THE SMALL VISCOSITY METHOD AND CRITERIA FOR THE EXISTENCE OF SHOCK WAVES IN RELATIVISTIC MAGNETIC HYDRODYNAMICS

V.I. ZHDANOV, M.S. BORSHCH

Taras Shevchenko Kyiv National University
(6, Academician Glushkov Prosp., Kyiv 03127, Ukraine; e-mail: zhdanov@observ.kiev.ua)

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We obtain criteria for the existence of shock waves (SWs) in relativistic magnetic hydrodynamics with no suppositions about the convexity of the equation of state. The method of derivation involves the consideration of a continuous SW profile in the presence of a Landau—Lifshits relativistic viscosity tensor with both nonzero viscosity coefficients η and ζ. We point out that the supposition of the existence of a viscous profile with only one nonzero coefficient ($\eta = 0$) appears to be too restrictive and leads to the loss of some physical solutions.

1. Introduction

Relativistic SWs arise in such powerful astrophysical phenomena as supernova explosions and gamma-ray bursts. The theoretical analysis of these processes involves the criteria of existence and stability of discontinuous solutions that describe SWs in a superdense matter. The consideration of these criteria is complicated in the case of a general equation of state (EOS) (cf., e.g., the classical results [1] and those of relativistic hydrodynamics [2, 3]).

Note that, in the case of a normal fluid, we deal with a convex EOS (this means the convexity of Poisson adiabats). Therefore, the only condition is needed to study the discontinuous solutions of hydrodynamical equations: the well-known entropy growth criterion (see [1,4] for classical hydrodynamics and [5,6] in relativistic MHD). However, in the case of a general EOS, the convexity condition may be violated, and neither customary entropy criterion nor the evolutionarity criterion [4] are not sufficient to single out physical solutions in a correct way. Moreover, the rarefaction shocks and the compression simple waves, as well as the complicated configurations of shocks and simple waves moving in the same direction, are possible. This situation is well known in classical hydrodynamics; it was first studied by H.Bethe [1]. In relativistic theory, such anomalous equations of state arise, e.g., when dealing with a superdense matter in the neighborhood of phase transitions (see, e.g., [2, 3]).

One of the most effective methods to study the SW existence in case of the general EOS is the investigation of the SW viscous profile. According to this method, a generalized (discontinuous) solution is treated as the small viscosity limit of corresponding continuous solutions. The shock transition is admissible, if the corresponding continuous solution (viscous profile) exists for any nonzero viscosity. In the case of a normal fluid (in the sense of Bethe and Weyl [1]), the results of this method are the same as those of the evolutionarity criterion. In relativistic hydrodynamics, the conditions for the existence of a viscous profile in the case of the general EOS have been derived and studied [2, 3] by using the Landau—Lifshits viscosity term in the relativistic energy-momentum tensor [4]. This term involves two viscosity coefficients ξ and η.

In relativistic magnetic hydrodynamics (MHD) [5,6], the investigation of SW viscous profiles becomes more complicated. Therefore, this problem has been first considered [7, 8] in a restricted version with one of the viscosity coefficients put equal to zero ($\eta = 0$) under the supposition that only one nonzero viscosity is sufficient to obtain a continuous SW profile. This was a technical supposition, and it is not evident. At least, the results of [7, 8] for $\eta = 0$ cannot be considered as necessary conditions.

In the present paper, we extend the results of [7, 8] to the case of an arbitrary ratio of positive viscosity coefficients and prove the conditions for the existence of the SW viscous profile under less restrictive requirements. We consider stationary viscous flows of a relativistic fluid with infinite conductivity. These solutions describe the MHD shock structure, the existence of SW being considered by means of the corresponding continuous solutions with nonzero viscosity. Our treatment shows that we may relax the conditions of [7, 8] to have the necessary and sufficient criteria.

2. Basic Equations

The equations of motion of an ideal relativistic fluid with infinite conductivity in a magnetic field follow from the conservation laws involving the energy-momentum tensor [5, 6]

$$T^{\mu\nu} = (p^* + \varepsilon^*)u^\mu u^\nu - p^* g^{\mu\nu} - \frac{\mu}{4\pi} h^\mu h^\nu, \qquad (1)$$

where u^μ is the four velocity (Greek indices run from 0 to 3), the flat space-time metric $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is used for raising and lowering the indices, $h^\mu = -\frac{1}{2} e^{\mu\alpha\beta\gamma} F_{\alpha\beta} u_\gamma$ is the magnetic field intensity, $e^{\alpha\beta\gamma\delta}$ is the fully antisymmetric symbol, $F_{\mu\nu}$ is the tensor of the electromagnetic field, $p^* = p + \frac{\mu}{8\pi}|h|^2$, $\varepsilon^* = \varepsilon + \frac{\mu}{8\pi}|h|^2$, $|h|^2 = -h^\alpha h_\alpha > 0$, μ is the permeability that is supposed to be constant; p is the pressure, and ε is the energy density (in the rest frame). We suppose EOS $p = p(\varepsilon, n)$ to be a sufficiently smooth function.

Following the small viscosity method [1, 3, 7, 8] in order to study the SW structure, we introduce dissipation effects that smear out discontinuities. Similarly to [2, 3, 7, 8], we use the Landau—Lifshits viscosity tensor [4]

$$\tau_{\mu\nu} = \eta(u_{\mu,\nu} + u_{\nu,\mu} - u_\mu u^\alpha u_{\nu,\alpha} - u_\nu u^\alpha u_{\mu,\alpha}) +$$

$$+(\xi - 2\eta/3) u^\alpha_{,\alpha}(g_{\mu\nu} - u_\mu u_\nu)$$

for the relativistic problem, where the commas stand for derivatives.

Now the fluid motion is constrained by the equations of energy-momentum conservation

$$\partial_\mu (T^{\mu\nu} + \tau^{\mu\nu}) = 0 \qquad (2)$$

and baryon charge conservation

$$\partial_\mu (n u^\mu) = 0, \qquad (3)$$

and one more equation follows from the Maxwell's equations [5, 6]

$$\partial_\mu (u^\mu h^\nu - u^\nu h^\mu) = 0. \qquad (4)$$

The discontinuous solutions follow from these equations in the limit, when ξ and η tend to zero. The questions is whether this limit depends on a relation between $\xi > 0$ and $\eta > 0$.

The viscous profile of a stationary SW may locally be represented in the proper reference frame of the shock front by a stationary continuous solution depending on the only variable x; here $\tau^{\mu\nu} \to 0$ and all the parameters of this viscous flow tend to constant values as $x \to \pm\infty$.

Without loss of generality, we suppose further that the limiting values of hydrodynamical parameters for $x \to -\infty$ correspond to the state ahead of the shock (denoted further by index "0") and the values for $x \to +\infty$ correspond to the state behind the shock (denoted further by index "1"), then we have $u^1 > 0$ behind and ahead of the shock.

Because all the values in (2)—(4) depend only on the variable x, these equations yield

$$T^{1\nu} + \tau^{1\nu} = T^{1\nu}_{(0)}, \qquad (5)$$

$$u^1 h^\nu - h^1 u^\nu = H^\nu \equiv u^1_{(0)} h^\nu_{(0)} - h^1_{(0)} u^\nu_{(0)}, \qquad (6)$$

$$n u^1 = n_{(0)} u^1_{(0)}. \qquad (7)$$

As a result of $\tau^{1\nu} \to 0$ for $x \to \pm\infty$, relations (5)—(7) must be fulfilled for the corresponding asymptotic values $T^{\mu\nu}, n$, and h^μ obtained from the continuous solutions of system (9)—(11). Similarly to classical hydrodynamics [1], we interpret the conditions for a shock transition from the state $u^\mu_{(0)}, h^\mu_{(0)}, n_0, p_0$ (ahead of the shock) into the state $u^\mu_{(1)}, h^\mu_{(1)}, n_1, p_1$ (behind the shock). As a consequence of (5)—(7), these states must satisfy the relations

$$T^{1\nu}_{(1)} = T^{1\nu}_{(0)}, \qquad (8)$$

$$u^1_{(1)} h^\nu_{(1)} - h^1_{(1)} u^\nu_{(1)} = H^\nu \equiv u^1_{(0)} h^\nu_{(0)} - h^1_{(0)} u^\nu_{(0)}, \qquad (9)$$

$$n_{(1)} u^{(1)}_{(1)} = n_{(0)} u^1_{(0)}. \qquad (10)$$

3. Dynamical System for the Shock Structure

In this section, we use some of the results of [7, 8]. Suppose that equations (8)—(10) are fulfilled.

Definition. We say that the shock transition $u^{\mu}_{(0)}, h^{\mu}_{(0)}, n_0, p_0 \to u^{\mu}_{(1)}, h^{\mu}_{(1)}, n_1, p_1$ has a viscous profile, if there is a continuous solution of (5)—(7) having the corresponding asymptotics for $x \to -\infty$ and $x \to +\infty$.

We use the reference frame such that $u^3 \equiv 0$ and $h^3 \equiv 0$; u^1, h^1 being normal components to the surface $x = const$, and u^2, h^2 being the tangential components; $u^2_{(0)} = 0$.

Following [7,8], we represent, due to (6), h^{μ} in terms of u^1 and u^2 as

$$h^{\mu} = \frac{1}{u^1}[H^{\mu} - u^{\mu} H^{\alpha} u_{\alpha}]. \tag{11}$$

Multiplying (5) by u_{ν} and taking into account that $\tau^{\mu\nu} u_{\nu} = 0$, we get

$$\varepsilon^* u^1 = T^{1\mu}_{(0)} u_{\mu}, \tag{12}$$

whence ε and h^{μ} can be expressed through u^1 and u^2.

Taking into account the explicit form of $\tau^{\mu\nu}$, relation (5) for $\nu = 1, 2$ yields

$$-\left(\xi + \frac{4}{3}\eta\right)\left[1 + \left(u^1\right)^2\right]\frac{\partial u^1}{\partial x} = T^{11}_{(0)} - T^{11}, \tag{13}$$

$$-\eta\left[1 + \left(u^1\right)^2\right]\frac{\partial u^2}{\partial x} - \left(\xi + \frac{\eta}{3}\right) u^1 u^2 \frac{\partial u^1}{\partial x} = T^{12}_{(0)} - T^{12}. \tag{14}$$

After the elimination of $\partial u^1/\partial x$ from (14) with the help of (13), the second equation transforms to

$$-\eta\left[1 + \left(u^1\right)^2\right]\frac{\partial u^2}{\partial x} =$$

$$= -\frac{u^1 u^2}{1 + (u^1)^2}\frac{(\xi + \eta/3)}{(\xi + 4\eta/3)}\left(T^{11}_{(0)} - T^{11}\right) + T^{12}_{(0)} - T^{12}.$$

It is convenient to introduce a new variable $v = u^2/\sqrt{1 + (u^1)^2}$ in (13) and (14); this yields the dynamical system for u^1 and v:

$$\left(\xi + \frac{4}{3}\eta\right)\frac{du^1}{dx} = F_1\left(u^1, v\right), \qquad \eta\frac{dv}{dx} = F_2\left(u^1, v\right). \tag{15}$$

Here,

$$F_1(u^1, v) = p - \frac{1}{1 + (u^1)^2}\left[T^{11}_{(0)} + \frac{\mu}{4\pi}(H^{\alpha} u_{\alpha})^2 - \right.$$

$$\left. - T^{1\mu}_{(0)} u_{\mu} u^1\right] + \frac{\mu}{8\pi (u^1)^2}[(H^{\alpha} u_{\alpha})^2 - H^{\alpha} H_{\alpha}], \tag{16}$$

$$F_2(u^1, v) = \frac{(T^{10}_{(0)} u^0 - T^{12}_{(0)} u^2) u^2 u^1 - (\mu/4\pi) u^2 (H^{\alpha} u_{\alpha})^2}{u^1 [1 + (u^1)^2]^{\frac{5}{2}}} +$$

$$+ \frac{(\mu/4\pi) H^2 (H^{\alpha} u_{\alpha}) - T^{12}_{(0)} u^1}{[1 + (u^1)^2]^{3/2} u^1}. \tag{17}$$

The continuous solutions of (15) describe the SW structure. It is worth to note that $p(\varepsilon, n)$ disappears from F_2.

4. Conditions for the Existence of a SW Viscous Profile

Let the state parameters $u^{\mu}_{(0)}, h^{\mu}_{(0)}, n_0, p_0$ ahead of the shock and $u^{\mu}_{(1)}, h^{\mu}_{(1)}, n_1, p_1$ behind the shock satisfy the conservation laws (8)—(10) that relate hydrodynamic quantities on both sides of SW. We denote $y = u^1$, $y_0 = u^1_{(0)}$, and $y_1 = u^1_{(1)}$.

Consider now the curves on (y, v) – plane where the right-hand sides (16), (17) of system (15) may change their signs. Let $\mathbf{V_1}$ be a locus of points (y, v) such that $F_1(y, v) = 0$ and let $\mathbf{V_2}$ be a locus of points (y, v) such that $F_2(y, v) = 0$. We suppose that $\mathbf{V_1}$ is represented by a smooth function $v = V_1(y)$. From the results of [7, 8], it follows that $\mathbf{V_2}$ is represented by a single-valued function $y = Y_2(v)$; this function may be not monotonous so it is more convenient to use this function instead of the inverse one. We suppose that "0" and "1" are connected by smooth components of $\mathbf{V_1}$ and $\mathbf{V_2}$ (see, e.g., Figs. 1 and 2).

We consider a part of (y, v), namely a plane between $\mathbf{V_1}$ and $\mathbf{V_2}$ such that there are two intersection points "0" and "1" corresponding to the states ahead and behind the shock: $v_1 = V_1(y_1)$, $y_1 = Y_2(v_1)$; $v_0 = V_1(y_0)$, $y_0 = Y_2(v_0)$, but the curves do not intersect between 0 and 1.

The points $(y_0, V_1(y_0))$, $(y_1, V_1(y_1))$ are the rest points of system (15).

We shall consider the following conditions.

A. The function $v = V_1(y)$ is a single-valued function on (y_1, y_0).

B. For all points of $\mathbf{V_2}$, $v \in (v_1, v_0)$, the following inequality is valid (cf. [7,8]):

$$(y_0 - y_1) F_1(Y_2(v), v) < 0. \tag{18}$$

Here, we do not consider the occurence of the Chapman-Joguet points, where the left-hand side of (18)

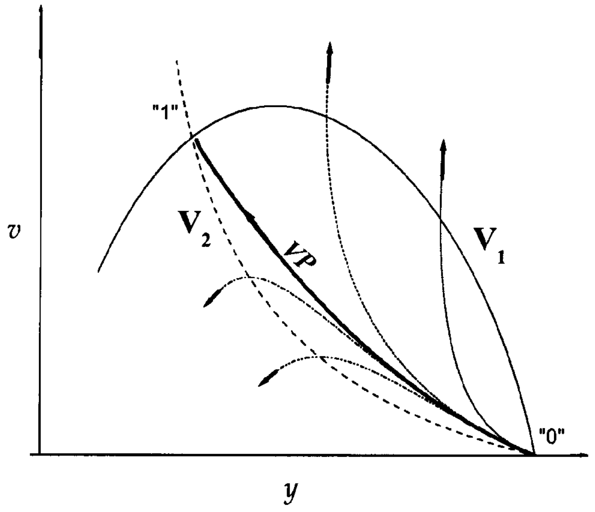

Fig. 1. Phase trajectories of system (15) in the case of DF; EOS is $p = \varepsilon/3$. The solid line ($\mathbf{V_1}$) corresponds to $v = V_1(y)$, the dashed line ($\mathbf{V_2}$) corresponds to $y = Y_2(v)$; the curve VP connecting "0" and "1" describes the viscous profile of the fast shock transition "0" → "1". The arrows show the direction of the phase flow

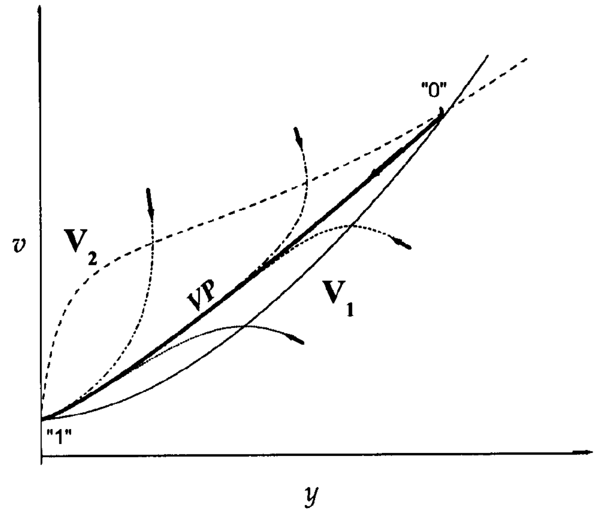

Fig. 2. Phase trajectories of system 15 in the case DS with the same EOS. In this case, the curve $\mathbf{V_1}$ (solid) is below $\mathbf{V_2}$ (dashed). The separatrix VP of the saddle point "0" describing the viscous SW profile of the slow shock transition "0"→ "1" goes to the final state "1"

equals to zero but does not change its sign. This case may be studied by taking the corresponding limit in (18).

C. We suppose that $h^1 h^2 \neq 0$ at point "0".

This is a technical requirement. Otherwise, we deal with a much simpler situation of parallel or perpendicular MHD SWs; this case is not considered here.

Let $u_{\rm sl}$, u_f, $u_{\rm A}$ stand for the speeds of relativistic slow, fast, and Alfven waves [5,6]; $u_{\rm sl}$, u_f being the roots of the polynomial $Q(y)$, where

$$Q(y) = (1 - c_S^2)(y^2 - u_f^2)(y^2 - u_{\rm sl}^2) =$$

$$= (1 - c_S^2) y^4 - y^2 \left(c_S^2 + \frac{\mu |h|^2}{4\pi\,(p+\varepsilon)} \right) + \frac{\mu c_S^2 (h^1)^2}{4\pi\,(p+\varepsilon)}, \quad (19)$$

$c_S^2 = (\partial \mathrm{p}/\partial\varepsilon)_S$ is the speed of sound.

The relativistic Alfven speed $u_{\rm A}$ is defined by the formula

$$u_{\rm A}^2 = \frac{\mu (h^1)^2}{4\pi (p^* + \varepsilon^*)}. \quad (20)$$

In [7, 8], one more parameter $u_{\rm A}^*$ which is a positive root of the equation $R^*(y) = 0$ has been introduced, where

$$R^*(y) = (p + \varepsilon)\, y^2 \left(1 + y^2\right) - \frac{\mu}{4\pi} \left(h^1\right)^2. \quad (21)$$

The above velocities satisfy the inequalities [8]

$$u_{\rm sl} < u_{\rm A} < u_{\rm A}^* < u_f. \quad (22)$$

We introduce one more requirement (one can show that it is consistent with A, B).

D. Either

(DF): $u^1 > u_f$ ahead of the shock at point "0",

or

(DS): $u_{\rm A} > u^1 > u_{\rm sl}$ ahead of the shock at point "0".

These inequalities correspond to the evolutionarity criteria of classical MHD for the velocities ahead of the shock [9]. The first inequality (DF) corresponds to a fast SW and the second one (DS) to a slow SW.

By means of (10)—(12), the variables p and ε can be expressed in terms of the velocity components u^1 and u^2.

Lemma 1. *If relations (7), (11)– (12) are satisfied, then, at point "0", we have*

$$\left.\frac{\partial S}{\partial u^1}\right|_0 = \left.\frac{\partial S}{\partial u^2}\right|_0 = 0,$$

where S is the entropy per baryon.

The proof uses the thermodynamical relation $TdS = pd(1/n) + d(\varepsilon/n)$, where T is the temperature. The statement of the lemma is obtained after the direct calculation of differentials dn and dp using (10)–(12).

Using Lemma 1, we get, after some calculations, at point "0" ($v_0 = 0$):

$$\left.\frac{\partial F_1}{\partial u^1}\right|_0 = \frac{(p+\varepsilon)}{u^1 (u^0)^4} D(u^1), \quad (23)$$

where

$$D(y) = (1 - c_S^2)\, y^4 - y^2 \left(2c_S^2 + \frac{\mu |h|^2}{4\pi\,(p+\varepsilon)} - 1 \right) +$$

$$+ \frac{\mu[(h^0)^2 - (h^2)^2]}{4\pi\,(p+\varepsilon)} - c_S^2.$$

The other derivatives of the right-hand sides (16), (17) of the dynamical system (15) at point "0" are

$$\left.\frac{\partial F_1}{\partial v}\right|_0 = \frac{\mu}{4\pi}\frac{h^1 h^2}{u^0 u^1}, \tag{24}$$

$$\left.\frac{\partial F_2}{\partial u^1}\right|_0 = \frac{\mu}{4\pi}\frac{h^1 h^2}{(u^0)^5 u^1}, \tag{25}$$

$$\left.\frac{\partial F_2}{\partial v}\right|_0 = \frac{1}{(u^0)^4 u^1} R^*(u^1). \tag{26}$$

Taking into account these relations in the vicinity of point "0", we have

$$F_1(y, V_2(y)) = \frac{(p+\varepsilon)^2}{u^1}\frac{\mathrm{Q}(u^1)}{R^*(u^1)}(y - y_0) \tag{27}$$

and

$$\left.\frac{dv}{dy}\right|_0 = -\frac{\mu h^1 h^2}{4\pi u^0 R^*(u^1)} \tag{28}$$

on the curve $\mathbf{V_2}$.

Lemma 2. *In the case DF, the rest point "1" of system (15) is a saddle point. In the case DS, the rest point "0" is a saddle point.*

Proof. The direct calculation yields

$$\frac{\partial F_1}{\partial u^1}\frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial F_2}{\partial u^1} = \frac{(p+\varepsilon)^2}{(u^1)^2\,(u^0)^4} Q(u^1).$$

In the case DS ahead of a SW, we have $Q < 0$ at "0" (see (19)). This yields the required statement according to the properties of the saddle point [10] . In the case DF, an analogous result at "1" can be checked directly. However, it is easier to use the same relations as (23)—(26) in the case of "0" by using the Lorentz transformation that preserves $u^1_{(0)}$ and transforms the transversal velocity component $u^2_{(1)}$ to zero at point "1".

Further, the coefficients of $Q(y)$ and $R^*(y)$ involving the magnetic field, energy density, and baryon density are taken only at point "0" ahead of the shock.

Now we proceed to prove the existence of a viscous profile.

Consider first the case DF for $y_1 < y_0$ and put, for definiteness, $h^1 h^2 > 0$ at "0". The consideration of the opposite sign of $h^1 h^2$ is completely analogous.

First of all, we note that condition (18) guarantees that the trajectories of system (15) in the plane $(y = u^1, v)$ cross the curve $\mathbf{V_2}$ from right to left (see Fig. 1).

On the curve $v = V_1(y)$, we have

$$F_2(y, V_1(y)) = -\frac{4\pi\,(p+\varepsilon)^2\,\mathrm{Q}(u^1)}{\mu h^1 h^2 u^1 (u^0)^3}(y - y_0) > 0 \tag{29}$$

in the vicinity of point "0", $y < y_0$.

Respectively, the trajectories of system (15) cross the curve V_1 bottom-up (Fig. 1). Evidently, this is true not only for the neighborhood of "0", but for the whole interval (y_1, y_0); otherwise, there must be additional rest points of system (15) between "0" and "1", which contradicts our suppositions.

Now we find out the relative disposition of V_1 and V_2. Let

$$\mathrm{tg}(\alpha_1) = \left.\frac{dV_1}{dy}\right|_0, \qquad \mathrm{tg}(\alpha_2) = \left.\frac{dV_2}{dy}\right|_0$$

at point "0". Using (23)–(24), we obtain that the ratio of tangents at "0" equals

$$\frac{\mathrm{tg}(\alpha_1)}{\mathrm{tg}(\alpha_2)} = \frac{16\pi^2 (p+\varepsilon)^2 (u^0)^2}{(\mu h^1 h^2)^2} Q(u^1) + 1 =$$

$$= \frac{16\pi^2\,(p+\varepsilon)}{(\mu h^1 h^2 u^0)^2} R^*(u^1) D(u^1). \tag{30}$$

Then, in the case of a fast SW ($Q(y_0) > 0$), this ratio is > 1. That is, $\mathbf{V_1}$ is above $\mathbf{V_2}$.

Therefore, the phase curves only can leave the domain between $\mathbf{V_1}$ and $\mathbf{V_2}$ (Fig. 1). Taking into account the sign of F_1, it is easy to see that, inside this domain, all the phase curves come out from point "0" ; and there exists a phase curve of (15) that comes from "0" to "1" . Because "1" is a saddle point (Lemma 2), this phase curve is unique. This conclusion does not depend upon a relation between the (positive) viscosity coefficients ξ and η.

Now we proceed to the case DS (see Fig. 2). It is now convenient to put $h^1 h^2 < 0$ at "0" (the opposite case is completely analogous). The sign of (29) remains the same as in the case DF and so is the direction of the phase curves crossing $\mathbf{V_1}$. The direction of the phase curves crossing $\mathbf{V_2}$ also remains due to (18). Taking into account DS and (22), we have $Q(y_0) < 0$, and, according to (30), we see that $\mathbf{V_2}$ is above $\mathbf{V_1}$ in the neighborhood of "0". As distinct from the case DF, here the phase curves can only enter the domain between "1" and “0”. Similarly to the previous consideration, there is a unique phase curve of system (15) passing from “0” to “1”; this is a separatrix of the saddle point “0”.

In the previous consideration, we supposed that $y_1 < y_0$; this corresponds to a usual compression SW. In the case of an anomalous EOS, the rarefaction shocks are also possible [1, 3, 7, 8]. Condition (18) is applicable in this case as well, and the consideration is completely similar.

Therefore, we proved that Eqs. (5)—(10) have a unique continuous solution that connects states “0” and “1”.

Theorem. *Let the states $u^{\mu}_{(0)}$, $h^{\mu}_{(0)}$, n_0, p_0 ahead of the shock and $u^{\mu}_{(1)}$, $h^{\mu}_{(1)}$, n_1, p_1 behind the shock satisfy the conservation equations (8)—(10). If conditions (A–D) are satisfied, then the MHD shock transition “0”→“1” has a unique viscous profile satisfying Eqs. (5)–(10).*

Note that the analogous criteria for the existence of a SW viscous profile obtained in [7, 8] appear to be too restrictive. These criteria have been obtained under condition that one of the viscosity coefficients is equal to zero ($\eta = 0$), and they rule out the shocks that satisfy the condition $u_{\mathrm{A}} < u^1 < u^*_{\mathrm{A}}$ at “1” after the shock front. However, these latter solutions are compatible with criteria (A)—(D) of the present paper. The explanation of this inconsistency is as follows. If the function $Y_2(v)$ is monotonous and $\eta \to 0$, then the phase curve of system (15) that goes from “0” to “1” tends to the curve $\mathbf{V_2}$. However, it may happen that $Y_2(v)$ is not monotonous; this just corresponds to $u_{\mathrm{A}} < u^1 < u^*_{\mathrm{A}}$ at “1”. In this case, the above phase curve that begins at “0” snuggles down to $\mathbf{V_2}$ only on some segment, and the corresponding limiting solution for $\eta \to 0$ has a discontinuity (see Fig. 3). This explains why such solutions have been rejected in [7,8], because the initial supposition of these papers was the existence of a regular viscous profile for $\eta = 0$.

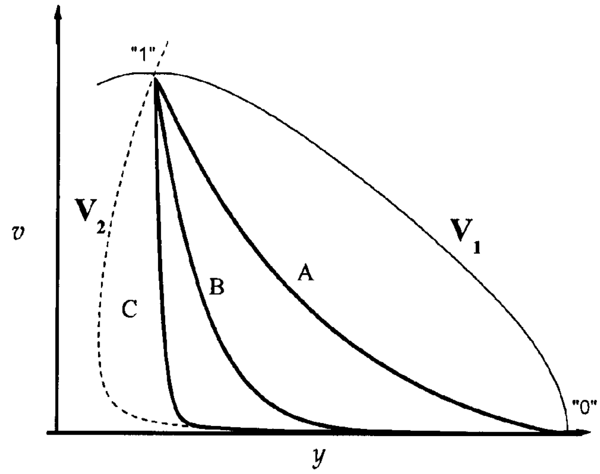

Fig. 3. Example of viscous SW profiles for three ratios of η/ξ in the case of DF (fast SW) with non-monotonous dependence $y = Y_2(v)$; $p = \varepsilon/3$. The disposition of curves $\mathbf{V_1}$ and $\mathbf{V_2}$ is as in Fig. 1. Curve A describes the profile with ratio $\eta/\xi = 1$, B corresponds to $\eta/\xi = 0.1$, and C — to $\eta/\xi = 0.01$. These curves corresponding to the smaller ratios snuggle down to $\mathbf{V_2}$ (after they go out from “0”) on some segment and then jump to “1”

5. Discussion

The SW existence conditions in relativistic MHD with a general equation of state has been analyzed in [7, 8] in the case of $\eta = 0$ in the Landau—Lifshits relativistic viscosity tensor. In the present paper, we obtained the criteria for the existence of a viscous SW profile dealing with both nonzero viscosity coefficients ($\xi > 0$, $\eta > 0$) in this viscosity tensor. If the additional limitations on EOS (e.g., convexity) are absent, our criteria are more restrictive than, e.g., evolutionarity conditions [9] or any other conditions that involve characteristics of the fluid only at the initial and final states. This is evident because condition (18) must be valid for the whole interval between states “0” and “1”. This situation is analogous to ordinary (non-magnetic) hydrodynamics [2, 3]; in this case, our criteria reduce to the criteria of these papers.

On the other hand, the criteria of the present paper are less restrictive than that of [7, 8] obtained in the case of $\eta = 0$, because the supposition of the existence of a regular viscous profile used in [7, 8] does not always hold in the case of $\eta = 0$ (even if $\xi \neq 0$), which rules out some physical solutions. It should be noted that such a situation is specific just of relativistic MHD; this does not appear neither in the nonrelativistic case nor in ordinary relativistic hydrodynamics.

Our criteria may be applied to an arbitrary smooth EOS. However, we must note that the requirement for $V_1(y)$ to be a continuous (single-valued) function is not trivial and may not be fulfilled in the case of certain equations of state (cf., e.g., [1]). Though the consideration of a viscous profile seems to be rather efficient for the investigation of the existence and stability of a SW, this method may not work in the case of a complicated EOS (cf. the remarks in [11]), which would require either a modification of the equations of motion of a fluid or the use of an additional physical information about solutions.

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МЕТОД МАЛОЇ В'ЯЗКОСТІ ТА КРИТЕРІЇ ІСНУВАННЯ УДАРНИХ ХВИЛЬ У РЕЛЯТИВІСТСЬКІЙ МАГНІТНІЙ ГІДРОДИНАМІЦІ

В.І. Жданов, М.С. Борщ

Р е з ю м е

Отримано критерії існування ударних хвиль (УХ) в релятивістській магнітній гідродинаміці без використання умови випуклості рівняння стану. Метод виведення грунтується на розгляді неперервного профілю УХ за наявності релятивістського тензора в'язкості Ландау—Ліфшица з обома ненульовими коефіцієнтами в'язкості η та ζ. Відзначено, що припущення про існування в'язкого профілю лише з одним ненульовим коефіцієнтом (при $\eta = 0$) виявляється дуже жорстким і приводить до втрати деяких фізичних розв'язків.