whose generators can be written conveniently in the form

$$\hat{s}_{\rho\sigma} = -\hat{s}_{\sigma\rho}: \quad \hat{s}_{01} = -\tfrac{i}{2}\gamma_2 C,$$

$$\hat{s}_{02} = \tfrac{i}{2}\gamma_2 C, \quad \hat{s}_{03} = -\tfrac{i}{2}\gamma_4, \quad \hat{s}_{jk} = \varepsilon^{jkl}\gamma_4\hat{s}_{0l}; \tag{3.11}$$

$$\gamma_4 \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.11a}$$

These matrices satisfy the equalities

$$\gamma_4 \cdot \gamma\partial = -\gamma\partial \cdot \gamma_4, \quad \hat{s}_{0k} \cdot \gamma\partial = -\gamma\partial \cdot \hat{s}_{0k},$$

$$\hat{s}_{jk} \cdot \gamma\partial = \gamma\partial \cdot \hat{s}_{jk}. \tag{3.12}$$

Further, as was shown by Theorem 1 in [16], the matrices $\hat{s}_{\rho\sigma}$ (3.11) satisfy the commutation relations (3.4) and generate the same $\mathcal{L}$-representation $\mathcal{L}^{\mathrm{S}}$ (3.9) as the matrices $s^{(\mathrm{S})}_{\rho\sigma}$ (3.8). However, the standard spin matrices (3.8), as distinct from matrices (3.11), are not the invariance transformations of Eq. (3.10). They are only the operators $j_{\rho\sigma} = m_{\rho\sigma} + s^{(\mathrm{S})}_{\rho\sigma}$ *of the total Lorentz moment*. In this connection, the following theorem is of interest.

T h e o r e m 2. *Matrices $\hat{s}_{\rho\sigma}$ (3.11) and $\mathrm{s}^{(\mathrm{S})}_{\rho\sigma}$ (3.8) are connected by the operator $\hat{C}$:*

$$\hat{s}_{\rho\sigma} = \hat{C}s^{(\mathrm{S})}_{\rho\sigma}\hat{C} = \tfrac{1}{4}[\hat{\gamma}_\rho, \hat{\gamma}_\sigma];$$

$$\hat{\gamma}_\rho \equiv \hat{C}\gamma_\rho\hat{C}; \quad \hat{C}^2 = I; \tag{3.13}$$

the explicit forms of the operator $\hat{C}$ and matrices $\hat{\gamma}_\rho$ in the PD-representation of the matrices γ_ρ (where $\gamma_{\rho\sigma} \equiv \gamma_\rho\gamma_\sigma$) are as follows:

$$\hat{C} = \begin{bmatrix} C & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \hat{\gamma}_0 = \gamma_0,$$

$$\hat{\gamma}_1 = -i\gamma_{02}C, \quad \hat{\gamma}_2 = \gamma_{02}C, \quad \hat{\gamma}_3 = i\gamma_{04}. \tag{3.14}$$

In the real CD algebra, the matrix $\hat{C}$ is unitary, and the matrices $\hat{\gamma}_\rho$ from (3.14) *satisfy the standard CD relations* (2.21).

P r o o f of this theorem is performed by the direct calculations of the corresponding equalities and relations with the use of the explicit form of the matrices γ_ρ in the PD-representation, **qed.**

Thus, the matrix $\hat{C}$ in (3.14) has the following sense: it transforms the operator of the $\mathcal{L}$-spin $s^{(\mathrm{S})}_{\rho\sigma}$(3.8) [this operator by itself is not an invariance transformation for Eq. (3.10)] into the operator of the internal $\mathcal{L}$-spin $\hat{s}_{\rho\sigma}$ (3.11) which is already, due to equalities (3.12), the *invariance transformation* for the Dirac massless equation (3.10). In this case, these invariance transformations are, firstly, of a purely matrix form (though they include the operator C of complex conjugation) and are not connected with transformations of the argument x of the spinor ψ (therefore, $\hat{s}_{\rho\sigma}$ (3.11) are called „internal" $\mathcal{L}$-spin). Secondly, most interesting is the fact that the matrices $\hat{s}_{\rho\sigma}$ and $\hat{\gamma}_\mu$ *commute* with all matrices $s^{(\mathrm{S})}_{\rho\sigma}$ and γ_μ:

$$\left[\hat{s}_{\rho\sigma}, s^{(\mathrm{S})}_{\mu\nu}\right] = 0 = [\hat{\gamma}_\mu, \gamma_\rho]. \tag{3.15}$$

This presents the possibility to construct the generators of local *Bose* $\mathcal{P}$-symmetries of the Dirac massless equation (3.10) with the use of $\mathcal{L}$-спінів $\hat{s}_{\rho\sigma}$ and $s^{(\mathrm{S})}_{\rho\sigma}$.

T h e o r e m [5]. *The matrices* $s^{(\mathrm{TS})}_{\rho\sigma}$ and $s^{(\mathrm{V})}_{\rho\sigma}$ *defined as*

$$s^{(\mathrm{TS})}_{0k} \equiv \tfrac{1}{2}(s^{(\mathrm{S})}_{0k} - \hat{s}_{0k}) = -s^{(\mathrm{TS})}_{k0},$$

$$s^{(\mathrm{TS})}_{jk} \equiv \tfrac{1}{2}(s^{(\mathrm{S})}_{jk} - \hat{s}_{jk}), \tag{3.16a}$$

$$s^{(\mathrm{V})}_{\rho\sigma} \equiv \tfrac{1}{2}(s^{(\mathrm{S})}_{\rho\sigma} + \hat{s}_{\rho\sigma}) \tag{3.16b}$$

satisfy relations (3.4) *and generate the tensor-scalar and, respectively, vector representations of the group $\mathcal{L}$:*

$$s^{(\mathrm{TS})}_{\rho\sigma} \in \mathcal{L}^{\mathrm{TS}} \equiv (0,1) \oplus (0,0); \tag{3.17a}$$

$$s^{(\mathrm{V})}_{\rho\sigma} \in \mathcal{L}^{\mathrm{V}} \equiv (\tfrac{1}{2}, \tfrac{1}{2}). \tag{3.17b}$$

Using this fact, it was shown (see Theorem 2 in [16]) that Eq. (3.10) is invariant not only relative to the Fermi (standard spinor) $\mathcal{P}$-representation, whose generators are set by the Lie operators (3.5) with $s^{(\mathcal{A})}_{\rho\sigma} = s^{(\mathrm{S})}_{\rho\sigma}$, but also relative to two Bose $\mathcal{P}$-representations, whose generators have the form (3.5) with $s^{(\mathcal{A})}_{\rho\sigma} = s^{(\mathrm{TS})}_{\rho\sigma}, s^{(\mathrm{V})}_{\rho\sigma}$.

The presented information about the Dirac massless equation (3.10) makes the analysis of symmetries of the CDK equation with arbitrary mass parameters κ_1, κ_2 to be transparent.

3.3 Matrix-involved and two relativistic symmetries of the CDK equation

Despite the fact that the 8-component field Ψ (together with its components ψ_1, ψ_2) is a field with nonzero mass $m = \sqrt{\kappa_1\kappa_2}$, the CDK equation turns out to be invariant relative to purely matrix transformations of the Pauli—Gürsey type.

T h e o r e m 3. *Seven independent* 8×8*-matrices*

$$\begin{aligned} &\tilde{s}_{\rho\sigma} = -\tilde{s}_{\sigma\rho}: \quad \tilde{s}_{0k} \equiv \begin{bmatrix} -\hat{s}_{ok} & 0 \\ 0 & \hat{s}_{ok} \end{bmatrix}, \\ &\tilde{s}_{jk} \equiv \begin{bmatrix} \hat{s}_{jk} & 0 \\ 0 & \hat{s}_{jk} \end{bmatrix}, \quad \tilde{\Gamma} \equiv \begin{bmatrix} -\gamma_4 & 0 \\ 0 & \gamma_4 \end{bmatrix}, \end{aligned} \tag{3.18}$$

where $\hat{s}_{\rho\sigma}$ are set by formulas (3.11), *together with the unit matrix are the generators of the purely matrix algebra $\tilde{A}_8$ of invariance of the CDK equation (2.24) at arbitrary values of the mass parameters κ_1, κ_2.*

The matrices $\tilde{s}_{\rho\sigma}$ in (3.18) and the matrices

$$\breve{s}_{\rho\sigma} = \tfrac{1}{4}[\Gamma_\rho, \Gamma_\sigma], \Gamma_\rho = g_{\rho\mu}\Gamma^\mu \tag{3.19}$$

(Γ^μ *are reducible matrices in (2.24a)) generate the same representation*

$$\tilde{\mathcal{L}}^{\mathrm{S}} = 2\mathcal{L}^{\mathrm{S}} \equiv$$

$$\equiv (0, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, 0) \tag{3.20}$$

of the group $\mathcal{L}$. Moreover, only the representation $\tilde{\mathcal{L}}^{\mathrm{S}}$ generated by the matrices $\tilde{s}_{\rho\sigma}$ in (3.18), as distinct from the representation $2\mathcal{L}^{\mathrm{S}}$ generated by the matrices $\breve{s}_{\rho\sigma}$, is the purely matrix group of invariance of the CDK equation (2.24) (the internal symmetry group of the CDK equation which is not related to transformations in the space-time M(1,3)).

P r o o f. Using the equalities

$$\tilde{\Gamma}\cdot M = -M\cdot\tilde{\Gamma}, \quad \tilde{s}_{ok}\cdot M = -M\cdot\tilde{s}_{ok},$$

$$\tilde{s}_{jk}\cdot M = -M\cdot\tilde{s}_{jk}, \tag{3.21}$$

which follow from (3.12) and definitions (3.18), we verify that, at arbitrary κ_1, κ_2, the equalities

$$\tilde{\Gamma} \cdot D = -D \cdot \tilde{\Gamma}, \quad \tilde{s}_{ok} \cdot D = -D \cdot \tilde{s}_{ok},$$

$$\tilde{s}_{jk} \cdot D = -D \cdot \tilde{s}_{jk}; \quad D \equiv \Gamma\partial - M \tag{3.22}$$

are valid. This testifies to the validity of the assertion of the first item of Theorem 3. The validity of the assertion of the second item of this theorem follows from Theorem 1 in [16], definitions (3.18), and the assertion of the first item, **qed.**

We now pass to the consideration of the Poincare-symmetry of the CDK equation. The reducible matrices Γ^μ in (2.24a) together with the matrix

$$\Gamma^4 \equiv \Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \begin{bmatrix} \gamma^4 & 0 \\ 0 & \gamma^4 \end{bmatrix} \tag{2.24b}$$

satisfy the CD relations (2.21). Therefore, the 8×8-matrices

$$\breve{s}_{\hat{\rho}\hat{\sigma}} = -\breve{s}_{\hat{\sigma}\hat{\rho}}: \quad \breve{s}_{\tilde{\rho}\tilde{\sigma}} = \tfrac{1}{4}[\Gamma_{\tilde{\rho}}, \Gamma_{\tilde{\sigma}}], \quad \breve{s}_{\tilde{\rho}5} = \tfrac{1}{2}\Gamma_{\tilde{\rho}} = -\breve{s}_{5\tilde{\rho}};$$

$$\tilde{\rho}, \tilde{\sigma} = \overline{0,4}; \quad \hat{\rho}, \hat{\sigma} = \overline{0,5} \tag{3.23}$$

satisfy relations (3.4) with the change of $\mu, \nu, \rho, \sigma = \overline{0,3}$ by $\hat{\mu}, \hat{\nu}, \hat{\rho}, \hat{\sigma} = \overline{0,5}$. This means that matrices (3.23) are the generators of a reducible representation of the covering $\mathcal{L}(1,5)$ of the proper orthochronous subgroup of the group $\mathrm{O}(1,5)$ of pseudoorthogonal transformations in the 6-dimensional Minkowski space $\mathrm{M}(1,5)$. Then, because all matrices (3.23) have the block-diagonal form, they commute with the mass operator M with arbitrary $\kappa_{1,2}$:

$$\left[\breve{s} \equiv \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}, \quad M \equiv \begin{bmatrix} 0 & \kappa_1 \\ -\kappa_2 & 0 \end{bmatrix}\right] = 0. \tag{3.24}$$

This means that M is the Casimir operator of a representation of the group $\mathcal{L}$(1,5) generated by the generators $\breve{s}_{\hat{\rho}\hat{\sigma}}$ (3.23), and Eq. (2.24) can be written as

$$(2i\breve{s}_{\rho 5}\partial^\rho - M)\Psi(x) = 0, \quad x \in \mathrm{R}^4. \tag{3.25}$$

Then, because the operator $D = \Gamma\partial - M$ in Eq. (2.24)=(3.25) commutes with operators (3.5), in which $s^{(\mathcal{A})}_{\rho\sigma} = \breve{s}_{\rho\sigma}$ (3.19), Eq. (2.24) is invariant relative to the $\mathcal{P}$-representation $\mathcal{P}^{\mathrm{S}}$ which is defined by the $\mathcal{L}$-representation $2\mathcal{L}^{\mathrm{S}}$ (3.20).

For the representation $\mathcal{P}^{\mathrm{S}}$, the Casimir operator (3.7) looks as

$$\hat{w} = \tfrac{1}{2}(\tfrac{1}{2} + 1)I_8\Box. \tag{3.26}$$

Thus, we have proved

T h e o r e m 4. *The CDK equation (2.24)=(3.25) with arbitrary mass parameters κ_1, κ_2 is the example of an equation of the Bhahba type [18] (in the sense discussed in [19]). It is a $\mathcal{P}$-invariant equation for the compound-field Ψ as a system of two coupled (at $\kappa_1 \neq \kappa_2 \neq 0$) 4-component Fermi fields ψ_1, ψ_2 with the same mass $m = \sqrt{\kappa_1\kappa_2}$ and spin $s = \frac{1}{2}$. Transformations of the invariance group $\mathcal{P}^{\mathrm{S}}$ which is defined by* (3.5) with $s^{(\mathcal{A})}_{\rho\sigma} = \breve{s}_{\rho\sigma}$ *(3.19) do not mix the components of different $\mathcal{P}$-covariants ψ_1, ψ_2. That is, they remain the notion of compound-field Ψ as a system of two coupled Fermi fields ψ_1, ψ_2 to be invariant.*

R e m a r k 2 (on the CDK equation in a 5-dimensional Minkowski space). In view of work [20] and the above-performed analysis, we conclude that the CDK equation in a 5-dimensional space, i.e. the $\mathcal{P}(1,4)$-invariant CDK equation, looks as

$$(\Gamma^{\tilde{\mu}}\partial_{\tilde{\mu}} - M)\Psi(\tilde{x}) \equiv (\Gamma^\mu\partial_\mu + \Gamma^4\partial_4 - M)\Psi(x, x^4) = 0,$$

$$\tilde{x} \equiv (x^{\tilde{\mu}}) \in \mathrm{M}(1,4), \quad \tilde{\mu} = \overline{0,4}. \tag{3.27}$$

The generators of the relevant representation of the group $\mathcal{P}(1,4)$, relative to which Eq. (3.27) is invariant, are given by (3.5) with $\rho, \sigma = \overline{0,3} \to \tilde{\rho}, \tilde{\sigma} = \overline{0,4}$ and with $s^{(\mathcal{A})}_{\rho\sigma} \to \breve{s}_{\tilde{\rho}\tilde{\sigma}}$ (3.19). Equation (3.27) is an 8-component equation of the Bhabha type in a 5-dimensional Minkowski space and can be used in the field theory in a 5-dimensional space.

We now present the assertion about the Bose $\mathcal{P}$-symmetry of the CDK equation.

T h e o r e m 5. *The CDK equation*(2.24) *is invariant also relative to the $\mathcal{P}$-representation $\mathcal{P}^{\mathrm{TSV}}$ which is set by the generators* (3.5) *with* $\mathcal{A} = \mathrm{TSV}$, *in which the spin 8×8-matrices*

$$s^{(\mathrm{TSV})}_{\rho\sigma} \equiv \tfrac{1}{2}(\breve{s}_{\rho\sigma} + \tilde{s}_{\rho\sigma}) = \begin{bmatrix} s^{(\mathrm{TS})}_{\rho\sigma} & 0 \\ 0 & s^{(\mathrm{V})}_{\rho\sigma} \end{bmatrix} \tag{3.28}$$

set a reducible Bose, namely, tensor-scalar-vector representation

$$\mathcal{L}^{\mathrm{TSV}} = (0,1) \oplus (0,0) \oplus (\tfrac{1}{2}, \tfrac{1}{2}) \tag{3.29}$$

of the Lorentz group $\mathcal{L}$. Transformations of the invariance group $\mathcal{P}^{\mathrm{TSV}}$ for Eq. (2.24)=(2.19) ∼(2.14)=(2.13)=(2.11) *do not mix components of different $\mathcal{P}$-covariants $\vec{\mathcal{E}}$ or $\mathcal{B}^{\mu\nu}$, ϕ, and V^μ, i.e.*

remain the notion of compound-field $\mathcal{F}$ as a system of three coupled Bose fields with the indicated spins to be invariant.

P r o o f. With regard for the mentioned theorem in [16] and the emplicit block-diagonal form of 8×8-matrices (3.28), in which $s^{(\mathrm{TS})}$ and $s^{(\mathrm{V})}$ are given by (3.16 a,b), it is clear that matrices (3.28) satisfy relations (3.4) and generate just the reducible Bose $\mathcal{L}$-representation (3.29). In this case, the terms $\breve{s}_{\rho\sigma}$(3.19) and $\tilde{s}_{\rho\sigma}$ (3.18) in (3.28) with arbitrary $\rho, \sigma = \overline{0,3}$ commute one with another. Therefore, operators (3.5) with $\mathcal{A}$ = TSV satisfy relations (3.6 a,b) for the $\mathcal{P}$-generators and thus generate the representation $\mathcal{P}^{\mathrm{TSV}}$ of this group. Finally, the direct calculations show that operators (3.5) with $\mathcal{A}$ = TSV are the invariance transformations of the CDK equation (2.24). Thus, the Bose representation $\mathcal{P}^{\mathrm{TSV}}$ corresponds to the invariance group of this equation.

It is clear that the assertion about the representations $\mathcal{P}^{\breve{\mathrm{F}}}$ and $\mathcal{P}^{\mathrm{TSV}}$ corresponding to the groups of $\mathcal{P}$-invariance of the CDK equation is true for it in all its forms. Let us clarify the assertion about the Bose invariance of the CDK equation. Because the transformation W (2.22), (2.23) does not change differential operators, we give the corresponding spin matrices, being the generators of the representation $\mathcal{P}^{\mathrm{TSV}}$, in their Bose representation $\overset{\mathrm{B}}{s} \equiv W^{-1}sW$ as

$$\overset{\mathrm{B(TSV)}}{s_{\rho\sigma}} = \begin{bmatrix} \overset{\mathrm{B(TS)}}{s_{\rho\sigma}} & 0 \\ 0 & \overset{\mathrm{B(V)}}{s_{\rho\sigma}} \end{bmatrix}; \quad \overset{\mathrm{B(TS)}}{s_{\rho\sigma}} = \begin{bmatrix} \overset{\mathrm{B(T)}}{s_{\rho\sigma}} & 0 \\ 0 & \overset{\mathrm{B(S)}}{s_{\rho\sigma}} \end{bmatrix}, \quad \overset{\mathrm{B(S)}}{s_{\rho\sigma}} \equiv 0, \tag{3.30}$$

where $3{\times}3$-matrices $\overset{B(T)}{s_{\rho\sigma}}$ are

$$\overset{\mathrm{B(T)}}{s_{01}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \overset{\mathrm{B(T)}}{s_{02}} = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \overset{\mathrm{B(T)}}{s_{03}} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overset{\mathrm{B(T)}}{s_{jk}} = i\varepsilon^{jkl}\overset{\mathrm{B(T)}}{s_{0l}}, \tag{3.31}$$

and the matrix elements of the 4.×4-matrices $\overset{\mathrm{B(V)}}{s_{\rho\sigma}}$ are set as

$$\left(\overset{\mathrm{B(V)}}{s_{\rho\sigma}} = s^{(\mathrm{V})}_{\rho\sigma}\right)^{\mu}_{\ \nu} = \delta^{\mu}_{\rho} g_{\sigma\nu} - \delta^{\mu}_{\sigma} g_{\rho\nu} \tag{3.32}$$

That is, they coincide with the matrices $s^{(\mathrm{V})}_{\rho\sigma}$ in (3.3). As clearly seen, the matrices $\overset{\mathrm{B(T)}}{s_{\rho\sigma}}$, $\overset{\mathrm{B(S)}}{s_{\rho\sigma}} \equiv 0$ and $\overset{\mathrm{B(V)}}{s_{\rho\sigma}}$ are the generators of (0,1), (0,0), and $(\frac{1}{2}, \frac{1}{2})$-representations of the group $\mathcal{L}$. That is, the fields, being $\mathcal{P}$-covariants in Eq. (2.14), are the complex irreducible tensor $\vec{\mathcal{E}}$ or $\mathcal{B}^{\mu\nu}$, scalar ϕ, and vector V^{μ} fields). It is clear that the invariance group $\mathcal{P}^{\mathrm{TSV}}$ of the CDK equation (2.14)=(2.13)=(2.11) does not mix the components of different $\mathcal{P}$-covariants $\vec{\mathcal{E}}$, ϕ, V, **qed.**

The fact that the 3-component complex field $\vec{\mathcal{E}}$ is the irreducible (namely (0,1)-) $\mathcal{P}$-covariant follows from the explicit form of the Casimir operator (3.7) with $s^{(\mathcal{A})}_{\rho\sigma} = s^{(\mathrm{T})}_{\rho\sigma}$ (3.31):

$$\hat{w} = 2\Box I_3. \tag{3.33}$$

We also note that

$$\overset{B}{C} \equiv W^{-1}\hat{C}W = C, \tag{3.34}$$

That is, the matrix $\hat{C}$ (3.14) in the B-representation is simply the operator C of complex conjugation.

4. Conclusions

The analyzed 8-component CDK equation (a complex analog of the 16-component DK equation) is written in its various Bose (2.11)=(2.13)=(2.14) and Fermi (2.19)=(2.24) forms which clarify the physical content of the equation. The presented group analysis of the CDK equation performed by the Bargman—Wigner method (Theorems 4, 5) yields that this equation in any form is invariant relative to two nonequivalent representations of the Poincare group $\mathcal{P}$: the Fermi representation $\mathcal{P}^{\mathrm{S}}$ and the Bose one $\mathcal{P}^{\mathrm{TSV}}$. Therefore, it describes both a system of two Fermi-fields $\psi_{1,2}$ (a doublet of particles with mass $m = \sqrt{\kappa_1\kappa_2}$ and with spin $s = \frac{1}{2}$) and a system of three $\mathcal{P}$-irreducible Bose-fields $\mathcal{B}^{\mu\nu} \sim \vec{\mathcal{E}}$, ϕ, and V^{μ} (particles with mass m and spins $s = 0, 1$). Moreover, the spin states of these fields (particles) are not doubled (as distinct from the case of the DK 16-component equation).

Then, we have found the algebra $\tilde{A}_8$ of the *internal symmetry* of the CDK equation with any m which is an analog of the Pauli—Gürsey algebra A_8 of invariance of the Dirac massless equation $\gamma\partial\psi = 0$. The special representation $\tilde{\mathcal{L}}^{\mathrm{S}}$ (3.12) of the Lorentz group $\mathcal{L}$, which is created by the generators $\{\tilde{s}_{\rho\sigma}\} \subset \tilde{A}_8$ with $\tilde{s}_{\rho\sigma}$(3.18), is connected with the standard spinor representation $2\mathcal{L}^{\mathrm{S}}$ (3.20) by the operator $\tilde{C} = \mathrm{diag}(\hat{C}, \hat{C})$ with $\hat{C}$ (3.14). However, it is not connected with transformations in the space-time M(1,3) and can be identify with a representation of the group SU(2) as the internal (isospin) group for the doublet $\Psi = (\psi_1, \psi_2)$.

The next nearest tasks are to find the consequences of the found symmetries, in particular, the main dynamical variables $(P_\mu, J_{\mu\nu})^{\mathrm{B,F}}$ as the functionals of the states of Bose or Fermi compound-fields, and to carry out the quantization of two types for the fields which are described by the CDK equation. Omitting the practical applications of the CDK equation, we point out the use of the results derived here. First, they can create a specific group basis of the theory of supersymmetric fields. Secondly, the specific limit $m^2 \to 0$ of the Bose version of the CDK equation can be used for the construction of a variant of electrodynamics in terms of only the tensor of intensities of the electromagnetic field without the use of the potential $A = (A^\mu)$ as the primary object of an electromagnetic field. Such a variant can be useful in the quasirelativistic approximations to the specific many-particle problems of atomic and nuclear physics.

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СИМЕТРІЇ 8-КОМПОНЕНТНОГО РІВНЯННЯ ТИПУ ДІРАКА—КЕЙЛЕРА

І.Ю. Кривський

Р е з ю м е

Запропоновано комплексний аналог рівняння Дірака—Кейлера — рівняння КДК — як систему 8 (а не 16) рівнянь для 8 незалежних комплексних компонент з відмінною від нуля масою $m = \sqrt{\kappa_1\kappa_2}$. Це рівняння записано у трьох бозонних (2.11), (2.13), (2.14) і двох ферміонних (2.19), (2.24) формах. Показано (теорема 3), що, незважаючи на $m \neq 0$, рівняння КДК інваріантне відносно алгебри $\tilde{A}_8$ чисто матричних перетворень, 8×8-матриці яких будуються з 4×4-матриць алгебри A_8 Паулі—Гюрші інваріантності безмасового рівняння Дірака $\gamma\partial\psi = 0$. Шість генераторів алгебри $\tilde{A}_8$ породжують групу внутрішньої симетрії рівняння КДК, яку можна ототожнити з ізоспіновою групою SU(2) компаунд-поля $\Psi = (\psi_1, \psi_2)$. Показано (теореми 4, 5), що рівняння КДК (у будь-якій формі) інваріантне відносно двох *нееквівалентних зображень* $\mathcal{P}^{\mathrm{S}}$ і $\mathcal{P}^{\mathrm{TSV}}$ групи Пуанкаре $\mathcal{P} \supset \mathcal{L}$, породжуваних спінорним $2\mathcal{L}^{\mathrm{S}}$ (3.20) і відповідно тензорно-скалярно-векторним $\mathcal{L}^{\mathrm{TSV}}$ (3.29) матричними зображеннями групи Лоренца $\mathcal{L}$. Знайдено оператор, який зв'язує бозонні й ферміонні форми рівняння КДК: за допомогою цього оператора ферміонне компаунд-поле $\Psi = (\psi_1, \psi_2)$ виражається через систему $\mathcal{F} = (\mathcal{B}^{\mu\nu}, \phi, V^\mu)$ трьох $\mathcal{P}$-незвідних бозонних полів. Виписано також (без обговорення) рівняння типу КДК у п'ятивимірному просторі Мінковського.