SOME PROPERTIES OF THE THREE-ANYON SPECTRUM

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The low-lying part of the spectrum of three anyons in a harmonic potential is found numerically, and some qualitative features of the spectrum are analyzed. The novel feature discovered is the presence of a significant number of states whose energy varies with the statistics parameter almost linearly. A semiclassical interpretation of such states is given.

1. Introduction

It is well known that particles can obey fractional statistics in two dimensions [1]; such particles are commonly referred to as anyons [2]. The defining property of anyons is that their wave function acquires a complex phase factor under particle permutations. If P_{jk} denotes an operation whereby the positions of two particles are continuously interchanged so that their relative radius vector rotates by an angle of π counterclockwise, then the anyonic wave function has to satisfy the condition

$$P_{jk}\Psi = \mathrm{e}^{\mathrm{i}\pi\alpha}\,\Psi\,, \tag{1}$$

where α, commonly called *statistics parameter*, is any real number. Clearly, $\alpha + 2$ is equivalent to α, and furthermore, $\alpha = 0$ (1) corresponds to bosons (fermions).

A multiboson or multifermion wave function, in the absence of an interaction, can be represented as a symmetrized or antisymmetrized, respectively, product of single-particle wave functions. A fundamental difficulty with fractional statistics is the fact that such a representation is not possible here. Indeed, the simplest structure of the kind that satisfies the interchange condition (1) is the fractional power $(z_j - z_k)^\alpha$, $z_j = x_j + \mathrm{i}y_j$ being the complex coordinate of particle j. It can be demonstrated that a function containing a fractional power of the Vandermonde determinant,

$$\Psi = \prod_{j<k=1}^{N} (z_j - z_k)^\alpha\, \Psi_{\mathrm{sym}}(z_1, \ldots, z_N)\,, \tag{2}$$

where Ψ_{sym} is any totally symmetric function, satisfies Eq. (1) for all pairs j, k. However, a fractional power of a binomial cannot be universally represented as a linear combination of products of monomials.

Therefore, the many-body problem with fractional statistics is intrinsically nontrivial. It is only the two-anyon problem that can be easily solved exactly [1, 2] by means of separating the center-of-mass motion. The three-anyon system is of particular interest: on the one hand, it possesses all the essential features of a multiparticle system; on the other hand, it is still simple enough, and many exact results can be obtained. It was found that there exists a set of states whose energies and wave functions can be evaluated analytically [3]. Later on, exact multiplicities of the spectrum were deduced from symmetry considerations and the continuity of states, and it was shown that the exactly known states constitute one third of the spectrum [4]. The remaining states, apparently, can only be found numerically. The first attempts were undertaken in [5, 6]. A systematic approach to the problem was worked out in [7]; an algorithm was devised, which lets one reduce the three-anyon problem to an effective one-body problem that is suitable for numeric computation. The results of the latter were subsequently used for a precise evaluation of the third virial coefficient of anyons [8], which has confirmed earlier exact and perturbative results [9—14] and disproved a hypothesis concerning its dependence on the statistics parameter [15].

In the present paper, we concentrate on the qualitative properties of the unknown part of the spectrum. We will place the particles in a harmonic potential, which is the most convenient model, since its rotational symmetry is compatible with the interchange conditions (1), and its single-particle spectrum is very simple. All the states fall into four groups defined by the average slope of their energy dependence $E(\alpha)$ or the difference $E(1) - E(0)$. For all the exact states, the dependence in question is linear, and the slope is ± 3 (in units of the harmonic frequency); for the remaining states, it is nonlinear, and the slope is ± 1. Based upon numerical computations, we analyze the qualitative features of these nonlinear states. It turns out that, for many of them, the α dependence of energy is almost linear. We demonstrate how such a behavior

follows from simple semiclassical considerations making use of the concept of anyon trajectories [16].

2. The Algorithm and Calculations

We consider the problem of three particles of unit mass in a harmonic potential of unit frequency. The single-particle Hamiltonian is

$$H_j = -2\partial_j\bar{\partial}_j + \frac{1}{2}z_j\bar{z}_j \tag{3}$$

($\partial_j = \partial/\partial z_j$), and the single-particle spectrum is

$$E = 2n + |L| + 1\,, \tag{4}$$

$n = 0, 1, 2, \ldots$, $L = 0, \pm1, \pm2, \ldots$. In the three-body problem, $H = H_1 + H_2 + H_3$, and the wave function is subject to the interchange conditions (1).

The key point to the solution is an appropriate choice of coordinates. Introduce a discrete Fourier transformation,

$$\begin{pmatrix} Z \\ u \\ v \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \eta & \eta^2 \\ 1 & \eta^2 & \eta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{5}$$

where $\eta = \exp(2\pi\mathrm{i}/3)$. The center-of-mass coordinate Z separates from the relative coordinates u, v, which are further rewritten as

$$u = \frac{rq\,\mathrm{e}^{\mathrm{i}(\varphi+\xi/6)}}{\sqrt{2(1+q^2)}}\,, \qquad v = \frac{r\,\mathrm{e}^{\mathrm{i}(\varphi-\xi/6)}}{\sqrt{2(1+q^2)}}\,. \tag{6}$$

In terms of the four new coordinates, the relative Hamiltonian is

$$H_{\rm rel} = -\frac{1}{2r^3}\frac{\partial}{\partial r}\,r^3\,\frac{\partial}{\partial r} - \frac{1+q^2}{2r^2}\left[\frac{1+q^2}{q}\frac{\partial}{\partial q}\,q\,\frac{\partial}{\partial q}\right.$$
$$\left. + \frac{1}{4q^2}\left(\frac{\partial}{\partial\varphi} + 6\frac{\partial}{\partial\xi}\right)^2 + \frac{1}{4}\left(\frac{\partial}{\partial\varphi} - 6\frac{\partial}{\partial\xi}\right)^2\right] + \frac{r^2}{2}\,, \tag{7}$$

while the relative angular momentum is

$$L_{\rm rel} = -\mathrm{i}\,\frac{\partial}{\partial\varphi}\,. \tag{8}$$

For three particles, all possible particle exchanges reduce to superpositions of two types of operations: cyclic permutation and pairwise exchange. Those, according to the anyonic interchange conditions, must supply the wave function with phase factors of $\mathrm{e}^{2\mathrm{i}\pi\alpha}$ and $\mathrm{e}^{\mathrm{i}\pi\alpha}$, respectively. It is easy enough to prove [7] that a cyclic permutation $(z_1, z_2, z_3) \mapsto (z_2, z_3, z_1)$ implies $\varphi \mapsto \varphi + m'\pi$ and $\xi \mapsto \xi + (6m' - 4)\pi$, where m' is some arbitrary integer. It is then straightforward to show that a wave function

$$\psi(r, q, \varphi, \xi) = r^\mu P_{n\mu}(r^2)\exp(-r^2/2)\,\mathrm{e}^{\mathrm{i}L\varphi}\times$$
$$\times \sum_{m=-\infty}^{\infty} \gamma_m g_m(q)\mathrm{e}^{-\mathrm{i}(m+\alpha'/2)\xi} \tag{9}$$

with

$$g_m(q) = q^{|j|}(1+q^2)^{-\mu/2}\times$$
$$\times {}_2F_1\left(\frac{|j|+|k|}{2} - \frac{\mu}{2}\,, \frac{|j|-|k|}{2} - \frac{\mu}{2}\,; 1+|j|\,; -q^2\right)\,, \tag{10}$$

$$j = \frac{L}{2} - 3\left(m + \frac{\alpha'}{2}\right), \quad k = \frac{L}{2} + 3\left(m + \frac{\alpha'}{2}\right)\,,$$

$$L = L_0 + 3\alpha\,, \quad \alpha' = \alpha + L_0 \bmod 2$$

where L_0 is an integer and $P_{n\mu}(r^2)$ is a polynomial of degree n whose explicit form is of no concern here, is an eigenfunction of $L_{\rm rel}$ with eigenvalue L and of $H_{\rm rel}$ with eigenvalue

$$E = \mu + 2n + 2 \tag{11}$$

and satisfies the cyclic permutation interchange condition. The radial quantum number n is responsible for a "tower structure" [5,9] with step 2, and it is enough to consider the "bottom states", with $n = 0$. There remains one more condition to be satisfied, namely the two-particle interchange condition which corresponds to $q \mapsto 1/q$ and $\xi \mapsto -\xi$, for $0 < \xi < 2\pi$. One can replace the condition at an arbitrary q with two conditions, for the function and its q derivative, at $q = 1$, namely

$$\sum_{m=-\infty}^{\infty} \gamma_m g_m(1)\,\mathrm{e}^{\mathrm{i}m\xi} =$$
$$= \mathrm{e}^{\mathrm{i}(\alpha\pi-\alpha'\xi)} \sum_{m=-\infty}^{\infty} \gamma_m g_m(1)\,\mathrm{e}^{-\mathrm{i}m\xi}\,, \tag{12}$$

$$\sum_{m=-\infty}^{\infty} \gamma_m g'_m(1)\,\mathrm{e}^{\mathrm{i}m\xi} =$$
$$= -\mathrm{e}^{\mathrm{i}(\alpha\pi-\alpha'\xi)} \sum_{m=-\infty}^{\infty} \gamma_m g'_m(1)\,\mathrm{e}^{-\mathrm{i}m\xi} \tag{13}$$

[$g'_m(q) \equiv dg_m(q)/dq$].

To use these boundary conditions, we introduce a linear operator A such that

$$[A\phi](\xi) = \mathrm{e}^{\mathrm{i}(\alpha\pi-\alpha'\xi)}\phi(2\pi - \xi)\,. \tag{14}$$

We take the function $\phi(\xi)$ to be defined for $\xi \in [0, 2\pi]$, and we represent it by the vector of its Fourier components $\{\phi_m\}$, so that $\phi(\xi) = \sum_{m=-\infty}^{\infty} \phi_m e^{im\xi}$. The matrix elements of A in this representation are

$$A_{mn} = \frac{\sin \pi\alpha}{\pi(m+n+\alpha')} \,. \tag{15}$$

Next, introduce two diagonal matrices

$$G_{mn} = g_m(1)\,\delta_{mn} \,, \quad G'_{mn} = g'_m(1)\,\delta_{mn} \,. \tag{16}$$

The two boundary conditions (12)—(13) then take the form

$$(I-A)G\gamma = 0 \,, \quad (I+A)G'\gamma = 0 \,, \tag{17}$$

where γ is the vector $\{\gamma_m\}$ of the unknown coefficients in the wave function (9). Now, since A is a real symmetric matrix and $A^2 = I$, the vectors on the left-hand sides of the two equations in (17) are mutually orthogonal. This makes it possible to replace the two equations with one,

$$[G + G' + A(G' - G)]\gamma = 0 \,. \tag{18}$$

A nontrivial solution for γ exists if and only if the determinant which depends on μ through (16) and (10) vanishes.

Thus, in order to find the energy eigenvalues, one fixes α, chooses a bosonic angular momentum L_0, and scans an interval of μ for the zeros $\mu_s(\alpha)$ of the determinant. The bottom levels are $E_s(\alpha) = \mu_s(\alpha) + 2$.

3. Results and Discussion

Obviously, in an actual numerical procedure, the series in Eq. (9) has to be truncated at some $|m| = \mathcal{N}$, so that there are $(2\mathcal{N}+1)$ unknown coefficients $\{\gamma_{-\mathcal{N}}, \gamma_{-\mathcal{N}+1}, \ldots, \gamma_{\mathcal{N}}\}$. To maintain consistency, one has then to restrict oneself to a discrete set $\{\xi_{-\mathcal{N}}, \xi_{-\mathcal{N}+1}, \ldots, \xi_{\mathcal{N}}\}$ instead of a continuous variable ξ, so that any set of function values $\gamma(\xi_n)$ can be represented by the Fourier components exactly. Choosing $\xi_n = \pi + 2\pi n/(2\mathcal{N}+1)$ and demanding (14) to hold at the points ξ_n, one gets the matrix elements of A as

$$A_{mn} = \frac{\sin \pi\alpha}{(2\mathcal{N}+1)\sin[\pi(m+n+\alpha')/(2\mathcal{N}+1)]} \,, \tag{19}$$

with the correct $\mathcal{N} \to \infty$ limit (15).

By analyzing the data $E_s(\alpha, \mathcal{N})$ obtained at different $\mathcal{N}$, we find that they are rather well described by an empirical formula

$$E_s(\alpha, \mathcal{N}) = E_s(\alpha) + \frac{\mathcal{A}_0}{\mathcal{N}^{2\alpha}} + \frac{\mathcal{A}_1}{\mathcal{N}^{2\alpha+1}} + \ldots \,, \tag{20}$$

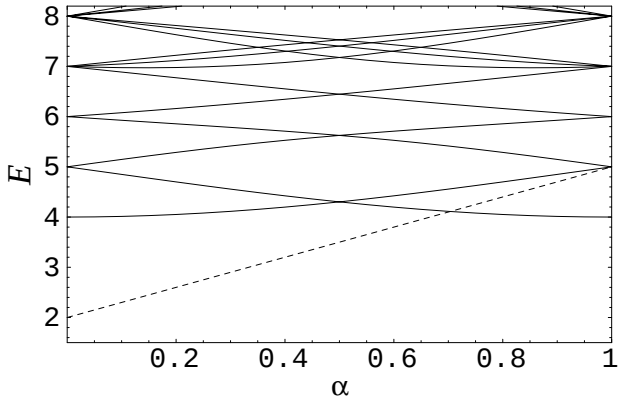

Fig. 1. Low-lying bottom states. The dashed line shows the linear state connecting to the bosonic ground state

which lets one to extrapolate to $\mathcal{N} = \infty$. Thanks to the useful property of "supersymmetry" [10], whereby each nonlinear state with the statistics parameter α has a "supersymmetric partner" with the same energy and the statistics parameter $1 - \alpha$, one can restrict the computation to the interval $\alpha \in [0.5, 1]$, where the $\mathcal{N}$ convergence is much better.

The values of $\mathcal{N} = 40, 80, 160, 320$ were used for the extrapolation. The known linear states are discounted, and the multiplicities of the states found are double-checked to coincide with the ones exactly known from [4]. Both the number of states and the precision attained are much higher than those in the previous work on the subject [5, 6].

Fig. 1 depicts the low-lying part of the spectrum. (Again, to obtain the full three-anyon spectrum, one supplements each of these states with a ladder of tower excitations, with step 2, and with center-of-mass excitations, whose spectrum is that of a single-particle two-dimensional oscillator.) The linear states have been omitted for clarity, save for the one which connects to the bosonic ground state. There is a crossing at $\alpha \simeq 0.7$, signifying an abrupt change of the structure of the three-anyon ground state.

As one moves higher up in the spectrum, the splitting of the nonlinear states increases (Fig. 2); at $E \geq 7$, there are states with a nonmonotonous $E(\alpha)$ dependence. At the same time, there emerge the bunches of states clearly visible in Fig. 2, for which this dependence is practically linear. At $E \simeq 20$, there are states with $E(\alpha) = E(0) \pm \alpha + \Delta E(\alpha)$ with $|\Delta E(\alpha)| < 10^{-5}$ for all $\alpha \in [0, 1]$. The multiplicity of such "quasilinear" states increases quickly with energy.

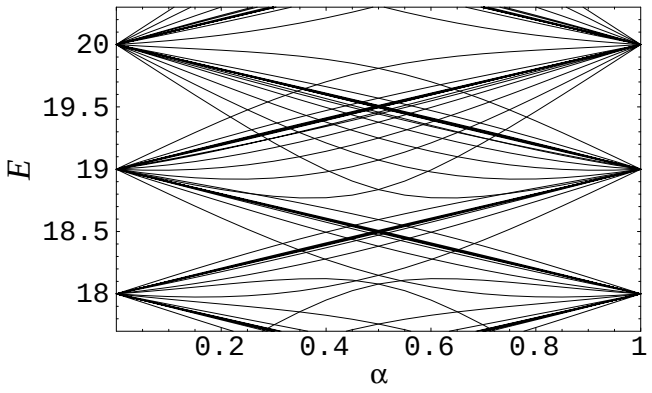

Fig. 2. A higher portion of the spectrum of nonlinear bottom states. Note the abundance of nearly linear states. The total number of states shown in the plot is 108

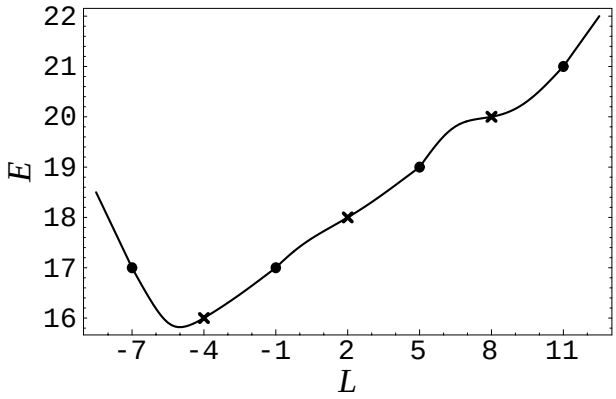

Fig. 3. A sample anyon trajectory. Circles and crosses depict Bose and Fermi points, respectively

There is a simple semiclassical argument which explains the presence of quasilinear states. Three particles form three pairs. If the motion of a pair were not influenced by the presence of the third particle, each pair would behave like a separate two-anyon system, its relative angular momentum contributing $+\alpha$ to the total angular momentum of the system. [In fact, the statement that the total angular momentum of N anyons is $L(\alpha) = L(0) + \frac{N(N-1)}{2}\alpha$ is exact, for any N.] Now, in the spirit of Eq. (4), increasing the algebraic value of a pair angular momentum either increases or decreases the energy, depending on the sign of the momentum. Each of the pairs, therefore, contributes either $+\alpha$ or $-\alpha$ to the total energy. Four possible combinations of signs yield four kinds of linear states with slopes ± 3 and ± 1, respectively. (An equivalent argument can be formulated for the N-body problem [17].) For the states with maximal slope, when all pairs rotate in one and the same direction, this semiclassical description turns out to yield the exact result. How can one tell when it is almost exact for the nonlinear states (i.e., when do the latter become quasilinear)?

The reason that the description in question is not exact is that, in a generic quantum-mechanical state, the quantum analogs of the three pairwise angular momenta do not have exact values (e.g., two positive and one negative), but only a probability distribution. In a sense, there is a linear combination of would-be exact semiclassical states, and, in different constituent states of that combination, one and the same pair rotates in different directions. It cannot therefore be said that each pair definitely contributes either $+\alpha$ or $-\alpha$ to the energy. One should thus expect that the quasilinear states are the quantum-mechanical counterparts of those semiclassical states where all three pairwise angular momenta are significantly different from zero, so that quantum fluctuations are not likely to change their signs.

One can proceed by invoking the concept of anyon trajectories [16]. The spectrum is periodic in α with period 2. For any given state, its dependence $E_s(\alpha)$ is a continuous function. However, one such function is not periodic in α; indeed, in general, $E_s(2) \neq E_s(0)$. Periodicity of the spectrum implies that $E_s(\alpha + 2) = E_{s'}(\alpha)$ with some $s' \neq s$. Thus, states s and s', perceived as different on the interval $\alpha \in [0, 2]$, are in fact one and the same continuous state. It is convenient to remove the ambiguity in the definition of α by considering the dependence $E(L)$ rather than $E(\alpha)$. Such dependences have been termed anyon trajectories (by allusion to Regge trajectories).

All trajectories are infinite, stretching from $L = -\infty$ to $+\infty$, and pass through infinitely many Bose and Fermi points, in turn. It has been demonstrated [16] that the average slope of any trajectory (the change of energy on a Fermi–Bose or Bose–Fermi interval) is a nondecreasing function of L, equal to -3 at large enough negative L and to $+3$ at large enough positive L. A semiclassical interpretation follows readily. At large negative L, all three pairwise angular momenta are negative. As α increases, one of those momenta turns into zero at some point; the slope changes from -3 to -1. The two remaining momenta turning into zero at some further points cause the slope to change to $+1$ and $+3$, respectively.

A simple heuristic rule follows from the abovesaid: Those nonlinear states should be close to being linear which correspond to parts of trajectories situated far enough from the points of slope changes. With a few exceptions, this is indeed what is observed; cf. Fig. 3.

As one moves further up in the spectrum, the distances between slope change points become greater, and more and more states turn quasilinear.

4. Conclusion

It has been demonstrated that many of the nonlinear states in the three-anyon spectrum are linear to a very high precision. One could be tempted to introduce an approximation in which all the nonlinear states are linearized and to assume that this approximation would adequately describe the system at high temperatures. However, this does not actually work. The partition function thus obtained leads to diverging third cluster and virial coefficients. A more precise treatment is necessary; it is possible to show that, for the third virial coefficient to come out finite [8], the levels should lie, on average, below the respective $E(0) \pm \alpha$ lines. This is quite compatible with the fact that, because the slope increases with L, the trajectories are, on the whole, concave (Fig. 3).

It should be possible to gain further insight into the problem by obtaining the wave functions numerically and analyzing them. It looks certain that, to the quasilinear states, there should correspond some approximate expressions for the wave functions, becoming asymptotically exact as one goes up by energy and corresponding to the semiclassical picture outlined above. Also, a set of variational wave functions suggested in [18] for the low-lying states should be analyzed within our approach. These issues will be addressed in future work.

1. *Leinaas J.M., Myrheim J.*// Nuovo cim. **37B** (1977) 1.
2. *Wilczek F.*// Phys. Rev. Lett. **49** (1982) 957.
3. *Wu Y.-S.*// Ibid. **53** (1984) 111.
4. *Mashkevich S.*//Phys. Lett. B 295 (1992) 233; Phys. Rev. D **48** (1993) 5946.
5. *Sporre M., Verbaarschot J.J.M., Zahed I.*// Phys. Rev. Lett. **67** (1991) 1813.
6. *Murthy M.V.N., Law J., Brack M., Bhaduri R.K.*// Ibid. P. 1817.
7. *Mashkevich S., Myrheim J., Olaussen K., Rietman R.*// Phys. Lett. B **348** (1995) 473.
8. *Mashkevich S., Myrheim J., Olaussen K.*// Ibid. **382** (1996) 124.
9. *Sporre M., Verbaarschot J.J.M., Zahed I.*// Nucl. Phys. B **389** (1993) 645.
10. *Sen D.*// Phys. Rev. Lett. **68** (1992) 2977; Phys. Rev. D **46** (1992) 1846.
11. *McCabe J., Ouvry S.*// Phys. Lett. B **260** (1991) 113.
12. *Comtet A., McCabe J., Ouvry S.*// Ibid. P. 372.
13. *Law J., Suzuki A., Bhaduri R.K.*// Phys. Rev. A **46** (1992) 4693.
14. *Law J., Khare A., Bhaduri R.K., Suzuki A.*// Phys. Rev. E **49** (1994) 1753.
15. *Myrheim J., Olaussen K.*// Phys. Lett. B **299** (1993) 267.
16. *Mashkevich S., Myrheim J., Olaussen K., Rietman R.*// Intern. J. Mod. Phys. A **11** (1996) 1299.
17. *Illuminati F., Ravndal F., Ruud J.Å.*// Phys. Lett. A **161** (1992) 323; *Ruud J.Å., Ravndal F.*// Phys. Lett. B **291** (1992) 137.
18. *Chin S.A., Hu C.-R.*// Phys. Rev. Lett. **69** (1992) 229.

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ДЕЯКІ ВЛАСТИВОСТІ СПЕКТРА ТРЬОХ ЕНІОНІВ

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Р е з ю м е

Чисельно розраховано низьколежачу частину спектра трьох еніонів у гармонічному потенціалі та проаналізовано деякі якісні характеристики спектра. Відкрито властивість спектра, яка полягає у присутності у спектрі значної кількості станів з енергією, що залежить від статистичного параметра майже лінійно. Запропоновано напівкласичну інтерпретацію таких станів.