ON THE VARIATIONAL PRINCIPLE AND EFFECTIVE ACTION FOR A SPHERICAL DUST SHELL IN GENERAL RELATIVITY

V.D. GLADUSH

UDC 530.12

Dnipropetrovsk National University
(13, Naukova Str., Dnipropetrovsk 49050, Ukraine; e-mail: gladush@ff.dsu.dp.ua)

The variational principle and the effective action for a thin spherical dust shell in a gravitational field are constructed. The variational principle is consistent with the boundary-value problem of the corresponding Euler–Lagrange equations, and leads to "natural boundary conditions" on the shell. These conditions and the field equations are used for performing the Lagrangian reduction of the system and eliminating the gravitational degrees of freedom. The equations of motion for the shell follow from the obtained reduced action. The modification of the variational procedure leads to two natural variants of the effective action. One of them describes the shell in the reference frame of a stationary interior observer, another in that of the exterior one. The isometric conditions of the exterior and interior faces of the shell lead to the momentum and Hamiltonian constraints.

Introduction

A spherically symmetric dust shell is among the simplest popular models of collapsing gravitating configurations. The equations of motion for a shell were obtained in [1], [2]. The construction of a variational principle for such systems was discussed in [3—5]. There are a number of problems here, the basic one is the choice of the evolution parameter (internal, external, proper). The choice of a time coordinate, in turn, affects the choice of a particular quantization scheme, leading, in general, to quantum theories which are not unitarily equivalent.

In most of these papers, the variational principle for shells is constructed in a comoving frame of reference, or in one of the variants of freely falling frames of reference. However, the use of such frames of reference leads sometimes to effects unrelated to the object under consideration.

The essential physics involves a picture of a gravitational collapse from the point of view of an infinitely remote stationary observer. In quantum theory, this point of view enables us to treat bound states in terms of asymptotic quantities and to correctly build the relevant scattering theory. In our opinion, the choice of exterior or interior stationary observers is most natural and corresponds to real physics. The natural Hamiltonian formulation of a self-gravitating shell was considered in [6,7]. However, this formulation was not obtained by a variational procedure from some initial action containing the standard Einstein–Hilbert term.

The general Lagrange approach to the theory of dust shells in General Relativity was developed in [8]. In the present paper, the effective action for a spherical dust shell in General Relativity is obtained, and its Lagrange and Hamiltonian formulations are constructed. We note that, in the case of spherical symmetry, it has some specific features in comparison with the general approach. The spherical configuration under consideration consists of two vacuum spherical regions D_- and D_+ with spherical boundary surface Σ. The total action is taken as the sum of actions of the York type $I_{\rm Y} = I_{\rm EH} + I_{\partial D}$ [9] for each of the regions $D_\pm$ and the action for the dust matter on the hypersurface Σ. The constructed variational principle is compatible with boundary-value problems of the corresponding Euler–Lagrange equations for each region of the configuration and, when we vary with respect to the metric, leads to the "natural boundary conditions" on the shell. These conditions together with the gravitation field equations are used for performing the reduction of the system and eliminating the gravitational degrees of freedom. The equation of motion for the shell is obtained from the reduced action by considering the normal variations of the hypersurface Σ.

Transforming the variational formula and applying the surface equations lead to two variants of effective action which describe the shell from the interior and exterior stationary observers' points of view, respectively. Using the isometry conditions of the exterior and interior faces of the shell generates momentum and Hamiltonian constraints. Here, c is the velocity of light, k is the gravitational constant, and $\chi = 8\pi k/c^2$. The metric tensor $g_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$ has signature $(+\ -\ -\ -)$.

1. Total Action for the Configuration, Bulk and Surface Equations

Consider a set of the regions $D = D_- \cup \Sigma \cup D_+ \subset V^{(4)}$ in a spherically symmetric space-time $V^{(4)}$. Here, D_- and D_+ are the interior and exterior regions, respectively, which are separated by the spherically symmetric infinitely thin dust shell Σ with the surface dust density σ. In $D_\pm$, we choose the general angle coordinates $x^i : \{x^2 = \theta,\ x^3 = \alpha\}$ $(i,k = 2,3)$ and individual space-time coordinates $x^a_\pm$ $(a,b = 0,1)$ for $D_\pm$, respectively. Then the gravitational fields in the regions $D_\pm$ are described by the metrics

$$ {}^{(4)}ds^2_\pm = {}^{(2)}ds^2_\pm - r^2 d\sigma^2 \,, \tag{1.1}$$

$$ {}^{(2)}ds^2_\pm = \gamma^\pm_{ab} dx^a_\pm dx^b_\pm \,, \quad d\sigma^2 = h_{ij}dx^i dx^j = d\theta^2 + \sin^2\theta d\alpha^2 \,, \tag{1.2}$$

where $\gamma^\pm_{ab} = \gamma^\pm_{ab}(x^a_\pm)$ are the two-dimensional metrics and $r = r(x^a_\pm)$.

Einstein's equations and the curvature scalar for each region $D_\pm$ can be represented in the form

$$ {}^{(4)}G_{ab} = -\frac{2}{r}\nabla_a\nabla_b r + \frac{1}{r^2}\left(2r\Delta\, r - (\nabla r)^2 + 1\right)\gamma_{ab} = 0 \,, \tag{1.3}$$

$$ {}^{(5)}G_{aj} \equiv 0 \,, \tag{1.4}$$

$$ {}^{(4)}G_{ij} = \frac{r}{2}\left(r\ {}^{(2)}R - 2\Delta r\right) h_{ij} = 0 \,, \tag{1.5}$$

$$ {}^{(4)}R = {}^{(2)}R - \frac{4}{r}\,\Delta r - \frac{2}{r^2}(\nabla r)^2 - \frac{2}{r^2} \,, \tag{1.6}$$

where $\Delta r = \nabla^a\nabla_a r = r_{;\,a}^{\ ;\,a}$, $(\nabla r)^2 = \gamma^{ab}\nabla_a r\nabla_b r = r^{,a} r_{,a}$, $\nabla_a \equiv {}_{;\,a}$ and ${}^{(2)}R$ are the covariant derivative and the curvature scalar with respect to the metric γ_{ab}, $r_{,\,a} \equiv \partial r/\partial x^a$. Here, we temporarily omit the signs "$_\pm$".

Now we introduce a general coordinate map $x^a \in D$, and metrics $\gamma^\pm_{ab}$ such that $\gamma^-_{ab|_\Sigma} = \gamma^+_{ab|_\Sigma} = \gamma_{ab}$. Then ${}^{(2)}ds_{+|_\Sigma} = {}^{(2)}ds_{-|_\Sigma} \equiv {}^{(2)}ds$, and the world line γ of the shell in this map is given by the equation $x^a = x^a(s)$. Let

$$ \{\vec u = u^a\partial_a \,,\ \vec n = n^a\partial_a\} \,,\ \{\omega = u_a dx^a \,,\ \eta = n_a dx^a\} \tag{1.7}$$

be some general orthonormal vector and covector bases in D. The components u^a, n^a satisfy the conditions $u_a u^a{}_{|\pm} = -n_a n^a{}_{|\pm} = 1$, $u_a n^a{}_{|\pm} = 0$ $(\partial_a = \partial/\partial x^a)$. Hence, to within a general factor $\epsilon = \pm 1$, we obtain

$$ n_0 = \sqrt{-\gamma}\, u^1, \qquad n_1 = -\sqrt{-\gamma}\, u^0 \,, \tag{1.8}$$

The metric γ_{ab} and the Kronecker delta with respect to the basis $\{\vec u, \vec n\}$ have the form

$$ \gamma_{ab} = u_a u_b - n_a n_b \,, \quad \gamma^{ab} = u^a u^b - n^a n^b \,, \quad \delta^a_b = u^a u_b - n^a n_b \,. \tag{1.9}$$

Further, we suppose that the vector field $\vec u$ at points $p \in \Sigma$ is tangential to the world line of a shell γ so, that $u^a_\pm|_\Sigma = dx^a/{}^{(2)}ds$. The vector field $\vec n$ at points $p \in \Sigma$ is normal to Σ and is directed from D_- to D_+. Inside the regions $D_\pm$, the dyad $\{u^a,\ n^a\}$ is arbitrary.

We note the relations

$$ u^a_{\ ;\,b}u^b = n^a n^b_{\ ;\,b} \,, \quad u^a_{\ ;\,b}n^b = n^a u^b_{\ ;\,b} \,, \quad n^a_{\ ;\,b}n^b = u^a u^b_{\ ;\,b} \,, \quad n^a_{\ ;\,b}u^b = u^a n^b_{\ ;\,b} \,, \tag{1.10}$$

which are specific to the two-dimensional case.

We define the one-forms $d\Sigma_a$ as

$$ dx^a \wedge d\Sigma_b = \delta^a_b\, dx^0 \wedge dx^1 = \delta^a_b\, d^2x \,, \tag{1.11}$$

where the symbol "$\wedge$" denotes the exterior product. It is also useful to define the one-forms

$$ d\Sigma_u = u^a d\Sigma_a \,, \quad d\Sigma_n = -n^a d\Sigma_a \,, \tag{1.12}$$

which are dual to the one-forms $\omega\,, \eta$, so that

$$ \sqrt{-\gamma}\,\omega \wedge d\Sigma_u = \sqrt{-\gamma}\,\eta \wedge d\Sigma_n = \sqrt{-\gamma}\, d^2x = \omega \wedge \eta \,, \tag{1.13}$$

$$ \omega = -\sqrt{-\gamma}\, d\Sigma_n \,, \qquad \eta = \sqrt{-\gamma}\, d\Sigma_u \,. \tag{1.14}$$

In addition, we have $g = \det|g_{\mu\nu}| = \gamma r^4 \sin^2\theta$, $\gamma = \det|\gamma_{ab}|$.

Now we introduce the tensors of extrinsic curvature

$$ K_{\mu\nu} = -n_{\mu\,;\,\rho}(n^\rho n_\nu + \delta^\rho_\nu) \,, \quad K = K^\mu_\mu = -n^\mu_{;\,\mu} \,, \tag{1.15}$$

$$ D_{\mu\nu} = u_{\mu\,;\,\rho}(u^\rho u_\nu - \delta^\rho_\nu) \,, \quad D = D^\mu_\mu = -u^\mu_{;\,\mu} \tag{1.16}$$

of the local subspaces Σ_n and Σ_u which are orthogonal to the vectors $n^\mu = \{n^a, 0, 0\}$ and $u^\mu = \{u^a, 0, 0\}$, respectively. Here, "${}_{;\,\rho}$" is the covariant derivative with respect to the metric $g_{\mu\nu}$. On the shell, the tensor $K_{\mu\nu}$ is the tensor of extrinsic curvature of the hypersurface Σ.

From definitions (1.15) and (1.16), we can obtain

$$ K_{i\,k} = r(\vec n r)h_{ik} \,, K_{ai} = K_{ia} = 0 \,, \quad K_{ab} = K_{uu}\, u_a u_b \,, \tag{1.17}$$

$$ K_{uu} = K_{ab}u^a u^b = -n^a_{\ ;\,a} \,, \quad K = -\frac{1}{r^2}\left(r^2 n^a\right)_{;\,a} = K_{uu} - \frac{2(\vec n r)}{r} \,, \tag{1.18}$$

$$D_{ik} = r(\vec{u}r)h_{ik}\,, \quad D_{ai} = D_{ia} = 0\,,$$
$$D_{ab} = D_{nn}\, n_a n_b\,, \tag{1.19}$$

$$D_{nn} = D_{ab}n^a n^b = u^a_{\ ;\,a}\,,$$
$$D = -\frac{1}{r^2}\left(r^2 u^a\right)_{;\,a} = -D_{nn} - \frac{2(\vec{u}r)}{r}\,, \tag{1.20}$$

where $\vec{n}r = n^a r_{,\,a}$, $\vec{u}r = u^a r_{,\,a}$.

We take the total action for the spherically symmetric configuration under consideration in the form

$$I_{\rm tot} = I_{\rm EH} + I_m + I_\Sigma + I_{\partial D} + I_0\,, \tag{1.21}$$

where

$$I_{\rm EH} = -\frac{c}{2\chi}\int\limits_{D_-\cup D_+}\sqrt{-g}\;{}^{(4)}R\; d^2x\wedge d\theta\wedge d\alpha \tag{1.22}$$

is the sum of Einstein–Hilbert actions for the regions $D_\pm$.

The dust on the singular shell Σ is described by the action

$$I_m = c\int\limits_\Sigma \sigma\sqrt{-g}\; d\Sigma_n\wedge d\theta\wedge d\alpha\,. \tag{1.23}$$

The third term on the right-hand side of (1.21) is the matching term

$$I_\Sigma = -\frac{c}{\chi}\int\limits_\Sigma\sqrt{-g}\;[K]\, d\Sigma_n\wedge d\theta\wedge d\alpha\,, \tag{1.24}$$

where the symbol $[A] = A|_+ - A|_-$ denotes the jump of the quantity A on the shell Σ. The signs "$|_\pm$" indicate that the marked quantities are calculated as the limit values when we approach Σ from inside and outside, respectively.

The fourth term on the right-hand side of (1.21)

$$I_{\partial D} = \frac{c}{\chi}\oint\limits_{\partial D}\sqrt{-g}\,(Du^a - Kn^a)d\Sigma_a\wedge d\theta\wedge d\alpha \tag{1.25}$$

contains the surface terms similar to the Gibbons–Hawking surface term, which are introduced to fix the metric on the boundary ∂D of the region D. Note, that the boundary ∂D consists of the pieces of timelike as well as spacelike hypersurfaces. The normalizing term I_0 in (1.21) is needed when the exterior boundary ∂D_+ of the region D_+ is situated on the timelike infinitely remote hypersurface.

The first and fourth terms in (1.21) form the action of the York type $I_{\rm Y} = I_{\rm EH} + I_{\partial D}$ [9]. It is used in variational problems with a fixed metric on the boundary ∂D of the configuration D. This action can also be used in variational problems with the general relativistic version of the "natural boundary conditions" for "a free edge" [10], when the metric on the boundary is arbitrary and the corresponding momenta vanish. In our case of the compound configuration, we also fix the metric on the boundary ∂D. However, in addition, we have the boundary surface Σ inside the system, with the singular distribution of matter on it. The sum of the actions $I_{\rm Y}$ for $D_\pm$, the action for matter I_m, and normalizing term I_0 does lead to the action $I_{\rm tot}$. If there is no dust, $\sigma = 0$, the common boundary is not "loaded". Then the requirement $\delta I_{\rm tot} = 0$ at arbitrary everywhere continuous variations of the metric, gives a generalization of the above "natural boundary conditions" for the free hypersurface Σ. They coincide with the ordinary continuity conditions for the extrinsic curvature on Σ. If the matched edges "are loaded" by some surface matter distribution, then we obtain the surface equations or the boundary conditions for $D_\pm$. They are the analog of the generalized "natural boundary conditions" for "loaded edges". The initial action $I_{\rm tot}$ was chosen so that the surface equations on Σ which follow from the requirement $\delta I_{\rm tot} = 0$ coincide with the matching conditions on hypersurfaces [1]. In this case, the variational principle for the action $I_{\rm tot}$ will be compatible with the boundary-value problem of the corresponding Euler–Lagrange equations [11], [12].

Integrating with respect to angles and taking into account relations (1.6) and (1.14), we can write actions (1.22) and (1.23) as

$$I_{\rm EH} = -\frac{c^3}{4k}\int\limits_{D^{(2)}_-\cup\, D^{(2)}_+}\sqrt{-\gamma}\Big({}^{(2)}R\,r^2 - 4r\Delta r -$$
$$-2(\nabla r)^2 - 2\Big)\, d^2x\,, \tag{1.26}$$

$$I_m = mc\int\limits_{\Sigma^{(1)}}\sqrt{-\gamma}\, d\Sigma_n = -mc\int\limits_\gamma \omega\,, \tag{1.27}$$

where $m = 4\pi\sigma r^2 = \text{const}$ is the shell mass.

The matching (1.24) and boundary surface (1.25) terms can be written as

$$I_\Sigma = \frac{c^3}{2k}\int\limits_\gamma r^2\,[K]\,\omega = \frac{c^3}{2k}\int\limits_\gamma r\,[rK_{uu} - 2(\vec{n}r)]\;\omega\,, \tag{1.28}$$

$$I_{\partial D} = \frac{c^3}{2k}\oint\limits_{\partial D} r^2\sqrt{-\gamma}\,(Du^a - Kn^a)d\Sigma_a\,. \tag{1.29}$$

In order to simplify the total action $I_{\rm tot1}$, we reduce action (1.26) to the form including only the first-order derivatives. To this end, we use the fact that, in the two-dimensional space, a curvature scalar can be reduced to the divergence of a vector (see Appendix)

$$ {}^{(2)}R = 2V^a_{;\,a}\,, \tag{1.30}$$

$$V^a = n^b_{;\,b}n^a - u^b_{;\,b}u^a = -K_{uu}n^a - D_{nn}u^a\,. \tag{1.31}$$

Then, using the formulae

$$\sqrt{-\gamma}\,r^2\ {}^{(2)}R = 2\sqrt{-\gamma}\,r^2V^a_{;\,a} =$$
$$= 2(\sqrt{-\gamma}\,r^2V^a)_{,\,a} - 4rr_{,\,a}V^a\,, \tag{1.32}$$

$$r\sqrt{-\gamma}\Delta r = (\sqrt{-\gamma}\,rr^{,\,a})_{,a} - \sqrt{-\gamma}\,(\nabla r)^2\,, \tag{1.33}$$

the Einstein–Hilbert actions (1.26) can be rewritten as

$$I_{\rm EH} = I_g - I_\partial\,, \tag{1.34}$$

where

$$I_g = \int\limits_{D_-^{(2)}\cup\, D_+^{(2)}} L_g d^2x\,. \tag{1.35}$$

is the gravitational action for a gravitational field with the Lagrangian, which includes only the first-order derivatives

$$L_g = \frac{c^3}{2k}\sqrt{-\gamma}\,(2rr_{,\,a}V^a - r_{,\,a}r^{,\,a} + 1)\,. \tag{1.36}$$

Here $r^{,\,a} = \gamma^{ab}r_{,\,a}\,,\ \ r_{,\,a} = \partial r/\partial x^a = (\vec{u}r)u_a - (\vec{n}r)n_a\,,$ $r_{,\,a}r^{,\,a} = (\vec{u}r)^2 - (\vec{n}r)^2\,.$

The second term in (1.34) is the surface term

$$I_\partial = \frac{c^3}{2k}\oint\limits_{\partial D_-^{(2)}} r\sqrt{-\gamma}\ W^a d\Sigma_a +$$
$$+\frac{c^3}{2k}\oint\limits_{\partial D_+^{(2)}} r\sqrt{-\gamma}\ W^a d\Sigma_a\,. \tag{1.37}$$

It includes the integration over total boundaries of the regions D_- and D_+, where

$$W^a = rV^a - 2r^{,\,a}\,. \tag{1.38}$$

Further, taking into account (1.31), (1.18), and (1.20), we find

$$W^a = (rD_{nn} - 2(\vec{u}r))\,u^a - (rK_{uu} - 2(\vec{n})r)\,n^a =$$
$$= r\,(Du^a - Kn^a)\,. \tag{1.39}$$

Now the term (1.37) can be rewritten as the sum of two addends

$$I_\partial = \tilde{I}_\Sigma + \tilde{I}_{\partial D}\,. \tag{1.40}$$

The addend $\tilde{I}_{\partial D}$ includes the integration only over that part of boundaries ∂D_+ and ∂D_- of the regions D_- and D_+ which coincides with the boundary ∂D of the configuration $D = D_-\cup\Sigma\cup D_+$. In the addend $\tilde{I}_\Sigma$, we integrate over the remaining parts of the boundaries $\partial D_\pm$, which means the integration over the exterior and interior sides of the common boundary Σ of the regions D_+ and D_- , i. e. over the exterior and interior faces of the dust shell. Taking into account (1.37), (1.39), (1.28), and (1.29), it is easy to see that $\tilde{I}_{\partial D} = I_{\partial D}$ and $\tilde{I}_\Sigma = I_\Sigma$. After substitution of (1.34) and (1.40) into (1.21), the surface terms are reduced and the complete action acquires the ordinary and natural form

$$I_{\rm tot} = I_g + I_m + I_0\,, \tag{1.41}$$

where the action I_g contains Lagrangian (1.36) with first-order derivatives only.

Forms (1.21) and (1.41) of the action $I_{\rm tot}$ are equivalent. Applying action (1.21), we can evaluate the value of $I_{\rm tot}$ on the extremals, whereas we use formula (1.41) for finding the extremals. The total action $I_{\rm tot}$ is the functional of the metric γ_{ab}, the radius r, and the shell world line γ: $I_{\rm tot}\equiv I_{\rm tot}[\gamma_{ab},\ r,\ \gamma]$.

Now we find the variation $\delta I_{\rm tot}$ generated by varying r and γ^{ab}. Using relations (1.9) and

$$\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{ab}\delta\gamma^{ab} =$$
$$= \sqrt{-\gamma}(n_a\delta n^a - u_a\delta u^a)\,, \tag{1.42}$$

it is convenient to express the variations of the metric $\delta\gamma^{ab}$ through the variations of the vectors u^a and n^a in the final formulas.

To calculate the variation δI_g, we use the formulae

$$\delta(r_{,\,a}r^{,a}) = 2(r^{,\,a}\delta r)_{;\,a} +$$
$$+2(\dot{r}r_{,\,a}\delta u^a - r'r_{,\,a}\delta n^a - r_{;\,a}^{\ \,a}\delta r)\,,$$

$$\delta(\sqrt{-\gamma}\,rr_{,a}V^a) = -\sqrt{-\gamma}\,rV^a_{;\,a}\delta r +$$
$$+\sqrt{-\gamma}(rr_{,a}n^b_{;b} + (r\dot{r})^{\cdot}n_a - (rr')^{\cdot}u_a)\delta n^a -$$
$$-\sqrt{-\gamma}(rr_{,a}u^b_{;b} + (r\dot{r})'n_a - (rr')'u_a)\delta u^a +$$
$$+\{r\sqrt{-\gamma}\,V^a\delta r + rr'\delta(\sqrt{-\gamma}\,n^a) - r\dot{r}\delta(\sqrt{-\gamma}\,u^a)\}_{,\,a}\,,$$

where $r_{,a}\equiv\partial r/\partial x^a = \dot{r}u_a - r'n_a$ and $\dot{r} = r_{,a}u^a = \vec{u}\,r\,,\ r' = r_{,a}n^a = \vec{n}\,r$. We assume $\delta d\Sigma_a|_{\partial D} = 0$, $\delta r|_{\partial D} = 0$, $\delta u^a|_{\partial D} = \delta n^a|_{\partial D} = 0$. In addition, $[\gamma^{ab}] = [\ \delta\gamma^{ab}] = 0$, $[n^a] = [\delta n^a] = [u^a] = [\delta u^a] = 0$. According to (1.27), for the variation δI_m, we have $\delta I_m = -mc\,\delta\int_\gamma\omega = -mc\int_\gamma\delta\omega_\gamma$. The symbol "$\omega_\gamma$" denotes the restriction of the one-form ω on the shell world line γ: $\omega_\gamma = (u_a dx^a)_\gamma = (u_a dx^a/{}^{(2)}ds)\,{}^{(2)}ds = u_a u^a\,{}^{(2)}ds = {}^{(2)}ds$. Therefore, we have

$$\delta\omega_{|\gamma} = \delta\;{}^{(2)}ds = \frac{1}{2}u^a u^b\;{}^{(2)}ds\;\delta\gamma_{ab} =$$
$$= -\frac{1}{2}u_a u_b \omega_{|\gamma}\;\delta\gamma^{ab} = -u_a\omega_{|\gamma}\delta u^a\,. \tag{1.43}$$

The requirement of stationarity, $\delta I_{\rm tot} = 0$, with respect to variations δu^a, δn^a leads to the equations

$$\dot r' - r' u^b_{;b} = 0\,, \quad \acute{r}^{\cdot} - \dot r n^b_{;b} = 0\,, \tag{1.44}$$
$$2r\ddot r - 2rr' n^b_{;b} + \dot r^2 - r'^2 + 1 = 0\,, \tag{1.45}$$
$$2r\acute{r}' - 2r\dot r u^b_{;b} - \dot r^2 + r'^2 - 1 = 0\,. \tag{1.46}$$

In deriving these formulae, we used Eqs. (1.10). It can be shown that Eqs. (1.44) - (1.46) are equivalent to Eqs. (1.3) written in the basis $\{u^a, n^a\}$.

The variations of $I_{\rm tot}$ with respect to r lead to the equation

$$rV^a_{;a} - \Delta r = 0\,, \tag{1.47}$$

which, in view of (1.30), is equivalent to the rest of the Einstein equations (1.5). Besides Eqs. (1.44)-(1.47), we also obtain the the surface equations for jumps

$$[\,r'] - r[n_a V^a] = 0\,, \tag{1.48}$$
$$c^2 r[\,r'] + km = 0\,. \tag{1.49}$$

We note that, by virtue of (1.31) and (1.10), the formulae

$$n_a V^a = -n^a_{;a} = K_{uu} = n_a u^a_{;b} u^b = n_a f^a\,, \tag{1.50}$$

are valid, where $f^a = u^a_{;b}u^b$ is the acceleration vector. Therefore, formula (1.48) can be written as

$$[\acute r] = r[K_{uu}] = r[n_a f^a]\,. \tag{1.51}$$

In order to obtain the equations of motion for the shell, we consider the normal variations of the hypersurface Σ. Let each point $p \in \Sigma$ be displaced at a coordinate distance $\delta x^a(p) = n^a\delta\lambda(p)$ in the direction of the normal. As a result of the displacement, we obtain a new hypersurface $\tilde\Sigma$. The initial and final positions of the shell are fixed. Therefore, we have $\delta\lambda(p) = 0$, $\forall p \in \Sigma \cap \partial D = \tilde\Sigma_t \cap \partial D$. In addition, we fix the metric γ_{ab} and also all quantities on Σ, so that $\delta I_m = 0$. As a result of the displacement of the hypersurface Σ, the original regions D_+ and D_- are transformed into new regions $\tilde D_+$ and $\tilde D_-$, such that $\tilde D_- \cup \tilde\Sigma \cup \tilde D_+ = D_- \cup \Sigma \cup D_+ = D$. Then the variation of the region D_- can be represented as $\delta D_- = \tilde D_- \backslash D_- = D_+ \backslash \tilde D_+$. The change of action (1.35) induced by a displacement of Σ is given by

$$\delta I_{\rm tot} = \delta I_g = \int\limits_{\tilde D_- \cup \tilde D_+} L_g\; d^2x - \int\limits_{D_- \cup D_+} L_g\; d^2x$$
$$\cong -\int\limits_{\delta D_-} \left(L_g^+ - L_g^-\right) d^2x\,. \tag{1.52}$$

Here, L_g^+ and L_g^- are determined by relation (1.36) and calculated to the right and to the left of Σ, respectively. Under the infinitesimal displacement of Σ, the variation of the total action takes the form

$$\delta I_{\rm tot} = -\int\limits_\Sigma \left(L_g^+ - L_g^-\right)\delta x^a d\Sigma_a = \int\limits_\Sigma [L_g]\delta\lambda d\Sigma\,. \tag{1.53}$$

Hence, owing to the arbitrariness of $\delta\lambda$ and the requirement $\delta I_{\rm tot} = 0$, we find

$$[L_g] = \left(L_g^+ - L_g^-\right)|_\Sigma =$$
$$= -\frac{c^3}{2\gamma}\sqrt{-\gamma}\;[2r(\vec n r)K_{uu} - (\vec n r)^2] = 0\,. \tag{1.54}$$

Using the formulas such as $[AB] = \bar A\,[B] + \bar B\,[A]$, where $\bar A = (A_{|+} + A_{|-})/2$, we obtain

$$r\overline{(\vec n r)}\,[K_{uu}] + r[\vec n r]\overline{K_{uu}} - \overline{(\vec n r)}\,[\vec n r] = 0\,. \tag{1.55}$$

Taking into account (1.51), the equations of motion for the shell can be written as

$$\overline{K_{uu}} = \frac{1}{2}\left(K_{uu|+} + K_{uu|-}\right) = \overline{n_a f^a} = 0\,. \tag{1.56}$$

Relations (1.49), (1.51), and (1.56) form the necessary complete set of the boundary algebraic conditions on Σ. In particular, these equations imply

$$K_{uu\pm} = n^a u_{a;b}u^b|_\pm = n^a \frac{Du_a}{ds}\bigg|_\pm = \mp\frac{km}{2c^2r^2}\,, \tag{1.57}$$

where $Du_a = u_{a;b}\,dx^b$ is the covariant differential. This equation can be written as

$$u^a_{;b}u^b{}_{|\pm} = \frac{Du^a}{ds}\bigg|_\pm = \pm\frac{km}{2R^2}\,n^a\,. \tag{1.58}$$

Eqs. (1.57) yield the two-dimensional spherically symmetric analog of the well-known Israel equations [1]

$$n^a\frac{Du_a}{ds}\bigg|_+ + n^a\frac{Du_a}{ds}\bigg|_- = 0\,, \tag{1.59}$$
$$n^a\frac{Du_a}{ds}\bigg|_+ - n^a\frac{Du_a}{ds}\bigg|_- = -\frac{km}{c^2r^2} = -\frac{\chi\sigma}{2}\,. \tag{1.60}$$

2. The Reduced and Effective Actions for a Dust Spherical Shell

Now we can realize a Lagrangian reduction of the system and eliminate the gravitational degrees of freedom from the action $I_{\rm tot}$. Thus, we will construct the reduced action for a shell. For this purpose, we calculate the action $I_{\rm tot}$ in the form (1.21) on the solutions of the vacuum Einstein equations (1.3)-(1.5) (or (1.44)-(1.47)). In addition, we take into account the surface equations (1.49), (1.51) and (1.56). On this stage, we use the following consequences of these equations

$$^{(4)}R = 0\,, \quad [\vec{n}r] = r[K_{uu}]\,, \quad r^2[K_{uu}] = -\frac{km}{c^2}\,. \qquad (2.1)$$

Substituting these relations into (1.21) and taking into account (1.27), we find

$$I_{{\rm tot}\,|\{{\rm Eqs.\ (2.1)}\}} = J_{\rm sh} + I_{\partial D} + I_0\,, \qquad (2.2)$$

where

$$J_{\rm sh} = -\left\{\int\limits_\gamma \left(mc + \frac{c^3}{2k}r^2[K_{uu}]\right)\omega\right\}_{|\{{\rm Eqs.\ (2.1)}\}} = $$
$$= -\frac{1}{2}\int\limits_\gamma mc\,\omega \qquad (2.3)$$

is the reduced action for the dust shell. This action must be considered together with the boundary conditions (1.49), (1.51) and (1.56). The reduced action is the functional of the shell world line γ: $J_{\rm sh} = J_{\rm sh}[\gamma]$. The action $J_{\rm sh}$ is quite certain if, in the neighborhood of Σ, the gravitational fields are assigned as the solutions of the vacuum Einstein equations (1.3)—(1.5).

Note that one usually comes to the action for the shell in the other form. In our approach, this form of the action can be found by partial reduction of the initial action $I_{\rm tot}$ when condition (1.51) is not taken into account. As a result, we arrive at the action similar to the expression in braces in (2.3) or at some of its modifications. Hence, one can obtain the Lagrangian of the shell in the frame of reference of the comoving observer. However, the quantity $K_{uu} = n_a f^a = n_a u^a_{;b}u^b$ contains the second derivatives of coordinates x^a with respect to the proper time of the shell. When these derivatives are eliminated by integrating by parts, we obtain rather complicated Lagrangians and Hamiltonians.

Now we introduce independent coordinates $x^a_\pm$ in each of the regions $D_\pm$. Then, the reduced action is the functional of embedding functions $x^a_\pm(s)$ of the shell: $J_{\rm sh} \equiv J_{\rm sh}[x^a_-(s), x^a_+(s)]$. Consider a variation of the integrand in $J_{\rm sh}$ with respect to these functions. We have

$$\delta\omega^\pm_{|\gamma} = \delta\,{}^{(2)}ds_\pm = \delta\left(\sqrt{\gamma_{ab}dx^a dx^b}\,\right)_\pm =$$
$$= -(u_{a;\,b}u^b\delta x^a\,{}^{(2)}ds)_\pm + d(u_a\delta x^a)_\pm\,. \qquad (2.4)$$

Hence, applying the formulas $\delta^a_b = u^a u_b - n^a n_b$ and $u^a u_{a;\,b} = 0$, we obtain

$$\left(\delta\,{}^{(2)}ds - n^c u_{c;\,b}u^b n_a\delta x^a\,{}^{(2)}ds\right)_\pm = d(u_a\delta x^a)_\pm\,, \qquad (2.5)$$

or, considering the boundary conditions (1.57),

$$\left(\delta\,{}^{(2)}ds \pm \frac{km}{2c^2r^2}n_a\delta x^a\,{}^{(2)}ds\right)_\pm = d(u_a\delta x^a)_\pm\,. \qquad (2.6)$$

Further, using conditions (1.8), we have [1]

$$(n_a\delta x^a ds)_{|\pm} = \{\sqrt{-\gamma}\,(u^1\delta x^0 - u^0\delta x^1)ds\}_{|\pm} =$$
$$= \{\sqrt{-\gamma}\,(dx^1\delta x^0 - dx^0\delta x^1)\}_{|\pm}\,. \qquad (2.7)$$

Therefore, the variational formula (2.5) takes the form

$$\left\{\delta\,{}^{(2)}ds \pm (km/2c^2r^2)\sqrt{-\gamma}\,(dx^1\delta x^0 - dx^0\delta x^1)\right\}_\pm =$$
$$= d(u_a\delta x^a)_\pm\,. \qquad (2.8)$$

Now we introduce the vector potential $B_a = B_a(x^0, x^1)$ with the relation

$$d\wedge(B_a dx^a) \equiv G_{01}dx^0\wedge dx^1 =$$
$$= -\frac{km}{2c^2r^2}\sqrt{-\gamma}\,dx^0\wedge dx^1\,, \qquad (2.9)$$

where $G_{ab} \equiv B_{b,a} - B_{a,b}$. Note that the integrability condition for the just introduced relation holds identically. With this definition in mind and owing to the fact that

$$\delta(B_a dx^a) - d(B_a\delta x^a) = G_{10}(dx^0\delta x^1 - dx^1\delta x^0)\,, \qquad (2.10)$$

the variational formula (2.8) can be written in the form

$$\delta\left\{{}^{(2)}ds \pm\ B_a dx^a\right\}_\pm = d\{(u_a \pm B_a)\delta x^a\}_\pm\,. \qquad (2.11)$$

Thus, if we introduce the actions as

$$I^\pm_{\rm sh} = -mc\int\limits_\gamma \left({}^{(2)}ds \mp B_a dx^a\right)_{|\pm}\,, \qquad (2.12)$$

[1] In paper [8], the factor $\sqrt{-\gamma}$ was lost in these formulas. However, in the particular case of the Schwarzschild solution, it is of no importance since $\sqrt{-\gamma} = 1$

then, owing to the variational formula (2.11), we obtain the stationarity condition $\delta I^{\pm}_{\rm sh} = 0$ for the fixed initial and final positions of the shell. The just obtained actions are the natural modification of action (2.3) which is compatible with the boundary conditions. The stationarity condition $\delta I^{\pm}_{\rm sh} = 0$ for an arbitrary variation of coordinates $x^a_{\pm}$ yields the equations of motion for the shell for the external or internal coordinates. Therefore, formula (2.12) is the general form of the effective actions for a dust spherical shell in General Relativity, where the vector potential B_a is found from Eq. (2.9).

Now let us construct the effective actions for a shell in the Schwarzschild gravitational field. Using curvature coordinates, we choose common spatial spherical coordinates $\{r,\ \theta,\ \alpha\}$ in $D_{\pm}$, and individual time coordinates $t_{\pm}$ in $D_{\pm}$, respectively. Then the world sheet of the shell in the coordinates $\{t_{\pm}\,, r\}$ is given by the equations $r = R_{\pm}(t_{\pm})$. The gravitational fields in the regions $D_{\pm}$ are described by the metrics

$$ {}^{(4)}ds^2_{\pm} = f_{\pm} c^2 dt^2_{\pm} - f^{-1}_{\pm} dr^2 - r^2(d\theta^2 + \sin^2\theta d\alpha^2)\,, \quad (2.13) $$

$$ f_{\pm} = 1 - \frac{2kM_{\pm}}{c^2 r}\,, \quad (2.14) $$

where M_+ and M_- are the Schwarzschild masses ($M_+ > M_-$). In our case, we have

$$ B_a dx^a = c\varphi(t_{\pm}, R)dt_{\pm} + U_R(t_{\pm}, R)dR\,. \quad (2.15) $$

Using the gauge condition $U_R(t_{\pm}, R) = 0$, action (2.12) can be written as

$$ I^{\pm}_{\rm sh} = -mc \int\limits_{k} \left({}^{(2)}ds \mp c\,\varphi dt \right)_{|\pm}\,. \quad (2.16) $$

Further, formula (2.9) implies

$$ \frac{\partial\varphi}{\partial R} = \frac{km}{2c^2R^2}\,. \quad (2.17) $$

From here, up to an additive constant, we find

$$ \varphi = -\frac{km}{2c^2R}\,. \quad (2.18) $$

Eventually, the effective action for the shell can be represented as

$$ I^{\pm}_{\rm sh} = \int\limits_{\gamma} L^{\pm}_{\rm sh} dt_{|\pm} = -\int\limits_{\gamma} \left(mc\ {}^{(2)}ds \pm \frac{km^2}{2R} dt \right)_{|\pm}\,, \quad (2.19) $$

where

$$ L^{\pm}_{\rm sh} = -mc^2 \sqrt{f_{\pm} - f^{-1}_{\pm} R^2_{t\pm}/c^2} \pm U \quad (2.20) $$

are the effective Lagrangians of the shell in the frames of reference of stationary observers in the regions $D_{\pm}$ ($R_{t\pm} = dR/dt_{\pm}$), respectively, and

$$ U = -\frac{km^2}{2R} \quad (2.21) $$

is the effective potential energy of gravitational self-action of the shell.

Note that the actions $I^{\pm}_{\rm sh}$ can be considered quite independently. The regions $D_{\pm}$ together with the gravitational fields (2.13) can also be regarded separately and independently as manifolds with edges $\Sigma_{\pm}$. These edges acquire the physical sense of the different faces of the dust shell with the world sheet Σ if the regions $D_{\pm}$ are joined along these boundaries. The last can be realized only if the conditions of isometry,

$$ f_+ c^2 dt^2_+ - f^{-1}_+ dR^2 = f_- c^2 dt^2_- - f^{-1}_- dR^2 = c^2 d\tau^2\,, \quad (2.22) $$

are fulfilled for the edges $\Sigma_{\pm}$ (or if the curves $\gamma_{\pm}$ coincide), where τ is the proper time of the shell. Consider some consequences following from the conditions of isometry for the edges. First of all, we have the relationships between the velocities

$$ c^2 \frac{f_+}{R^2_{t+}} - \frac{1}{f_+} = c^2 \frac{f_-}{R^2_{t-}} - \frac{1}{f_-}\,, \quad (2.23) $$

$$ R^2_{\tau} \equiv \left(\frac{dR}{d\tau}\right)^2 = \frac{c^2 R^2_{t\pm}}{c^2 f_{\pm} - f^{-1}_{\pm} R^2_{t\pm}}\,, $$
$$ R^2_{t\pm} \equiv \left(\frac{dR}{dt_{\pm}}\right)^2 = \frac{c^2 f^2_{\pm} R^2_{\tau}}{c^2 f_{\pm} + R^2_{\tau}}\,. \quad (2.24) $$

Further, from the Lagrangians $L^{\pm}_{\rm sh}$ (2.20), we find the momenta and Hamiltonians for the shell as

$$ P_{\pm} = \frac{\partial L^{\pm}_{\rm sh}}{\partial R_{t\pm}} = \frac{mR_{t\pm}}{f_{\pm}\sqrt{f_{\pm} - f^{-1}_{\pm} R^2_{t\pm}/c^2}} = \frac{m}{f_{\pm}} R_{\tau}\,, \quad (2.25) $$

$$ H^{\pm}_{\rm sh} = \frac{mc^2 f_{\pm}}{\sqrt{f_{\pm} - f^{-1}_{\pm} R^2_{t\pm}/c^2}} \mp U = $$
$$ = mc^2 f_{\pm} \frac{dt_{\pm}}{d\tau} \mp U \quad (2.26) $$

or

$$ H^{\pm}_{\rm sh} = c\sqrt{f_{\pm}(m^2c^2 + f_{\pm}P^2_{\pm})} \mp U = $$
$$ = mc^2 \sqrt{f_{\pm} + R^2_{\tau}/c^2} \mp U = E_{\pm}\,, \quad (2.27) $$

where $E_{\pm}$ are the shell energies which are conjugated to the coordinate times $t_{\pm}$ and are conserved in the frames of the reference of respective stationary observers (interior or exterior one). Eliminating the velocity R_τ from (2.25) and (2.27), the conditions of isometry for the edges can be written as

$$f_+P_+ = f_-P_- \,, \tag{2.28}$$

$$(E_+ + U)^2 - m^2c^4 f_+ = (E_- - U)^2 - m^2c^4 f_- \,. \tag{2.29}$$

Substituting U and $f_\pm$ from (2.14) and (2.21) for those in the last relation and equating the coefficients at the same power of R, one obtains the relations between the Hamiltonians $H^\pm_{\rm sh}$ and the Schwarzschild masses $M_\pm$:

$$H^+_{\rm sh} = H^-_{\rm sh} = (M_+ - M_-)c^2 = E \,. \tag{2.30}$$

Here, $E = E_\pm$ denotes the total energy of the shell, which is conjugated of the coordinate times $t_\pm$ and whose value is independent of the stationary observer's position (inside or outside of the shell). Thus, the dynamical systems with Lagrangians $L^\pm_{\rm sh}$ are not independent. They satisfy momentum and Hamiltonian constraints (2.28), (2.30) which ensure the isometry of the shell faces.

The Lagrangian $L^-_{\rm sh}$ can be used when $R > 2kM_-/c^2$, and $L^+_{\rm sh}$ for $R > 2kM_+/c^2$ $(M_+ > M_-)$. A generalization of these Lagrangians, which is valid in all the regions R^+, T^-, R^-, T^+ of the Kruskal—Szekeres diagram [8], can be easily constructed.

For a self-gravitating shell, we have $M_- = 0$. Denote $M_+ = M$. Then, in the coordinates of the region D_+, the Lagrangian and Hamiltonian of the shell have the forms

$$L^+_{\rm sh} = -mc^2\sqrt{1 - \frac{2\gamma M}{c^2R} - \left(1 - \frac{2\gamma M}{c^2R}\right)^{-1}\frac{R^2_{t+}}{c^2}} - \frac{\gamma m^2}{2R} \,, \tag{2.31}$$

$$H^+_{\rm sh} = c\,\sqrt{1 - \frac{2\gamma M}{c^2R}}\,\sqrt{m^2c^2 + \left(1 - \frac{2\gamma M}{c^2R}\right)P^2_+} + \frac{\gamma m^2}{2R} \,. \tag{2.32}$$

In the coordinates of the interior region D_-, the same shell is described by the Lagrangian and Hamiltonian

$$L^-_{\rm sh} = -mc^2\sqrt{1 - R^2_{t-}/c^2} + \frac{\gamma m^2}{2R} \,, \tag{2.33}$$

$$H^-_{\rm sh} = c\,\sqrt{m^2c^2 + P^2_-} - \frac{\gamma m^2}{2R} \,. \tag{2.34}$$

The dynamical systems with Lagrangians $L^\pm_{\rm sh}$ are not independent. They obey the momentum and Hamiltonian constraints $P_- = f_+P_+$, $H^+_{\rm sh} = H^-_{\rm sh} = Mc^2$, and are canonically equivalent in the extended phase space [8]. However, they are not canonically equivalent to the dynamical system which is obtained at a choice of proper time as an evolutionary parameter.

The effective Hamiltonian $H^-_{\rm sh}$ was postulated in [6] and was used for finding the energy spectrum of quantum states of a dust shell with bare mass m that was less than the Planck mass $m_{\rm Pl}$. In [13], the Hamiltonians $H^\pm_{\rm sh}$ were used for constructing a quasiclassical model of collapsing spherical configuration, for describing a tunneling spherical dust shell, and for constructing a model of creation and annihilation of the pairs of shells.

APPENDIX A:
The Curvature Scalar in the Two-dimensional Space

By definition, we have

$$u_{a\,;b\,;c} - u_{a\,;c\,;b} = R^d_{abc}u_d \,, \tag{A.1}$$

$$n_{a\,;b\,;c} - n_{a\,;c\,;b} = R^d_{abc}n_d \,. \tag{A.2}$$

Multiplying Eq. (A.1) and Eq. (A.2) by u^f and n^f, respectively, and applying the formula $u_du^f - n_dn^f = \delta^f_d$, we obtain, in the two-dimensional space,

$$R^f_{abc} = u^f(u_{a\,;b\,;c} - u_{a\,;c\,;b}) - n^f(n_{a\,;b\,;c} - n_{a\,;c\,;b}) \,. \tag{A.3}$$

From here, we find

$$R_{ac} = u^b(u_{a\,;b\,;c} - u_{a\,;c\,;b}) - n^b(n_{a\,;b\,;c} - n_{a\,;c\,;b}) \,, \tag{A.4}$$

$$R = u^b(u^a_{\,;b\,;a} - u^a_{\,;a\,;b}) - n^b(n^a_{\,;b\,;a} - n^a_{\,;a\,;b}) =$$
$$= (u^a_{\,;b}u^b - u^b_{\,;b}u^a - n^a_{\,;b}n^b + n^b_{\,;b}n^a)_{;a} \,. \tag{A.5}$$

With the help of Eqs. (1.10), this formula can be rewritten as

$$R = 2(n^b_{\,;b}n^a - u^b_{\,;b}u^a)_{;a} = 2V^a_{\,;a} \,, \tag{A.6}$$

where the vector V^a is defined through the vectors $\{u^a, n^a\}$ according to formula (1.31).

1. *Israel W.*// Nuovo cim. **44B** (1966) 1; Ibid. **48B** (1967) 463.
2. *Kuchař K.*// Czech. J. Phys. **B18** (1968) 435.
3. *Visser M.*// Phys. Rev. **D43** (1991) 402; *Kraus P., Wilczek F.*// Nucl. Phys. **B433** (1995) 403; *Ansoldi A., Aurilia A., Balbinot R., Spallucci E.*// Phys. Essays **9** (1996) 556; Class. Quantum Grav. **14** (1997) 2727.
4. *Hájíček P., Bičák J.*// Phys. Rev. **D56** (1997) 4706; *Hájíček P., Kijowski J.*// Ibid. **D57** (1998) 914; *Hájíček P.*// Ibid. **D57** (1998) 936.
5. *Berezin V.A., Boyarsky A.M., Neronov A.Yu.*// Phys. Rev. **D57** (1998) 1118.
6. *Hájíček P., Kay B.S., Kuchař K.*// Phys. Rev. **D46** (1992) 5439.

7. *Dolgov A.D., Khriplovich I.B.*// Phys. Lett. **B400** (1997) 12.

8. *Gladush V.D.*// J. Math. Phys. **42** (2001) 2590.

9. *York J.W., Jr.*// Phys. Rev. Lett. **28** (1972) 1082.

10. *Hayward G., Louko J.*// Phys. Rev. **D42** (1990) 4032.

11. *Courant R., Hilbert D.* Methoden der Mathematischen Physik. — Springer, 1931 (Vol. I), 1937 (Vol. II).

12. *Ponomarev V.N., Barvinsky A.O., Obukhov Yu.N.* Geometrodynamical Methods and the Gauge Approach to the Theory of Gravitational Interactions. — Moscow: Energoatomizdat, 1985 (in Russian).

13. *Gladush V.D.*// Intern. J. Mod. Phys. **D11** (2002) 367.

Received 21.08.03

ПРО ВАРІАЦІЙНИЙ ПРИНЦИП І ЕФЕКТИВНУ ДІЮ ДЛЯ СФЕРИЧНОЇ ПИЛОВОЇ ОБОЛОНКИ У ЗТВ

В.Д. Гладуш

Р е з ю м е

Побудовано варіаційний принцип і ефективну дію для тонкої сферичної пилової оболонки в гравітаційному полі. Варіаційний принцип узгоджений з крайовою задачею відповідних рівнянь Лагранжа–Ейлера і приводить до “природних граничних умов” на оболонці. Ці умови та рівняння поля використовуються для лагранжевої редукції системи та виключення гравітаційних ступенів вільності. З одержаної редукованої дії випливають рівняння руху оболонки. Модифікація варіаційної процедури приводить до двох природних варіантів ефективної дії. Один із них описує оболонку з точки зору внутрішнього, а другий — зовнішнього стаціонарного спостерігача. Умови ізометричності внутрішньої й зовнішньої поверхонь оболонки приводять до імпульсної та гамільтонової в’язок.