STATIC PERFECT FLUID BALLS WITH GIVEN EQUATION OF STATE AND WITH COSMOLOGICAL CONSTANT

C.G. BÖHMER[1]

Institut für Theoretische Physik, Technische Universität Wien
(8-10, Wiedner Hauptstr., Wien A-1040, Austria; e-mail: boehmer@hep.itp.tuwien.ac.at)

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The solutions of Einstein's field equations with cosmological constant for a static and spherically symmetric perfect fluid are analyzed. After showing the existence and uniqueness of a regular solution at the center, the extension of this solution is discussed. Then the existence of global solutions with a given equation of state and the cosmological constant bounded by $4\pi\rho_b$, where ρ_b is the boundary density (given by the equation of state) of a perfect fluid ball, is proved.

1. Introduction

This paper analyzes static, spherically symmetric perfect fluid solutions to Einstein's field equations with cosmological constant for a given monotonic equation of state $\rho = \rho(P)$. The choice of the central pressure (central energy density) and cosmological constant uniquely determines the pressure function. The aim of this work is to extend the results presented in [12] to include the cosmological constant.

The existence and uniqueness of a global solution with a given equation of state can be proved for cosmological constants satisfying $\Lambda < 4\pi\rho_b$, which is given by the equation of state since $\rho_b = \rho(P = 0)$, following the line of arguments in [12].

The existence of such global solutions is quite important because Buchdahl [5] showed, assuming their existence, that the total mass of a fluid ball is bounded by its radius. He proved the strict inequality $M < (4/9)R$ which holds for fluid balls, in which the density does not increase outwards. It implies that the radii of fluid balls are always larger than the black-hole event horizon. It is expected that perfect fluid balls with cosmological constant are larger than the black hole event horizon but still be smaller than the cosmological event horizon.

This paper is organized as follows: In Section 2., Einstein's field equations and the Buchdahl variables are presented. At the beginning of Section 3., the uniqueness and existence of a regular solution at the center is shown. In theorem 3., the extension of the solution is discussed. For fluid balls, the Buchdahl inequality with the cosmological term is obtained in Theorem 4.. Solutions without singularities are constructed in Section 5.. Finally, remarks on the finiteness of solutions are given, and some conclusions are presented.

2. Field Equations and Buchdahl Variables

The most general static and spherically symmetric metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{a(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \qquad (1)$$

in Einstein's theory of gravity yields three independent field equations with cosmological constant

$$\frac{1}{r^2}e^{\nu(r)}\frac{d}{dr}\left(r - re^{-a(r)}\right) - \Lambda e^{\nu(r)} = 8\pi\rho(r)e^{\nu(r)}\,, \qquad (2)$$

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$$\frac{1}{r^2}\left(1 + r\nu'(r) - e^{a(r)}\right) + \Lambda e^{a(r)} = 8\pi P(r) e^{a(r)}\,, \tag{3}$$

$$-\frac{\nu'(r)}{2}(P(r)+\rho(r)) = P'(r), \tag{4}$$

for an isotropic perfect fluid. Note that one can either use the three field equations which imply the conservation of energy-momentum or two of the field equations together with the conservation equation which we do.

One function can be chosen freely since these are three independent ordinary differential equations for four unknown functions. The most physical assumption is to prescribe an equation of state $\rho = \rho(P)$. The integration of (2) gives

$$e^{-a(r)} = 1 - 2w(r)r^2 - \frac{\Lambda}{3}r^2\,, \tag{5}$$

where the constant of integration is put to zero demanding the regularity at the center, and $w(r)$ is the mean density up to r defined by

$$w(r) = \frac{m(r)}{r^3}\,, \qquad m(r) = 4\pi\int_0^r s^2\rho(s)ds\,. \tag{6}$$

Eliminating the function $\nu'(r)$ from the field equations yields the Tolman—Oppenheimer—Volkoff [11, 14, 16] equation (TOV-Λ)

$$P'(r) = -r\frac{\left(4\pi P(r) + w(r) - \frac{\Lambda}{3}\right)(P(r)+\rho(r))}{1 - 2w(r)r^2 - \frac{\Lambda}{3}r^2}\,. \tag{7}$$

With the given equation of state, the conservation equation leads to

$$\nu(r) = -\int_{P_c}^{P(r)} \frac{2dP}{P+\rho(P)}\,, \tag{8}$$

where P_c is the central pressure. With $m(r)$, Eqs. (6) and (7) form an integro-differential system for $\rho(r)$ and $P(r)$. However, differentiating the mean density $w(r)$ with respect to r implies

$$w'(r) = \frac{1}{r}\left(4\pi\rho(P(r)) - 3w(r)\right)\,. \tag{9}$$

Therefore, for a certain $\rho = \rho(P)$, Eqs. (7) and (9) form a system of first-order differential equations for $P(r)$ and $w(r)$.

To extract the TOV-Λ equation from Einstein's field equations, the metric function $\nu(r)$ was eliminated. On the other hand, one may eliminate the pressure, for which Buchdahl [5] introduced the new variables

$$y^2 = e^{-a(r)} = 1 - 2w(r)r^2 - \frac{\Lambda}{3}r^2\,, \tag{10}$$

$$\zeta = e^{\nu/2}\,, \tag{11}$$

$$x = r^2\,, \tag{12}$$

which we have supplemented by a cosmological constant. The second field equation (3) written in those variables leads to

$$8\pi P - \frac{2}{3}\Lambda = 4y^2\frac{\zeta_{,x}}{\zeta} - 2w\,. \tag{13}$$

After differentiating this with respect to x and using the conservation equation (4), the pressure can be eliminated. After some algebra, one arrives at

$$(y\zeta_{,x})_{,x} - \frac{1}{2}\frac{w_{,x}\zeta}{y} = 0\,, \tag{14}$$

which is surprisingly similar to the square of the Weyl tensor

$$C_{abcd}C^{abcd} = \frac{64}{3}\frac{y^2}{\zeta^2}\left((y\zeta_{,x})_{,x} + \frac{1}{2}\frac{w_{,x}\zeta}{y}\right)^2 x^2\,. \tag{15}$$

This implies that the constant density solutions are conformally flat [13] since $w_{,x} = 0$, and therefore, with (14), gives $(y\zeta_{,x})_{,x} = 0$.

3. Existence of Unique Regular Solutions

Since the field equation for a static and spherically symmetric perfect fluid with a given equation of state is reduced to a system of singular first-order differential equations, it is our first aim to apply Theorem 1 of [12] to the resulting system with cosmological constant.

Theorem 1 (Rendall and Schmidt, 1991). *Let V be a finite dimensional real vector space, $N : V \to V$ a linear mapping, $G : V \times I \to V$ a C^∞ mapping and $g : I \to V$ a smooth mapping, where I is an open interval in R containing zero. Consider the equation*

$$s\frac{df}{ds} + Nf = sG(s, f(s)) = g(s) \tag{16}$$

for a function f defined on a neighbourhood of 0 and I and taking values in V. Suppose that each eigenvalue of N has a positive real part. Then there exists an open interval J with $0 \in J \subset I$ and a unique bounded C^1 function f on $J \setminus 0$ satisfying (16). Moreover, f extends *to a C^∞ solution of (16) on J if N, G, and g depend smoothly on a parameter z, and the eigenvalues of N are distinct. Then the solution also depends smoothly on z.*

Note that (9) is singular at the center, whereas (7) is not. However, using $\rho = \rho_c + x\rho_1$, where ρ_c is the central density, we find

$$x\frac{dw}{dx} + \frac{3}{2}w = 2\pi\rho_c + 2\pi x\rho_1 \,, \tag{17}$$

$$x\frac{d\rho_1}{dx} + \rho_1 = -\frac{1}{2}\left(\frac{dP}{d\rho}\right)^{-1} \times$$

$$\times \frac{(4\pi P + w - \frac{\Lambda}{3})(P + \rho_c + x\rho_1)}{1 - 2wx - \frac{\Lambda}{3}x} \tag{18}$$

for the given equation of state. With the help of the corresponding pressure relation $P = P_c + xP_1(\rho_1)$ and by noting that

$$\left(1 - 2wx - \frac{\Lambda}{3}x\right)^{-1} = 1 + \left(2wx + \frac{\Lambda}{3}x\right) \times$$

$$\times \left(1 - 2wx - \frac{\Lambda}{3}x\right)^{-1} \,, \tag{19}$$

we find that the matrix N of (16) has the form

$$N = \begin{pmatrix} 3/2 & 0 \\ \frac{(P_c+\rho_c)}{2dP/d\rho(\rho_c)} & 1 \end{pmatrix} . \tag{20}$$

Since the eigenvalues of N are independent of the cosmological constant, the system has a unique bounded solution in the neighbourhood of the center, which is indeed C^∞. This implies the existence of a unique smooth solution to (7) and (9) or (17) and (18) near the center.

In [6], it is claimed that, for a fixed equation of state and a cosmological constant, the choice of central pressure (and, by virtue of the equation of state, of the central density) does not uniquely determine the solution. The existence and uniqueness theorem above disproves this statement.

The uniqueness of the solution at the center immediately implies:

Theorem 2. *Let an equation of state $\rho(P)$, a central pressure P_c, and the cosmological constant Λ be given such that*

$$4\pi P_c + \frac{4\pi}{3}\rho(P_c) - \frac{\Lambda}{3} = 0 \,, \tag{21}$$

where $(4\pi/3)\rho_c = (4\pi/3)\rho(P_c) = w_c$ so that also the central energy density is given by the equation of state. Then the unique solution is the Einstein static universe with $\Lambda = \Lambda_{\mathrm{E}}$.

For the well-defined right-hand sides of (17) and (18), the standard theorems for differential equations guarantee that the solution can be extended. This implies that the solution is extendible if the pressure is finite, $P < \infty$, and if the denominator of (18) is non-zero, i.e. $y = 1 - 2wx - (\Lambda/3)x > 0$. Since the second condition depends on the cosmological constant, new properties may arise. Moreover, it must be clarified whether $y = 0$ corresponds to a coordinate singularity of the space-time, as in the constant density case [4], or to a geometric singularity. In the Buchdahl variables (10)—(12), the line element (1) takes the form

$$ds^2 = -\zeta(x)^2 dt^2 + \frac{dx^2}{4xy(x)^2} + x(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{22}$$

which implies the following non-vanishing components of the Riemann tensor:

$$R_{\theta t}{}^{\theta t} = R_{\phi t}{}^{\phi t} = -2y(x)^2 \frac{\zeta'(x)}{\zeta(x)} \,, \tag{23}$$

$$R_{x\theta}{}^{x\theta} = R_{x\phi}{}^{x\phi} = -2y(x)y'(x) \,. \tag{24}$$

The remaining two ones are given by

$$R_{\theta\phi}{}^{\theta\phi} = \frac{1 - y(x)^2}{x} \,, \tag{25}$$

$$R_{xt}{}^{xt} = -4xy(x)^2 \frac{\zeta''(x)}{\zeta(x)} - 2y(x)^2 \frac{\zeta'(x)}{\zeta(x)} -$$

$$-4xy(x)y'(x)\frac{\zeta'(x)}{\zeta(x)} \,, \tag{26}$$

which indicates that $y \to 0$ corresponds to a coordinate singularity rather then a geometric singularity if $y'(x)$ and $1/\zeta(x)$ behave well as $y \to 0$.

The following theorem clarifies the extendibility of solutions.

Theorem 3. *Suppose the pressure is decreasing near the center, which means*

$$4\pi P_c + \frac{4\pi}{3}\rho(P_c) - \frac{\Lambda}{3} > 0 \,. \tag{27}$$

Then the solution is extendible, and the pressure is monotonically decreasing if

$$4\pi P + w - \Lambda/3 > 0\,. \tag{28}$$

Proof. Assume that $\rho = \rho(P)$, P_c, and Λ are given such that P is decreasing near the center, then $w(x)_{,x} \leq 0$. Since $y > 0$, Eq. (14) implies

$$\left(y\zeta_{,x}\right)_{,x} \leq 0\,. \tag{29}$$

Rewriting (13) gives

$$y\zeta_{,x} = \frac{\zeta}{2y}\left(4\pi P + w - \frac{\Lambda}{3}\right). \tag{30}$$

Then using the implication of (29) leads to

$$y\zeta_{,x} \leq \left(y\zeta_{,x}\right)(0). \tag{31}$$

Together with the explicit expression of $y\zeta_{,x}$ in (30), this finally shows

$$y \geq \frac{4\pi P + w - \frac{\Lambda}{3}}{4\pi P_c + w_c - \frac{\Lambda}{3}}. \tag{32}$$

Therefore, the Buchdahl variable y cannot vanish before the numerator does, and consequently the right-hand sides of (17) and (18) are well-defined. Hence, one can extend the solution if $4\pi P + w - \Lambda/3 > 0$.

Since $y > 0$ and $4\pi P + w - \Lambda/3 > 0$, the sign of the right-hand side of (18) is strictly negative. Therefore, the energy density and, because of the equation of state, the pressure are monotonically decreasing functions. □

The last theorem contained the extendibility condition of solutions, which also allows us to show the existence of global solutions.

Theorem 4. *Suppose an equation of state is given such that ρ is defined for $p \geq 0$, non-negative and continuous for $p \geq 0$, C^∞ for $p > 0$, and suppose that $d\rho/dp > 0$ for $p > 0$. Furthermore, assume that the cosmological constant is given such that $\Lambda < 4\pi\rho_b$.*[2]

Then the pressure is decreasing near the center, and there exists a unique inextendible static and spherically symmetric solution of Einstein's field equations with the cosmological constant having a perfect fluid source with the equation of state $\rho(P)$ for any positive value of the central pressure P_c.

If $\Lambda \leq 0$, the matter either occupies the whole space-time with ρ tending to ρ_∞ as r tends to infinity or the matter has finite extent. In the second case, a unique Schwarzschild-anti-de Sitter solution is attached as an exterior field.

If the cosmological constant satisfies $0<\Lambda<4\pi\rho_b$, the matter has always a finite extent, and a unique Schwarzschild—de Sitter solution is attached as an exterior field.

Proof. If the cosmological constant is given such that $\Lambda < 4\pi\rho_b$, then

$$0 < \frac{4\pi}{3}\rho_b - \frac{\Lambda}{3} < 4\pi P_c + \frac{4\pi}{3}\rho(P_c) - \frac{\Lambda}{3}\,,$$

and the pressure is decreasing near the center by (27).

Since the pressure is decreasing near the center, the denominator of (32) is some positive number. Furthermore, one can estimate the numerator of (32) by

$$y \geq \frac{4\pi P + w - \frac{\Lambda}{3}}{4\pi P_c + w_c - \frac{\Lambda}{3}}\,,$$

$$y \geq \frac{w_b - \frac{\Lambda}{3}}{4\pi P_c + w_c - \frac{\Lambda}{3}}\,, \tag{33}$$

$$y \geq \frac{\frac{4\pi}{3}\rho_b - \frac{\Lambda}{3}}{4\pi P_c + w_c - \frac{\Lambda}{3}} \tag{34}$$

and conclude that if

$$\Lambda < 4\pi\rho_b\,, \tag{35}$$

the Buchdahl variable y cannot vanish before the pressure does. The coordinate x_b where the pressure vanishes will be taken as the definition of the stellar object's radius R.

$\Lambda \leq 0$

If $\Lambda \leq 0$, the matter can occupy the whole space because (32) implies the positivity of y.

Suppose that $P(x_b) = 0$. At the corresponding radius R, the Schwarzschild-anti-de Sitter solution is joined uniquely by the condition $M = m(R)$. In this manner, the metric is C^0 only, because the density at the boundary may be non-zero. The metric is C^1 at $P(R) = 0$ if Gauss coordinates relative to the hypersurface $P(R) = 0$ are used. If the boundary density

[2]Assumptions on the equation of state could be weakened according to [2] and [8]. Moreover, the line of arguments presented in [8] can surely be applied for cosmological constants having the derived bound.

does not vanish, the Ricci tensor has a discontinuity. Thus, the metric is at most C^1. Without further assumptions on the boundary density, this cannot be improved.

Now assume that $P(x) > 0$ for all $x > 0$. $P(x)$ is monotonically decreasing, therefore $\lim_{x\to\infty} P(x) = P_\infty$ exists. This implies that the pressure gradient tends to zero as $x \to \infty$. Because $y^{-1} \to 0$ as $x \to \infty$, Eq. (7) does not imply that $P_\infty = 0$, which it does if $\Lambda = 0$. Thus, the equation of state only gives $\rho \to \rho_\infty = \rho(P_\infty)$ as $x \to \infty$.

$0 < \Lambda < 4\pi\rho_b$

If $0 < \Lambda < 4\pi\rho_b$, then one can estimate the pressure at the possible coordinate or geometric singularity $\hat{r}$ defined by $y(\hat{r}) = 0$ since

$$\frac{\Lambda}{3} < \frac{4\pi}{3}\rho_b \le w_b \le w(r)\,, \tag{36}$$

which holds for all r. Therefore.

$$P(\hat{r}) = \frac{1}{4\pi}\left(\frac{\Lambda}{3} - w(\hat{r})\right) < 0\,. \tag{37}$$

Hence, there exists R such that $P(R) = 0$. Since the pressure is decreasing and $P(\hat{r}) < 0$, it follows that $R < \hat{r}$.

Thus, if the cosmological constant is positive and $\Lambda < 4\pi\rho_b$, then the pressure always vanishes at some x_b. At the corresponding radius R, the Schwarzschild-de Sitter solution is joined uniquely by the same condition $M = m(R)$. The metric is at most C^1. Because the boundary density is larger than zero, this cannot be improved. □

It is quite remarkable that this upper bound $\Lambda < 4\pi\rho_b$ was found independently by considering the consistency of the Newtonian limit [9] and the gravitational equilibrium via the virial theorem [10] and by demanding the stability of circular orbits [15]. This "coincidence" deserves a closer inspection, which was already initiated in [1].

4. Buchdahl Inequality

The importance of Buchdahl's inequality was already discussed in the introduction. In what follows, its generalization with a cosmological constant is derived, see e.g. [3] without the cosmological term and [7] with a cosmological constant following the Buchdahl's original approach.

In the proof of the following theorem, the field equations with constant density, denoted with a tilde, are solved with the help of Buchdahl variables and then this solution is compared with a general decreasing solution. The boundary mean density of the general solution defines the constant density solution that is used for comparison.

Theorem 5. *Let the cosmological constant be given such that $\Lambda < 4\pi\rho_b$. Then, for the solutions having finite radius, there holds*

$$\sqrt{1 - 2w_b R^2 - \frac{\Lambda}{3}R^2} \ge \frac{1}{3} - \frac{\Lambda}{9w_b}\,. \tag{38}$$

Proof. Assume that $\rho(P)$, P_c and Λ are given such that the pressure is decreasing near the center. Then, by Theorem 3., the pressure and mean density are decreasing functions and Eq. (14) implies

$$(\tilde{y}\tilde{\zeta}_{,x})_{,x} = 0\,, \quad \tilde{y}\tilde{\zeta}_{,x} = \frac{1}{2}\tilde{w} - \frac{\Lambda}{6}\,, \tag{39}$$

where the constant of integration is obtained from (13) evaluated at the boundary which can be used to integrate (39). Let us rewrite the right-hand side of (39) as

$$\tilde{y}\tilde{\zeta}_{,x} = \left(2\tilde{w} + \frac{\Lambda}{3}\right)\left(\frac{1}{2}\tilde{w} - \frac{\Lambda}{6}\right)\left(2\tilde{w} + \frac{\Lambda}{3}\right)^{-1}\,, \tag{40}$$

substitute in $-2\tilde{y}\tilde{y}_{,x} = 2\tilde{w} + \Lambda/3$ for the first factor, and divide by $\tilde{y}$. This yields, after the integration,

$$\tilde{\zeta}(x) = -\frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}\tilde{y}(x) + \tilde{\zeta}(0) + \frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}\,. \tag{41}$$

From Eq. (29), we conclude $\tilde{y}\tilde{\zeta}_{,x} = (y\zeta_{,x})_b < y\zeta_{,x}$. So that with $\tilde{y} > y$, one finds

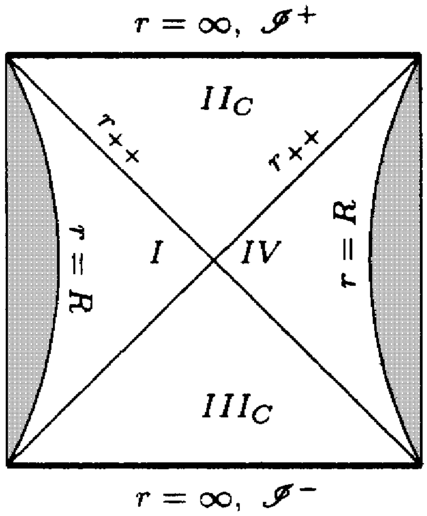
$$\zeta(x) \ge \tilde{\zeta}(x) = -\frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}\tilde{y}(x) + \tilde{\zeta}(0) + \frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}\,. \tag{42}$$

Evaluating this at the boundary and using that ζ can be normalized such that $\zeta_b = y_b$, we get that

$$y_b \ge -\frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}y_b + \tilde{\zeta}(0) + \frac{\tilde{w} - \frac{\Lambda}{3}}{2\tilde{w} + \frac{\Lambda}{3}}\,. \tag{43}$$

Since $\tilde{\zeta}(0)$ is a positive number, some algebra finally leads to the generalized Buchdahl inequality

$$y_b \ge \frac{1}{3} - \frac{\Lambda}{9\tilde{w}}\,. \tag{44}$$



Penrose—Carter diagram with two stellar objects having radii R which lie between the two horizons. Since the group orbits are increasing up to R, the vacuum part contains the cosmological event horizon r_{++}. This solution with two objects has no singularities. Because of regions II_C and III_C, this space-time is not globally static.

Now we compare the solutions with decreasing mean density and the solution with constant density, where the constant density corresponds to the boundary mean density of the general solution. Thus, (44) yields

$$y_b \geq \frac{1}{3} - \frac{\Lambda}{9w_b}\,, \tag{45}$$

which reads more explicitly as

$$\sqrt{1 - 2w_b R^2 - \frac{\Lambda}{3}R^2} \geq \frac{1}{3} - \frac{\Lambda}{9w_b}\,, \tag{46}$$

and holds for all monotonically decreasing densities. □

By using the definition of the boundary mean density $w_b = M/R^3$, one can reformulate (46) and arrive at the generalized Buchdahl inequality

$$3M < \frac{2}{3}R + R\sqrt{\frac{4}{9} - \frac{\Lambda}{3}R^2}\,, \tag{47}$$

that takes the cosmological constant into account. In [7], the $2M/R$ version of (47) was used, for example, to calculate the surface redshift with Λ. For comparison, note that the cosmological constant was rescaled by 8π in [7].

5. Solutions without Singularities

The last two sections have shown the existence of stellar models for cosmological constants $\Lambda < 4\pi\rho_b$. Equation (47) implies that the boundary of a stellar object has the lower bound given by the black-hole event horizon and the upper bound given by the cosmological event horizon. The upper bound is valid for positive cosmological constants.

Stellar models with $\Lambda \leq 0$ have the lower bound given by the black-hole event horizon. At the boundary, the Schwarzschild-anti-de Sitter solution is attached as an exterior field. Therefore, stellar models with $\Lambda \leq 0$ have no singularities. Solutions with cosmological constant satisfying $\Lambda \leq 0$ are globally static.

For positive cosmological constants, the situation is different. But one can also construct solutions without singularities.

At the boundary $r = R$ defined by $P(R) = 0$, the Schwarzschild-de Sitter solution joins C^1 by the usual procedure introducing Gauss coordinates. The surface $r = R$ can also be found in the vacuum region, where the time-like Killing vector is past directed. This means that a second stellar object can be put in the space-time:

Remarks on the Finiteness of the Radius

So far it has been shown that, given an equation of state, a central pressure, and a cosmological constant, there exists a unique model with finite or infinite extent. This depends on the given equation of state and on the cosmological constant. As was proved in Theorem 3., solutions are always finite for positive cosmological constants. If $\Lambda \leq 0$, either the pressure vanishes for some finite radius or the density is always positive and tends to ρ_∞ as r tends to infinity, which depends on the equation of state; in [12], the necessary and sufficient conditions can be found.

6. Conclusions and Outlook

In this paper, the existence and uniqueness of solutions in the case of a static and spherically symmetric perfect fluid with given equation of state for cosmological constants satisfying $\Lambda < 4\pi\rho_b$ were proven. It would be interesting to extend these proofs to any value of the cosmological constant. The existence of these solutions is conjectured in [1].

One could start studying this system numerically or by using generating function techniques. An investigation of the Riemann tensor as outlined might also be a good starting point.

Furthermore, the relation of the different theories that give rise to the same upper bound must be analyzed and understood.

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СТАТИЧНІ КУЛІ ІДЕАЛЬНОЇ РІДИНИ ІЗ ЗАДАНИМ РІВНЯННЯМ СТАНУ ТА КОСМОЛОГІЧНОЮ СТАЛОЮ

К.Г. Бомер

Р е з ю м е

Для ідеальної рідини аналізуються статичні і сферично-симетричні розв'язки польових рівнянь Ейнштейна з космологічною сталою. Після доведення існування та єдиності регулярного розв'язку в центрі, розглядається розширення цього розв'язку. Потім доводиться існування глобальних розв'язків для заданого рівняння стану і космологічної сталої, обмеженої величиною густини, що задається рівнянням стану на поверхні кулі з ідеальною рідиною.