GENERALIZATION OF THE γ-METRIC IN THE PRESENCE OF NON-GRAVITATIONAL FIELDS

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We consider a generalization of the well-known static axisymmetric vacuum solution of the Einstein equations in the case where the central naked singularity is also a source of some non-gravitational field or fields with the diagonal energy-momentum tensor (EMT). We find the condition for EMT components, which makes such a generalization possible. In the special case of massless field, when all components of the EMT have the same coordinate dependence, we obtain two generalized metrics and study the properties of these space-times, in particular, the type of singularities.

1. The γ-Metric and its Properties

The so-called γ-metric was obtained by Zipoy [1] and Voorhees [2]. After rewriting in the different coordinate system, it gets the form

$$ds^2 = \mathrm{tgh}^{2\mu}\frac{v}{2}dt^2 - \frac{L^2}{4}\mathrm{tgh}^{-2\mu}\frac{v}{2}\sinh^2 v\times$$
$$\times\left[\left(1+\frac{\cos^2 u}{\sinh^2 v}\right)^{1-\mu^2}\left(dv^2+du^2\right)+\cos^2 u d\varphi^2\right]. \qquad (1)$$

In this paper we use the system of units with $c = G = 1$. The vacuum static and axisymmetric γ-solution of the Einstein equations is one of the simplest examples of the Weyl metric

$$ds^2 = e^{\nu}dt^2 - \rho^2 e^{-\nu}d\varphi^2 - e^{\gamma-\nu}(d\rho^2+dz^2),$$
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\nu}{\partial\rho}\right)+\frac{\partial^2\nu}{\partial z^2}=0,$$
$$\frac{\partial\gamma}{\partial z}=\rho\frac{\partial\nu}{\partial\rho}\frac{\partial\nu}{\partial z}, \qquad \frac{\partial\gamma}{\partial z}=\frac{\rho}{2}\left[\left(\frac{\partial\nu}{\partial\rho}\right)^2-\left(\frac{\partial\nu}{\partial z}\right)^2\right]. \qquad (2)$$

One could get the γ-metric in the case where a harmonic function $\nu(\rho,z)$ has a source in the form of a thin rod with length L and constant linear mass density μ in a flat space with coordinates ρ, φ, z. The best way to solve the equation $\Delta\nu = 0$ in this case is to use the coordinates v, u, φ for the oblate spheroidal system with foci on the ends of the rod. One can get a solution $\nu = 2\mu\ \ln(\mathrm{tgh}(v/2))$ and finally obtain metric (1). At $\nu = 0$, it has a naked time-like singularity. If $\mu = 0$ or $\mu = 1$, it is just a coordinate singularity, otherwise it is a real one with curvature invariants diverging at $v \to 0$.

Both the structure and the properties of this singularity and the whole space-time (1) are quite complex. Their total analysis was made in [3]. It was shown that there are three main possibilities. At $\mu < 0$, we deal with a point-like singularity with a negative mass. In particular case $\mu = -1$, the γ-metric becomes equivalent to the Schwarzschild metric with a negative mass $M = -L/2$. At $\mu = 0$, we have the Minkowski space-time. At $0 < \mu < 1$, the central singularity is a linear one. At $\mu = 1$, metric (1) is also equivalent to the Schwarzschild one, but, in this case, its mass is positive $M = L/2$. The false singularity $v = 0$ corresponds to a black hole's horizon. At $\mu > 1$, we deal with a new type of singularity which cannot exist in a space-time with finite curvature. Its properties were described in [3]. Later, in [4], such singularities were named paradoxical. At $\mu \geq 2$, there are also two directional singularities at the ends of the "rod" ($v = 0$, $u = \pm\pi/2$), corresponding to two infinitely distant points connected by the paradoxical singularity. In this case, space-time (1) has three spatial infinities.

As shown in [3], all these types of singularity are typical of more general classes of naked singularities, such as Weyl singularities, Israel's simple linear sources, and generalized spatial Kasner singularities. That is the reason why the study of the γ-metric is so important for the investigation of the properties of naked singularities.

2. Generalization of the γ-Metric

There are many non-gravitational fields, both in nature and in theoreticians' speculations. If the naked singularity is a source of gravitational and non-gravitational fields, the latter can influence its type and properties. We can demonstrate this with a simple example of the γ-metric generalization in a case where the singularity $v = 0$ is also a source of some non-gravitational field. The properties of space-time as well as the possible types of singularities can be assigned to some more complex classes of metrics with time-like singularities.

Such a generalization was obtained in [4] for the cases of electrostatic and scalar fields acting both individually and together. All these generalized metrics have the form

$$ds^2 = \mathrm{tgh}^{2\mu}\frac{v}{2}e^{2\alpha}dt^2 - \frac{L^2}{4}\mathrm{tgh}^{-2\mu}\frac{v}{2}\sinh^2 v\times$$
$$\times\left[e^{2\beta}\left(1+\frac{\cos^2 u}{\sinh^2 v}\right)^{1-\mu^2}\left(dv^2+du^2\right)+e^{2\gamma}\cos^2 u d\varphi^2\right], \qquad (3)$$

where the new functions α, β, γ depend only on the coordinate v. The exact expressions for these functions and the complete analysis of the space-times obtained can be found in [4].

In the present paper, we consider a generalization of the γ-metric (1) in the form (3) if the singularity $v = 0$ is the source of an arbitrary non-gravitational field or some combination of such fields. Why the possibility of generalization in the form (3) is so important to us? This metric describes the simplest space-time with the source of both gravitational and non-gravitational fields with finite length. One can investigate the type and the properties of this time-like singularity using the method developed in [3]. By its comparison with the type and the properties of the ordinary Zipoy—Voorhees metric obtained in [3], one can study the influence of a non-gravitational field or fields on the properties of the time-like singularities of different types. The qualitative results obtained using solution (3) can be used for describing the more general classes of space-times.

We will find the conditions that must be imposed on the field's energy-momentum tensor in order to get a generalization in the form (3). In the special case where all components of T_i^k have the same coordinate dependence, we will get the generalized metric and make a brief analysis of its properties. We will use only two assumptions.

In order to discard the Schwarzschild metric, we assume that $\mu^2 \neq 1$.

We suggest that the field's energy-momentum tensor is diagonal in the coordinate system $x^i = (t, v, u, \varphi)$ used in (3). Naturally, this is not the general case. For example, a generalization in the form (3) is valid if the field under consideration is a massless conformal-invariant Penrose's scalar field with non-diagonal energy-momentum tensor in frame (3) [5,6]. Nevertheless, in the present paper, we restrict our attention to fields with diagonal energy-momentum tensors.

From the (01)-component of the Einstein equations, we get

$$0 = R_{12} = (\gamma' - \beta')\,\mathrm{tg}\,u -$$
$$-(\alpha' + \gamma')(1-\mu^2)\,\frac{\sin u \cos u}{\sinh^2 v + \cos^2 u}. \qquad (4)$$

This can be valid only if

$$\gamma' = \beta' = -\alpha'. \qquad (5)$$

Using this relationship we can write the diagonal components of the Ricci tensor in the form

$$R_0^0 = -R_2^2 = -R_3^3 = -g^{11}(\alpha'' + \alpha'\ \mathrm{ctg}\, v), \qquad (6)$$

$$R_1^1 = -g^{11}\left(\frac{4\mu\alpha'}{\sinh v} + 2\alpha'^2 - \alpha'' - \alpha'\ \mathrm{ctg}\, v\right). \qquad (7)$$

Substituting (6), (7) into the Einstein equations, we can find out the u-dependence of the energy-momentum tensor components

$$T_0^0 = f(v)\, g^{11}(u,v)/8\pi,$$
$$T_2^2 = T_3^3 = -T_1^1 = q(v)\, g^{11}(u,v)/8\pi. \qquad (8)$$

The functions f and q depend only on v and satisfy the equations

$$q = \frac{2\mu\alpha'}{\sinh v} + \alpha'^2, \qquad f = q - 2\left(\alpha'' + \alpha'\ \mathrm{ctgh}\, v\right). \qquad (9)$$

The fulfillment of conditions (8) and (9) means that the equations $T^k_{i;k} = 0$ are also met.

Conditions (8) are quite rigid and rule out a lot of hypothetical fields. A good example is the non-linear massless scalar field proposed by Korkina and Grigorev in [7]. Such a field ψ has the Lagrangian density, the energy-momentum tensor, and the field equation

$$L = -\left(\psi_{,l}\psi^{,l}\right)^2,$$

$$T_{ik} = -4\left(\psi_{,l}\psi^{,l}\right)\psi_{,i}\psi_{,k} + g_{ik}\left(\psi_{,l}\psi^{,l}\right)^2,$$

$$\psi_{,i}\psi^{,i}\psi^{;k}_{;k} + 2\psi^{,i}\psi^{,k}\psi_{;ik} = 0. \qquad (10)$$

Solving this equation in space-time (3), we get the dependence $T^0_0 \propto T^1_1 \propto T^2_2 \propto T^3_3 \propto \left(g^{11}\right)^2$. This means that it is impossible to generalize the γ-metric in the presence of field (10).

3. The Case of the Same Coordinate Dependence of the Energy-Momentum Tensor

Naturally, we cannot solve two equations (9) and get the three functions α, f, q without using an equation for a non-gravitational field. Nevertheless, we can get some further progress in the case where all components of the energy-momentum tensor have the same coordinate dependence and $q(v)$ is proportional to $f(v)$. Let us note that, in all interesting cases, we have $q \neq 0$. Really, substituting $q = 0$ into Eq. (9), one can get either the pure γ-metric (1) with α =const, or the flat space-time with α =const$-\mu$ tgh$(v/2)$, g_{00} =const. So let us put the additional condition of proportionality in the form

$$q - f = Aq, \quad A = \text{const}, \qquad (11)$$

$$T^k_i = \frac{q(v)\, e^{2\alpha(v)} \operatorname{tgh}^{2\mu}\frac{v}{2}}{2\pi L \sinh^2 v}\left(1 + \frac{\cos^2 u}{\sinh^2 v}\right)^{\mu^2-1} \times$$

$$\times \begin{pmatrix} A-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (12)$$

This is practically impossible if we consider the co-work of the several non-gravitational fields of different types, e.g. scalar and vector ones. Even in the case of the one massive field, condition (10) is not fulfilled. But exactly in the case of proportionality, we have dealing with one non-gravitational massless field. If such a field is electrostatic, we have $A = 2$. The case of the massless scalar field corresponds to $A = 0$. Thus, case (10) is worth to be considered.

Substituting (11) into (9) and integrating, we have

$$\alpha' = C\, e^{A\alpha/2} \operatorname{tgh}^{\mu A}\frac{v}{2} \sinh^{-1} v, \qquad C = \text{const}. \qquad (13)$$

If $A \neq 0$ and $\mu \neq 0$, the solution of (13) and the function q have the form

$$e^\alpha = C_1 F, \quad e^\beta = C_2 F^{-1}, \quad e^\gamma = C_3 F^{-1},$$

$$C_1, C_2, C_3 = \text{const}, \qquad (14)$$

$$F = \left(1 - \lambda \operatorname{tgh}^{\mu A}\frac{v}{2}\right)^{-2/A},$$

$$q = \frac{4\mu^2}{\sinh^2 v}\frac{\lambda \operatorname{tgh}^{\mu A}\frac{v}{2}}{\left(1 - \lambda \operatorname{tgh}^{\mu A}\frac{v}{2}\right)^2},$$

$$\lambda = \text{const}, \quad C_1 = \left(\frac{2\mu\lambda}{C}\right)^{2/A}. \qquad (15)$$

If μ=0, we obtain (14) with

$$F = \left(1 - \lambda \ln\operatorname{tgh}\frac{v}{2}\right)^{-2/A},$$

$$q = \frac{4\lambda^2}{A^2 \sinh^2 v}\left(1 - \lambda \ln\operatorname{tgh}\frac{v}{2}\right)^{-2},$$

$$\lambda = \text{const}, \quad C_1 = \left(\frac{2\lambda}{CA}\right)^{2/A}. \qquad (16)$$

If $A = 0$, we have

$$e^\alpha = \text{const} \operatorname{tgh}^C\left(\frac{v}{2}\right), \qquad q = \left(2\mu C + C^2\right)\sinh^{-2} v. \qquad (17)$$

This is the case of the massless scalar field considered in [4].

In order to avoid conical singularities at the axis $u = \pm\pi/2$, $v \neq 0$, we must put $C_2 = C_3$. Hereafter, we will hide this constant by denomination $L_{\text{new}} = L|C_2|$. We will also hide the constant C_1 by time rescaling: $t_{\text{new}} = |C_1|t$. Further, we will omit the "new" indices. As a result, we obtain two generalizations of the γ-metric. In the general case, the metric has the form

$$ds^2 = \left|\operatorname{tgh}^{-\mu A/2}\frac{v}{2} - \lambda \operatorname{tgh}^{\mu A/2}\frac{v}{2}\right|^{-4/A} dt^2 -$$

$$-\left|\mathrm{tgh}^{-\mu A/2}\frac{v}{2}-\lambda\mathrm{tgh}^{\mu A/2}\frac{v}{2}\right|^{4/A}\times$$
$$\times\frac{L^2}{4}\sinh^2 v\left[\left(1+\frac{\cos^2 u}{\sinh^2 v}\right)^{1-\mu^2}\times\right.$$
$$\left.\times\left(dv^2+du^2\right)+\cos^2 u d\varphi^2\right]. \qquad (18)$$

In the special case $\mu=0$, we have

$$ds^2=\left|1-\lambda\ln\mathrm{tgh}\frac{v}{2}\right|^{-4/A}dt^2-\frac{L^2}{4}\left|1-\lambda\ln\mathrm{tgh}\frac{v}{2}\right|^{4/A}\times$$
$$\times\left[\left(\sinh^2 v+\cos^2 u\right)\left(dv^2+du^2\right)+\sinh^2 v\,\cos^2 u\,d\varphi^2\right]. \qquad (19)$$

These solutions at $\lambda\to 0$ tend to the γ-metric and the flat space-time, correspondingly. It is obvious that the parameter λ is associated with the charge of the considered non-gravitational field coupled with the singularity $v=0$. There is no other parameter that could act as a mass of the field's quantum. That is the reason why we think of the non-gravitational field as of a massless one.

4. Analysis of the Space-Times Obtained

Let us consider the properties of space-time (18). The energy density in (12) must be positive. This leads us to the condition $(A-1)\lambda>0$ following from (15). If $A<1$, we have $\lambda<0$ and the space-time (18) is asymptotically flat at $v\to\infty$. Introducing the asymptotic radial co-ordinate $r=L^*e^v/4$ with $L^*=L(1-\lambda)^{2/A}$ and rescaling the time co-ordinate $t^*=t(1-\lambda)^{-2/A}$, we get an asymptotic behaviour $g_{t^*t^*}\to 1-2M/r$ with

$$M=\frac{L^*\mu(1+\lambda)}{2(1-\lambda)}=L\mu(1+\lambda)|1-\lambda|^{2/A-1}/2. \qquad (20)$$

This is the total mass of the singularity and the non-gravitational field including the mass defect.

In this case, we have only one singularity $v=0$. Using the method of determination of a naked singularity type developed in [3], we can study its properties. At arbitrary $\lambda\neq 0$, the singularity type depends only on μ. At $0<|\mu|<1$, this is a linear singularity and, at $|\mu|>1$, this is a paradoxical one. At $|\mu|\geq 2$, we have two directional singularities at $v=0$, $u=\pm\pi/2$ corresponding to infinitely distant points connected by the paradoxical singularity.

In the case $A>1$, $\lambda>0$, the type of the singularity $v=0$ is the same, but there are three possibilities for the structure of the space-time. At $\lambda<1$, the properties of the space-time are the same as at $\lambda<0$ and Eq. (20) is also valid. At $\lambda>1$, we have an additional singularity at $v=v_0$ with $\mathrm{tgh}(v_0/2)=\lambda^{-1/\mu A}$. It separates two independent solutions described by metric (18) in the regions $0<v<v_0$ and $v>v_0$. Considering the first region, we see the above-described naked singularity $v=0$. At the finite distance from it, another naked singularity lies at $v=v_0$. It is point-like, has a negative mass, and appears due to the self-gravitation of the non-gravitational field. The volume of the space is finite. Considering the region $v>v_0$, we see the asymptotically flat space-time with an infinite volume and the total mass (20) with the naked point-like singularity $v=v_0$ with negative mass. At $\lambda=1$, space-time (18) contains two singularities at $v=0$ and $v=\infty$. The properties of the latter depends on A. At $A>2$, this is a point-like singularity with positive mass at an infinite distance from the $v=0$ singularity. At $A<2$, this is a point-like singularity with negative mass at a finite distance from the $v=0$ singularity. The case $\lambda=1$, $A=2$ was described in [4].

In the same way, one can investigate space-time (19). At $v=0$, there is the naked linear singularity. At $v\to\infty$, we have the asymptotically flat space-time with the total mall $M=\lambda L/A$. The positivity of the energy density means $A>1$. If $\lambda<0$, we have an additional singularity at $v=v_0$ with $\mathrm{tgh}(v_0/2)=\exp(1/\lambda)$. It is point-like and lies at a finite distance from the $v=0$ singularity.

Conclusions

We have found conditions (8), (9), which make possible a generalization of the γ-metric in the form (3) if the central singularity is also a source of some non-gravitational field or fields with diagonal energy-momentum tensor. In the special case of the same coordinate dependence of components of this tensor, we have arrived at the generalized solutions (18) and (19) and made a brief analysis of their properties. So if anybody will study the influence of any non-gravitational field and will find it satisfying the conditions of this case, she or he will have the possibility to use the analysis of the properties of the space-time from this paper.

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УЗАГАЛЬНЕННЯ γ-МЕТРИКИ
В ПРИСУТНОСТІ НЕГРАВІТАЦІЙНИХ ПОЛІВ

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Р е з ю м е

Розглянуто узагальнення відомого статичного аксіально-симетричного вакуумного розв'язку рівнянь Ейнштейна у випадку, коли центральна гола особливість є також джерелом деякого негравітаційного поля або полів з діагональним тензором енергії-імпульсу (ТЕІ). Знайдено умови, за яких можливе таке узагальнення. В особливому випадку безмасового поля, коли всі компоненти ТЕІ мають однакову координатну залежність, отримано дві узагальнені метрики та проаналізовано властивості їхніх просторів-часів, зокрема тип сингулярностей.