DRIFT AND ION-ACOUSTIC WAVES IN MAGNETIZED PLASMAS, SYMMETRIES, AND INVARIANT SOLUTIONS

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A $3D$ model for coupled drift and ion-acoustic waves in the inhomogeneous magnetized plasma is considered. Symmetries of the model in the presence of the magnetic shear as well as in the shearless case are found. Some of the most symmetric solutions, exact and perturbative, are presented. In particular, solutions describing the zonal flow generation by initially monochromatic waves are obtained.

Introduction

Low frequency drift and ion-acoustic waves play an important role in the transport processes in magnetized plasmas [1]. The main problem in their treatment is the presence of nonlinear effects even at relative small amplitudes. In this situation, the symmetry analysis can help us to find exact or perturbative solutions (see, e.g. [2]).

Only in the particular case of pure ion-acoustic waves (the well-known Korteweg — de Vries equation in one dimension) dispersion and nonlinear steepening can balance to form coherent structures called solitons. In the two-dimensional case of pure drift waves (Hasegawa—Mima model), anisotropic dispersion fails to balance degenerate vortex non-linear terms [2, 3].

In the present work, a more general spatially three-dimensional model [4] is considered for the coupled drift and ion-acoustic waves. Symmetry analysis for this model is performed, and the magnetic shear influence on symmetry properties is studied. The form of the most symmetric localized and spatially periodic waves is determined. For the waves of small but finite amplitude, the perturbation theory based on the multiple-time-scale formalism is built. Some exact and perturbative solutions describing higher harmonics generation, frequency shifts, and zonal flow generation by initially monochromatic waves are presented.

1. Model

Let us consider an inhomogeneous plasma slab with the background plasma density

$$n_o \sim \exp(x/L_n)$$

in the constant external magnetic field

$$\mathbf{B} = B_o(\mathbf{e}_z + \mathbf{e}_y(x/L_{sh}))$$

with shear length L_{sh}. The condition of quasi-neutrality relates the ion density to the electron density, $n_i \approx n_e$. Electrons, unlike ions, are magnetized, smoothing an electrostatic potential Φ along the magnetic field lines,

$$n_e = n_o \exp(e\Phi/T_e),$$

where e and T_e mean the electron charge and electron plasma component temperature, respectively.

In this case, the well-known $3D$ generalization (described in detail in [4]) of the Hasegawa—Mima model equations holds:

$$d\Psi/dt + dv/dz = \partial\Phi/\partial y, \quad dv/dt + d\Phi/dz = 0,$$

$$\Psi = \Phi - \Delta_{tr}\Phi, \tag{1}$$

where v is the ion velocity along the main magnetic field direction $0z$. $\Psi \equiv \Psi_z$ is the only non-zero component of the generalized vorticity determined by the potential Φ according to the last equation of system (1). The operators in (1) are

$$d/dt = \partial/\partial t + J[\Phi, \ldots],$$

$$J[F,G] \equiv \partial F/\partial x \partial G/\partial y - \partial F/\partial y \partial G/\partial x,$$

$$d/dz = \partial/\partial z + Sx\partial/\partial y, \quad S = L_{sh}/L_n,$$

$$\Delta_{tr}\Phi \equiv \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2.$$

Transverse ion velocity components are determined by the potential Φ,

$$v_x = -\partial\Phi/\partial y, v_y = \partial\Phi/\partial x,$$

while the temporal evolution of the longitudinal ion velocity component $v_z \equiv v$ is governed by system (1).

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System (1) is written in dimensionless variables $\varepsilon\omega_{\rm B}t, x/r_{\rm B}, y/r_{\rm B}, \varepsilon z/r_{\rm B}, e\Phi/T_e\varepsilon$, where the ion cyclotron frequency $\omega_{ci} = eB_o/Mc$ and ion sound speed $c_s = (T_e/M)^{1/2}$ determine the characteristic dispersion length $r_{\rm B} = c_s/\omega_{ci}$. The small parameter ε is equal to the ratio $r_{\rm B}/L_n$.

As a consequence of system (1), the quantities $\langle\Phi\rangle$ and $\langle v\rangle$ averaged over the coordinates x, y (transverse to the main external magnetic field) are the conjugated solutions of the linear ion-acoustic wave equation in t and the longitudinal variable z:

$$\partial\langle\Phi\rangle/\partial t + \partial\langle v\rangle/\partial z = 0,$$

$$\partial\langle v\rangle/\partial t + \partial\langle\Phi\rangle/\partial z = 0, \quad \langle\Psi\rangle = \langle\Phi\rangle, \tag{2}$$

where the averaging procedure means the integration over the spatial period $2L$,

$$\langle\Phi\rangle \equiv (1/4L^2)\int \Phi(t,x,y,z)dxdy,$$

for the solutions periodic in x, y, and $L \to \infty$ for the localized solutions.

2. Symmetries

First, to express the symmetries in a more simple form, let us perform the simplifying transformation

$$\Phi = \Phi(t,x,y+t,z) - x, \quad v = v(t,x,y+t,z) \tag{3}$$

which removes the term $\partial\Phi/\partial$y from the first equation of system (1). On the other hand, the boundary conditions become more complicated. For example, the homogeneous boundary condition for the RHS functions $\Phi(t,x,y+t,z)$ and $v(t,x,y+t,z)$ takes the form

$$\Phi = x, \quad v = 0 \quad \text{as} \quad |r| \to \infty, \quad r = (x^2+y^2+z^2)^{1/2}. \tag{4}$$

Periodical boundary conditions also are shifted by x.

In the shearless case, $S = 0$, we obtain a Lie group of symmetry by the standard procedure. The infinitesimal operators of this group are

$$X_1 = \partial/\partial t, \quad X_2 = \partial/\partial x, \quad X_3 = \partial/\partial y,$$

$$X_4 = \partial/\partial z, \quad X_5 = -y\partial/\partial x + x\partial/\partial y,$$

$$X_6 = t\partial/\partial t + z\partial/\partial z - \Phi\partial/\partial\Phi - \Psi\partial/\partial\Psi - v\partial/\partial v,$$

$$X = F(t,z)(\partial/\partial\Phi + \partial/\partial\Psi) + G(t,z)\partial/\partial v, \tag{5}$$

where F and G are the conjugated solutions of the linear wave equation in variables t and z:

$$\partial F/\partial t + \partial G/\partial z = 0, \quad \partial G/\partial t + \partial F/\partial z = 0. \tag{6}$$

Model (1) also admits the reflection symmetries

a) $\{x, \Phi, \Psi, v\} \to \{-x, -\Phi, -\Psi, -v\}$,

b) $\{t, y, v\} \to \{-t, -y, -v\}$,

c) $\{z, v\} \to \{-z, -v.\}$. (7)

In the presence of magnetic shear, $S \neq 0$, the symmetry group is reduced, and the remaining infinitesimal operators are

$$X_1, \ X_3, \ X_4, \ \text{and} \ X. \tag{8}$$

A lesser number of the reflection symmetries is admitted:

a) $\{x, z, \Phi, \Psi\} \to \{-x, -z, -\Phi, -\Psi\}$,

b) $\{t, y, z\} \to \{-t, -y, -z\}$. (9)

The physical reason for this symmetry reduction is the explicit and anisotropic dependence of the external magnetic field on the x-coordinate, i.e. along the background plasma density gradient.

Let us consider now the symmetry transformation generated by the operator X in (5), (8). This gauge transformation of the potential Φ and ion velocity v allows us to add to any solution of (1), arbitrary conjugate solutions $F(t,z)$, $G(t,z)$ of the pure ion acoustic linear wave equation in t, z variables:

$$\Phi' = \Phi + F(t,z), \quad \Psi' = \Psi + F(t,z), \quad v' = v + G(t,z) \tag{10}$$

where $F(t,z)$ and $G(t,z)$ are conjugated by condition (6). This symmetry is not affected by the simplifying transformation (3), so it is valid for both initial and simplified systems. In other words, the solution of (1) is determined up to the addition of an arbitrary pure ion-acoustic wave. This fact allows us to impose the following conditions on any solution of (1):

$$\langle\Phi\rangle = \langle\Psi\rangle = 0, \quad \langle v\rangle = 0. \tag{11}$$

Here, the brackets mean the averaging over the variables x, y transverse to the main external magnetic field.

When conditions (11) are imposed, only X_1 to X_6 symmetries remain in the shearless case (5) and only X_1, X_3, X_4 if the magnetic shear is present (8). Reflection symmetries are not affected by conditions (11).

The physical reason for symmetry (10) is the neglect of the ion-acoustic potential non-linearity in model (1), since the corresponding nonlinear term produces only next-order (i.e., ε^2) effects. On the other hand, the presence of a linear pure ion-acoustic wave as a symmetry transformation of model (1) is not trivial.

3. Solutions

It is clear from (10) that an arbitrary linear pure ion-acoustic wave is an exact solution of the non-linear system (1) as a consequence of the gauge symmetry. This result does not depend on the presence of magnetic shear. Only time t and the longitudinal variable z are involved in this solution. That is, the non-linear terms describing the transverse (to the main external magnetic field) dynamics exactly vanish in this case.

Let us review now the most symmetric solutions of model (1) involving both longitudinal and transverse variables, in the shearless case ($S = 0$). Without loss of generality, we impose condition (11) on the solutions considered. Among symmetries (5), only X_1 to X_4 are compatible with the homogeneous boundary condition (4) or a periodic one. Combining these symmetries with the reflection ones (7), we see that the most symmetric solutions must have the form (u being an arbitrary constant)

$$\Phi = f(x, y + ut, z), \quad v = g(x, y + ut, z), \tag{12}$$

where, according to (7), the functions f and g must have the following properties:

$$f(-x, y + ut, z) = -f(x, y + ut, z),$$

$$f(x, -(y + ut), z) = f(x, y + ut, z),$$

$$f(x, y + ut, -z) = f(x, y + ut, z),$$

$$g(-x, y + ut, z) = -g(x, y + ut, z),$$

$$g(x, -(y + ut), z) = -g(x, y + ut, z),$$

$$g(x, y + ut, -z) = -g(x, y + ut, z). \tag{13}$$

It should be noted that the constant velocity u is an essential parameter, since the similarity transformation generated by the infinitesimal operator X_6 of the symmetry group (5) is not compatible with the boundary condition (4).

a) First, let us consider pure drift waves, $\partial/\partial z = 0$, $v = 0$.

In this particular case, the symmetry conditions (12), (13) are simplified [2]:

$$\Phi = f(x, y + ut), \quad f(-x, y + ut) = -f(x, y + ut),$$

$$f(x, -(y + ut)) = f(x, y + ut). \tag{14}$$

There exist two exact solutions of this kind periodic in the variables x and y, namely the periodic zonal flow of plasma

$$\Phi = \sin(k_1 x) \tag{15}$$

and a monochromatic standing wave

$$\Phi = \sin(k_1 x) \cos(\omega_1 t + k_2 y). \tag{16}$$

The amplitudes of these solutions are arbitrary, since the nonlinear term $J[\Phi, \Psi]$ exactly vanishes. So the frequency ω_1 in the second solution is determined by the expression of the linear wave theory:

$$\omega_1 = k_2/(1 + k_1^2 + k_2^2).$$

Upon the combination of the exact solutions (15) and (16), the nonlinear term $J[\Phi, \Psi]$ is not zero. To study the non-linear interaction of the zonal flow (15) with the standing wave (16), let us choose the initial condition

$$\Phi(0, x, y) = \alpha(1 + \beta \cos(k_2 y)) \sin(k_1 x),$$

where the constant $\beta \approx 1$ is the weight parameter of the standing wave relative to the zonal flow. Supposing that wave amplitude α is small but finite, we obtain the perturbative solution

$$\Phi = \alpha\Phi_1 + \alpha^2\Phi_2 + \dots. \tag{17}$$

In this way, we obtain

$$\Phi_1 = (1 + \beta \cos(\omega_1 t + k_2 y + \delta\omega t)) \sin(k_1 x), \tag{18}$$

where the frequency shift is equal to $\delta\omega = (\alpha^2/12) k_2^3 (3k_1^2 + k_2^2)$.

In the second order, we obtain

$$\Phi_2 = \beta k_2^2 (1 + k_1^2 + k_2^2)(\cos(\omega_2 t + k_2 y) -$$

$$- \cos(\omega_1 t + k_2 y)) \sin(2k_1 x)/(6k_1), \tag{19}$$

where $\omega_2 = k_2/(1 + 4k_1^2 + k_2^2)$.

As a result of the interaction, higher harmonics are generated, beginning from the second order α^2 of the amplitude. The frequency shift $\delta\omega$ of the main harmonic appears as the third-order effect ($\sim \alpha^3$). Pulsations of the zonal flow also appear in the third order, the correspondent complicated expressions are omitted here.

b) $3D$ nonlinear drift-ion acoustic standing wave, whose potential Φ and ion velocity v have the form (12) and obey the symmetry conditions (13). The additional non-linear term $J[\Phi, v]$ appears in this case. To describe the dynamics of the periodic waves, we must build up the perturbative solution

$$\Phi = \alpha\Phi_1 + \alpha^2\Phi_2 + \dots, \quad v = \alpha v_1 + \alpha^2 v_2 + \dots \tag{20}$$

for the initial conditions

$$\Phi(0,x,y,z)=\sin(k_1x)(\omega_1\cos(\theta)+$$

$$+\omega_2\sin(\theta))\cos(k_2y)\cos(k_3z),$$

$$v(0,x,y,z)=k_3\sin(k_1x)(\cos(\theta)+$$

$$+\sin(\theta))\sin(k_2y)\sin(k_3z),$$

where the parameter θ determines the relative weight of the components with the frequencies (in linear approximation) ω_1 and ω_2:

$$\omega_{1,2}=(k_2\pm$$

$$\pm(k_2^2+4(1+k_1^2+k_2^2)k_3^2)^{1/2})/(2(1+k_1^2+k_2^2)).$$

In the first order, we obtain

$$\Phi_1=\sin(k_1x)(\omega_1\cos(\theta)\cos(\omega_1t+k_2y)+$$

$$+\omega_2\sin(\theta)\cos(\omega_2t+k_2y))\cos(k_3z),$$

$$v_1=k_3\sin(k_1x)(\cos(\theta)\sin(\omega_1t+k_2y)+$$

$$+\sin(\theta)\sin(\omega_2t+k_2y))\sin(k_3z). \tag{21}$$

In the second order of the amplitude, higher harmonics are generated:

$$\Phi_2=(k_1k_2/16)\sin(2k_1x)\cos(2k_3z)\times$$

$$\times(2(\omega_1+\omega_2+(\omega_1-\omega_2)\cos(2\theta)\sin^2(\omega_3t/2)-$$

$$-((\omega_1+\omega_2)\omega_3^2(\cos((\omega_1-\omega_2)t)-$$

$$-\cos(\omega_3t))\sin(2\theta))/((\omega_1-\omega_2)^2-\omega_3^2)),$$

$$v_2=-(k_1k_2/16)(1+4k_1^2)^{1/2}\sin(2k_1x)\sin(2k_3z)\times$$

$$\times((\omega_1+\omega_2+(\omega_1-\omega_2)\cos(2\theta))\sin(\omega_3t)+$$

$$+(\omega_1+\omega_2)\omega_3\sin(2\theta)((\omega_1-\omega_2)(\sin((\omega_1-\omega_2)t)-$$

$$-\omega_3\sin(\omega_3t))/((\omega_1-\omega_2)^2-\omega_3^2)). \tag{22}$$

Here, $\omega_3=2k_3/(1+4k_1^2)^{1/2}$. It is interesting that, for any combination of waves determined by the weight factor θ, the second-order (α^2) contribution does not depend on the drift direction coordinate y. Thus, in the second order, a pure zonal flow is generated, and the correspondent ion velocity components are

$$v_x=-\partial\Phi_2/\partial y=0,$$

$$v_y=\partial\Phi_2/\partial x=(k_1^2k_2/8)\cos(2k_1x)\cos(2k_3z)\times$$

$$\times(2(\omega_1+\omega_2+(\omega_1-\omega_2)\cos(2\theta)\sin^2(\omega_3t/2). \tag{23}$$

In this way, the result obtained in [5] for the particular case $\theta=0$ is generalized to the temporal evolution of an arbitrary combination of the waves with frequencies (in linear approximation) ω_1 and ω_2. This solution describes the zonal flow generation by the combination of the initially monochromatic coupled drift and ion-acoustic waves. The generation is self-consistent as compared to the zonal flow generation by a drift-wave pump considered in [6].

In the third order (α^3), the shifts $\delta\omega_{11}$ and $\delta\omega_{12}$ of the main frequency ω_1 appear in Φ_1 and v_1, respectively (the $\theta=0$ expressions are presented for the sake of simplicity):

$$(\delta\omega_{11}/\omega_1)=(\alpha^2/16)k_1^2k_2^2(k_2^2-3k_1^2)/(1+k_1^2+k_2^2),$$

$$(\delta\omega_{12}/\omega_1)=-(\alpha^2/16)k_1^2k_2^2. \tag{24}$$

Conclusions

A continuous symmetry group is found for a $3D$ generalization of the Hasegawa—Mima model (1) in the shearless case (5) and in the presence of the magnetic shear (8). The reflection symmetries (6), (9) are taken into account as well.

A pure ion-acoustic linear wave is the exact solution of these model equations. Moreover, it can be added to any other solution by the gauge symmetry transformation (10). On the other hand, this gauge transformation allows us to impose conditions (11) on the transverse averaged potential and longitudinal ion velocity.

Pure drift waves are present in the model as the invariant ($\partial/\partial z=0$) solutions, and the model equations are reduced to the usual $2D$ Hasegawa—Mima equation in this case. The symmetry conditions (14) determine the form of the most symmetric solutions. Among the exact solutions of this kind, the zonal flow (15) and the standing wave (16) are present. The non-linear term exactly vanishes on these solutions, but the interaction of the zonal flow with the standing wave is non-trivial, the corresponding approximate solution was found by the perturbation theory based on the multiple-time-scale formalism (17)—(19). The higher harmonics generation, frequency shifts and zonal flow pulsations are the main non-linear effects in this case.

Finally, the coupled periodic drift and ion-acoustic waves are considered. The symmetry conditions (12) and (13) which determine the form of solutions were obtained, and then perturbation theory was applied (20). As usual, higher harmonics are generated, but the second-order terms of the amplitude do not depend on the drift direction coordinate y. So it is shown that the main non-linear effect in this case is the zonal flow generation by an arbitrary combination of two basic monochromatic standing waves (23). Moreover, frequency shifts are determined (24) which represent one of the third-order effects in the amplitude.

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ДРЕЙФОВІ ТА ІОННО-АКУСТИЧНІ ХВИЛІ
У МАГНЕТИЗОВАНІЙ ПЛАЗМІ, СИМЕТРІЇ
ТА ІНВАРІАНТНІ РОЗВ'ЯЗКИ

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Р е з ю м е

Розглянуто тривимірну модель зв'язаних дрейфових та іонно-акустичних хвиль у магнетизованій неоднорідній плазмі. Знайдено перетворення симетрії цієї моделі як у присутності магнітного ширу, так і без нього. Наведено деякі найбільш симетричні розв'язки, точні та одержані з допомогою теорії збурень. Зокрема, отримано розв'язки, що визначають генерацію зональних потоків початково монохроматичною хвилею.