THE FRIEDMAN MODELS WITH THE PRESSURE AND THE COSMOLOGICAL CONSTANT

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A model of the Universe in the presence of different types of matter and the cosmological constant is considered. For the Universe with radiation and $\Lambda \neq 0$, the exact analytical solution is found. The cosmological parameters are derived, and the scale factor dependence on time is constructed.

Introduction

At present, the interest in the problem of the cosmological constant Λ has arisen significantly. At first, the cosmological constant was introduced by A.Einstein in 1917 in order to obtain the static homogeneous solution for the equations of general relativity in the presence of matter, i.e. to provide the appropriate equation of state of matter [1]. However, after the expansion of the Universe was discovered, the necessity of cosmological constant input disappeared, because the Friedman solutions describing the expanding Universe have the physical equation of state without involving Λ [2].

According to the last sufficiently precise observational data, the cosmological constant plays an important dynamical role in our Universe [3–6]. In addition, the physical sense of this value became obvious. Now it is generally accepted that the cosmological constant can be treated as the average energy density of vacuum [7, 11].

The vacuum characteristics are essentially different from ones of all the other kinds of matter. It has the energy density constant in time and to be the same in the whole space for any coordinate system. In this connection, it seems to be interesting to study the model of the Universe, in which the vacuum energy is taken into account as a source except the ordinary matter. By the ordinary matter, cosmology usually means a dust. But, in the early stages of evolution of the Universe, one should also take into account other kinds of matter, the most important of which is a radiation.

In this paper, a Friedman-like model is constructed, i.e. a homogeneous isotropic model, where the different kinds of matter are taken into account as a sources, including the curvature k and vacuum. The models with radiation, curvature, dust, and vacuum are considered in more details.

A Model with Different Forms of Matter

Let us take the metric interval as

$$ds^2 = dt^2 - a^2(t)(dR^2 + f^2(R)d\sigma^2), \tag{1}$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2$, $a(t)$ is the scale factor, $f^2(R) = \sin^2(R), \sinh^2(R)$, and R^2 for the elliptic, hyperbolic, and parabolic solutions, respectively. The speed of light is taken to be equal to 1.

The Friedman equations [8] have the form:

$$\begin{cases} \varepsilon(t) = \dfrac{\dot a^2(t)}{a^2(t)} + \dfrac{k}{a^2(t)}, \\ \dot\varepsilon(t) = -3\dfrac{\dot a(t)}{a(t)}(\varepsilon(t) + p(t)). \end{cases} \tag{2}$$

Here, $\varepsilon(t)$ is the energy density in length units (cm^{-2}), $p(t)$ is the pressure, $\dot\varepsilon(t) = d\varepsilon(t)/dt$, $\dot a = da(t)/dt$, $k = \pm 1; 0$ is the spatial curvature for elliptic, hyperbolic, and parabolic solutions, respectively.

If we have different kinds of matter, for example, the radiation, the dust and others, we should put $\sum \varepsilon_i$ instead of ε into the first equation of (2), where ε_i is the energy density of the i-th kind of matter and the second equation in (2) becomes

$$\sum \dot\varepsilon_i = -3\frac{\dot a(t)}{a(t)} \sum(\varepsilon_i + p_i), \tag{3}$$

p_i being the relevant pressure. If we neglect the interaction between different kinds of matter, Eq. (3) will turn to the system of i equations

$$\dot\varepsilon_i = -3\frac{\dot a(t)}{a(t)}(\varepsilon_i + p_i). \tag{4}$$

Thus, the Friedman system of equations can be written in the following form:

$$\begin{cases} \sum \varepsilon_i(t) = \dfrac{\dot a^2(t)}{a^2(t)} + \dfrac{k}{a^2(t)}, \\ \dot{\varepsilon_i}(t) = -3\dfrac{\dot a(t)}{a(t)}(\varepsilon_i(t)+p_i(t)). \end{cases} \tag{5}$$

Let us find the solution of system (5) under the assumption that

$$p_i = n_i \varepsilon_i, \tag{6}$$

where n_i=const. For the energy density of the i-th kind of matter. we have:

$$\varepsilon_i = \frac{a_i^{\beta_i-2}}{a^{\beta_i}}. \tag{7}$$

Here, $\beta_i = 3(n_i+1)$, and a_i is the appropriate constant for each kind of matter.

In our case, the dominant energy conditions [3] give $\varepsilon_i + p_i \geqslant 0$ and $|p_i| \leqslant |\varepsilon_i|$. Hence, we obtain the following restriction on β_i:

$$0 \leqslant \beta_i \leqslant 6.$$

The solutions of system (5) for each value of the parameter $\beta_i = 0,1,2,3,4,5,6$ are known [10]. $\beta_\Lambda = 0$ corresponds to the de Sitter solution for vacuum [8] with the energy density $\varepsilon_\Lambda = 1/a_\Lambda^2$, where $a_\Lambda = \sqrt{3/\Lambda}$; $\beta_{\text{dust}} = 3$ corresponds to the Friedman solution with $\varepsilon_{\text{dust}} = a_{\text{dust}}/a^3$; $\beta_{\text{rad}} = 4$ corresponds to the radiation with $\varepsilon_{\text{rad}} = a_{\text{rad}}^2/a^4$. The solution with $\beta = 6$ corresponds to a massless scalar field with the equation of state $p = \varepsilon$. The case of $\beta = 2$ represents the known static Einstein solution with the equation of state $p = -\frac{1}{3}\varepsilon$ [1].

The physical sense of the equations of state $p = \frac{2}{3}\varepsilon$ ($\beta = 5$) and $p = -\frac{2}{3}\varepsilon$ ($\beta = 1$) is not obvious yet. However, we note that the model built on the basis of the lattice of cosmic strings and domain walls implies the effective equation of state $p = -\frac{1}{3}\varepsilon$ ($\beta = 2$) for the strings and $p = -\frac{2}{3}\varepsilon$ ($\beta = 1$) for the walls [1,3].

The forms of matter with $\beta = 5$ and $\beta = 6$ play an important part only near the beginning, under very small $a(t)$, when, as will be shown below, the cosmological constant influence on the evolution is negligible. So, these forms of matter will not be taken into consideration in what follows.

Thus, in the case of our interest, the general energy density is

$$\varepsilon = \frac{a_{\text{dust}}}{a^3} + \frac{a_{\text{rad}}^2}{a^4} + \frac{1}{a_\Lambda^2}. \tag{8}$$

For the pressure, relations (6) and (8) yield

$$p = \frac{1}{3}\frac{a_{\text{rad}}^2}{a^4} - \frac{1}{a_\Lambda^2}. \tag{9}$$

System (5) for the energy density (8) cannot be integrated analytically. However, using the first equation from (5) and (8), it is possible to write the metric interval (1) in terms of the scale factor as

$$ds^2 = \left(\tfrac{a_{\text{rad}}^2}{a^2} + \tfrac{a_{\text{dust}}}{a} - k + \tfrac{a^2}{a_\Lambda^2}\right)^{-1} da^2 - \\ -a^2(dR^2 + f^2(R)d\sigma^2). \tag{10}$$

If the Universe is filled with matter of only the i-th form and if $\Lambda \neq 0$, then it is possible for the metric with the flat spatial part ($k = 0$) to find the exact solution of system (5) for the scale factor in the general form [10]

$$a(t) = a_i\left[\frac{a_\Lambda}{a_i}\sinh\frac{\beta_i t}{2a_\Lambda}\right]^{2/\beta_i}. \tag{11}$$

It is also possible to find the general expression for the deceleration parameter

$$q = -\frac{\ddot a a}{\dot a^2} = \frac{\beta - 1 - \cosh\frac{\beta_i t}{a_\Lambda}}{1 + \cosh\frac{\beta_i t}{a_\Lambda}}. \tag{12}$$

For the density parameters Ω_i and Ω_Λ, we have

$$\Omega_i = \frac{\varepsilon_i}{\varepsilon_{\text{cr}}} = \frac{1}{\cosh^2\frac{\beta_i t}{2a_\Lambda}}, \tag{13}$$

$$\Omega_\Lambda = \text{tg}^2\frac{\beta_i t}{2a_\Lambda}. \tag{14}$$

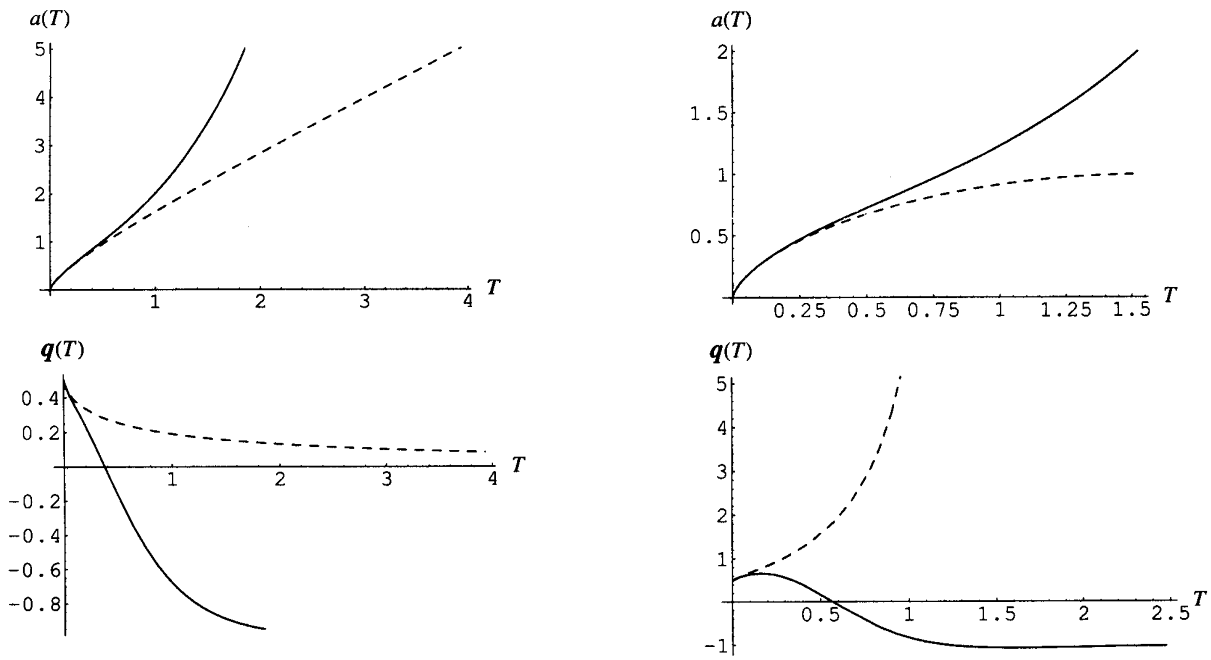
Here, $\varepsilon_{\text{cr}} = H^2$ is the critical energy density, and H is a Hubble constant.

In the case of nonzero spatial curvature ($k = \pm1$,) it is possible to write the cosmological parameters in terms of the scale factor $a(t)$:

$$q = \frac{\frac{\beta_i-2}{2}\left(\frac{a_i}{a}\right)^{\beta_i-2} - \left(\frac{a}{a_\Lambda}\right)^2}{\left(\frac{a_i}{a}\right)^{\beta_i-2} + \left(\frac{a}{a_\Lambda}\right)^2 - k}, \tag{15}$$

$$H = \frac{1}{a}\sqrt{\left(\frac{a_i}{a}\right)^{\beta_i-2} + \left(\frac{a}{a_\Lambda}\right)^2 - k}; \tag{16}$$

$$\Omega_i = \frac{a_i^{\beta_i-2}}{a^{\beta_i-2}\left[\left(\frac{a_i}{a}\right)^{\beta_i-2} + \left(\frac{a}{a_\Lambda}\right)^2 - k\right]}, \tag{17}$$



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$$\Omega_\Lambda = \frac{a^2}{a_\Lambda^2 \left[\left(\frac{a_i}{a} \right)^{\beta_i - 2} + \left(\frac{a}{a_\Lambda} \right)^2 - k \right]}. \tag{18}$$

Let us consider some important particular cases of solution (10).

In the absence of the radiation when $a_{\rm rad} = 0$, it is possible to find the exact solution of system (5) for the flat space ($k = 0$) [10]. This solution was first obtained by Lemaitre in 1927 [9] and represents a generalization of the Friedman solution with the flat spatial part to the case of nonzero cosmological constant.

The results concerning the studying of this solution were published, for example, in [12].

For the case of nonzero spatial curvature $k = \pm 1$ the solution was studied numerically. In Fig. 1, the obtained dependences $a(T)$ and $q(T)$ in dimensionless units ($a = a/a_{\rm dust}$, $T = t/a_{\rm dust}$) are represented for hyperbolic ($k = -1$), elliptic ($k = 1$), and parabolic ($k = 0$) cases, respectively. The dashed curves show the correspondent Friedman solution with $\Lambda = 0$. From the figures, we can conclude that the behavior of the Universe with nonzero Λ differs from that in the Friedman models with $\Lambda = 0$. While an open Friedman model always turns to the flat space-time in the limit $t \to \infty$, the open model with nonzero cosmological constant turns to that with the de Sitter metric at large times. For a closed model with $\Lambda \neq 0$, there is a possibility to have the de Sitter space-time as the end stage of evolution, while a closed Friedman model always collapses.

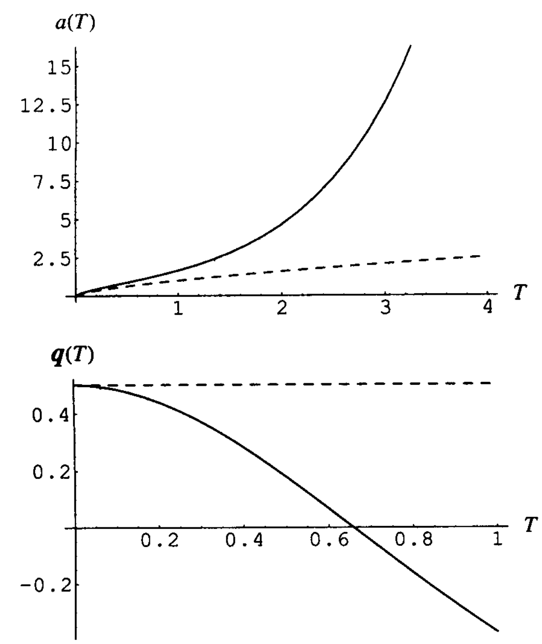

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Fig. 1. Dependences $a(T)$ and $q(T)$ in dimensionless units ($a = a/a_{\rm dust}$, $T = t/a_{\rm dust}$) for hyperbolic ($k = -1$) (a), elliptic ($k = 1$) (b), and parabolic ($k = 0$) (c) cases

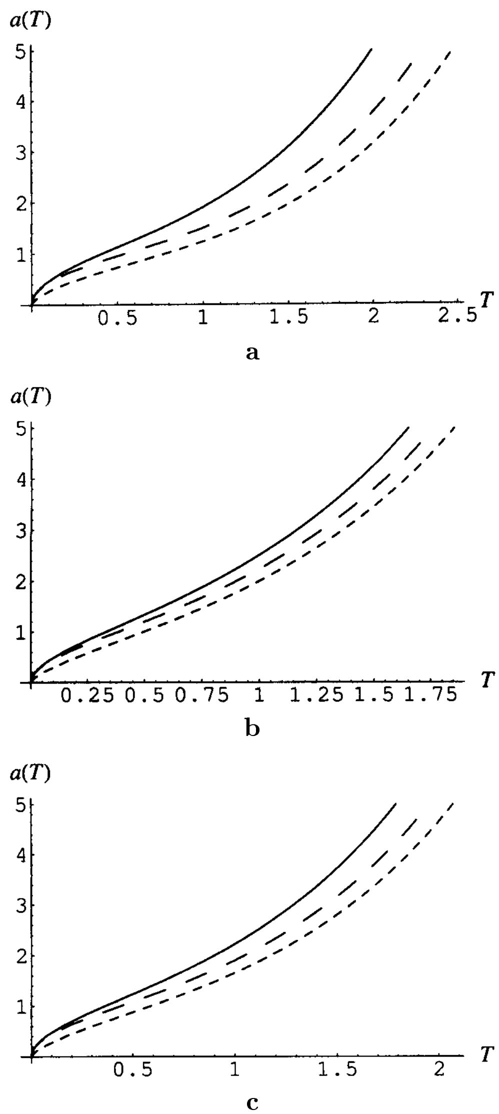

Fig. 2. Dependences of the scale factor on time for different models for $k=+1$ (a), -1 (b), 0 (c) (in dimensionless units)

In addition, one can see that, in the models with $\Lambda \neq 0$, the deceleration parameter curve $q(T)$ crosses the time axis. This means that the deceleration of the Universe expansion turns into the acceleration, which is in accordance with modern observational data [3] and early theoretical results [7], while the Universe expansion is always decelerated for the Friedman models without Λ.

At small times, both dashed and solid curves have the similar behavior. This means that, near the singularity, the cosmological constant does not essentially influence the Universe evolution process.

It is of special interest to consider the Universe with the radiation only and a nonzero cosmological constant, i.e. when $a_{\rm dust}=0$ in (10). In this case, it turns out to be possible to obtain an exact solution in the general form for all three types of spatial curvature. The Friedman equation from (5) for the scale factor will have the form:

$$\frac{a_{\rm rad}^2}{a^2}+\frac{a^2}{a_\Lambda^2}=\dot{a}^2+k. \qquad (19)$$

The solution of Eq. (19) is

$$a^2(t)=a_{\rm rad}a_\Lambda\sinh\frac{2t}{a_\Lambda}-\frac{ka_\Lambda^2}{2}(\cosh\frac{2t}{a_\Lambda}-1). \qquad (20)$$

In solution (20), the constants of integration are taken so that $a(t)=0$ if $t=0$. In the limit $t\to 0$, this solution goes into the known solution for the Universe filled with radiation only, general for all types of the spatial curvature,

$$a^2(t)=2a_{\rm rad}t-kt^2. \qquad (21)$$

It follows from (21) that, at small t, when it is possible to neglect t^2 in comparison with t, all three types $k=0,\pm1$ behave in the same way.

The case of particular interest is the Universe with positive spatial curvature. For the elliptic case, let us rewrite (20) in the following form:

$$a^2(t)=a_\Lambda\sinh\frac{t}{a_\Lambda}\cosh\frac{t}{a_\Lambda}(2a_{\rm rad}-a_\Lambda{\rm tg}\frac{t}{a_\Lambda}). \qquad (22)$$

In this case, a collapse is possible under the condition $a_{\rm rad}<\frac{1}{2}a_\Lambda$. Under $a_{\rm rad}>\frac{1}{2}a_\Lambda$, the evolution is analogous to that in the open model, i.e. the later stage of evolution corresponds to the de Sitter metric. Under $a_{\rm rad}=\frac{1}{2}a_\Lambda$, the Universe goes to the static Einstein Universe with the effective equation of state $\varepsilon+3p=0$. This follows from the fact that $\tanh\frac{t}{a_\Lambda}$ reaches 1 at infinite time, and the scale factor $a(t)\to\frac{a_\Lambda}{\sqrt{2}}$ as $t\to\infty$.

The cosmological parameters in this case in general form for all three k can be obtained from (15) – (18) putting there $\beta_i=4$ and $a(t)$ from (20). They look as follows:

$$q=\frac{\frac{1}{2}\left(\frac{a_{\rm rad}}{a}\right)^2-\left(\frac{a}{a_\Lambda}\right)^2}{\left(\frac{a_{\rm rad}}{a}\right)^2+\left(\frac{a}{a_\Lambda}\right)^2-k}; \qquad (23)$$

$$H=\frac{1}{a}\sqrt{\left(\frac{a_{\rm rad}}{a}\right)^2+\left(\frac{a}{a_\Lambda}\right)^2-k}; \qquad (24)$$

$$\Omega_{\rm rad}=\frac{a_{\rm rad}^2}{a^2\left[\left(\frac{a_{\rm rad}}{a}\right)^2+(\frac{a}{a_\Lambda})^2-k\right]}; \qquad (25)$$

$$\Omega_\Lambda = \frac{a^2}{a_\Lambda^2\left[\left(\frac{a_{\rm rad}}{a}\right)^2 + \left(\frac{a}{a_\Lambda}\right)^2 - k\right]}. \qquad (26)$$

Dependences of the scale factor on time for different models for $k = +1, -1, 0$ is represented (in dimensionless units) in Fig. 2.

In all these figures, solid curves show the behavior of the scale factor for the general case $a_{\rm dust} \neq 0, a_{\rm rad} \neq 0$ and $a_\Lambda \neq 0$, dashed curves represent the case $a_{\rm rad} = 0$, and dashed curves with long dashes are for the case $a_{\rm dust} = 0$. We can conclude from these figures that the behavior of the scale factor for all models is similar, and the Universe is open in all three types of the spatial curvature, except the particular cases like $a_{\rm rad} \leqslant \frac{1}{2}a_\Lambda$.

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ФРІДМАНІВСЬКІ МОДЕЛІ З ТИСКОМ
ТА КОСМОЛОГІЧНОЮ СТАЛОЮ

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Р е з ю м е

Розглянуто модель Всесвіту за наявності різних типів матерії в присутності космологічної сталої. Для Всесвіту з випромінюванням та $\Lambda \neq 0$ знайдено точний аналітичний розв'язок, космологічні параметри та досліджено залежність масштабного фактора від часу.