INFLUENCE OF THE KORKINA—GRIGOR'EV SCALAR FIELD ON PROPERTIES OF TIME-LIKE SINGULARITIES IN GENERAL RELATIVITY THEORY

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We find and analyze the exact solution of the Einstein equations which generalize the Kasner metric for the case where the singularity is also a source of the massless nonlinear conformal-invariant Korkina—Grigor'ev scalar field. It is shown that such a field does not affect the type of the singularity.

Introduction

This paper completes a small cycle of works devoted to the study of the influence of a scalar field with nonminimal connection on the type and properties of time-like singularities in general relativity theory. The cycle was started by work [1], in which we found a generalization of the well-known Kasner metric,

$$ds^2 = x^{2p_1} dt^2 - dx^2 - x^{2p_2} dy^2 - x^{2p_3} dz^2, \tag{1}$$

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1, \tag{2}$$

for the case where a naked singularity at $x = 0$ is a source of not only the gravitational field, but of a scalar one as well. It turns out that, for any nonzero coupling of a scalar field, properties of the singularity are the same and differ qualitatively from those of the generalization obtained earlier in [2] to the case of a scalar field with zero coupling. Along with the latter, most frequently is used the other field, namely the massless conformally invariant scalar one with the coupling parameter equal to $1/6$ which was introduced by Penrose and is employed in the problems of quantization in a curved space-time.

For the more reliable investigation of the properties and types of singularities, we generalized [3] the known solution given by Zipoy—Voorhees to the case where the central singularity is also the source of a conformally invariant scalar field. The space-time which is described with the obtained metric contains a naked singularity, whose type differs considerably from that of a singularity for an analogous generalization with a massless scalar field with minimal coupling [4].

For the description of a scalar field in general relativity theory, there exists one more variant different from the standard one. In [5], Korkina and Grigor'ev proposed a conformally invariant massless nonlinear scalar field with the Lagrangian density

$$L = -\left(\psi_{,i}\psi^{,i}\right)^2 . \tag{3}$$

It satisfies the equation

$$\psi_{,i}\psi^{,i}\psi^{;k}_{;k} + 2\psi^{,i}\psi^{,k}\psi_{;ik} = 0 \tag{4}$$

and has the energy-momentum tensor

$$T_{ik} = -4\left(\psi_{,l}\psi^{,l}\right)\psi_{,i}\psi_{,k} + g_{ik}\left(\psi_{,l}\psi^{,l}\right)^2 . \tag{5}$$

Korkina and Grigor'ev [5] obtained a spherically symmetric solution with source which possesses a charge of such a field. It describes a naked singularity rather than a black hole.

However, to study the influence of field (3)—(5) on the type and properties of time-like singularities, it is worth to consider the static solution with cylindrical symmetry which generalizes metric (1). The present work is devoted to its derivation and investigation. It would be better to construct a generalization of the Zipoy—Voorhees metric for the case where a singularity is the source of the Korkina—Grigor'ev field, but, as shown in [6], a similar generalization is impossible.

1. An Infinite Massive Thread as a Source of the Korkina—Grigor'ev Scalar Field

We seek for a space-time metric in the following form:

$$ds^2 = e^{2\alpha(\rho)} dt^2 - d\rho^2 - e^{2\beta(\rho)} d\varphi^2 - e^{2\gamma(\rho)} dz^2. \tag{6}$$

Denote $\alpha + \beta + \gamma = 3f$. The field equation (4) is easily integrated, and we get

$$\psi_{,\rho} = \lambda e^{-f}, \lambda = \text{const}. \tag{7}$$

Substituting this formula in (5), we get

$$T_0^0 = T_2^2 = T_3^3 = \lambda^4 e^{-4f}, \quad T_1^1 = -3\lambda^4 e^{-4f}. \tag{8}$$

Three components of the Einstein equations, namely (00)-, (22)-, and (33)-yield

$$\alpha_{,\rho\rho} + 3f_{,\rho}\,\alpha_{,\rho} = \beta_{,\rho\rho} + 3f_{,\rho}\,\beta_{,\rho} =$$

$$= \gamma_{,\rho\rho} + 3f_{,\rho}\,\gamma_{,\rho} = \lambda^4 e^{-4f}. \tag{9}$$

By subtracting these relations from one another and integrating, we obtain

$$\alpha_{,\rho} = f_{,\rho} + q_1 r; \quad \beta_{,\rho} = f_{,\rho} + q_2 r; \gamma_{,\rho} = f_{,\rho} + q_3 r; \tag{10}$$

$$q_1, q_2, q_3 = \text{const}, \quad q_1 + q_2 + q_3 = 0, \quad r = e^{-3f}.$$

In this case, equalities (9) are reduced to

$$f_{,\rho\rho} + 3\left(f_{,\rho}\right)^2 = \lambda^4 e^{-4f}, \tag{11}$$

and the (11)-component of the Einstein equations yields

$$3f_{,\rho\rho} + 3f_{,\rho}^2 + qe^{-6f} = -3\lambda^4 e^{-4f}, \tag{12}$$

where we denoted $q = q_1^2 + q_2^2 + q_3^2$. Eliminating the second derivative from system (11), (12), we get

$$f_{,\rho}^2 = \lambda^4 e^{-4f} + \frac{q}{6} e^{-6f}. \tag{13}$$

It is easy to verify by the direct differentiation that (13) guarantees the validity of both relations (11), (12).

Then we intend to pass from the coordinate ρ to a new radial coordinate f, by using

$$d\rho = \frac{df}{\sqrt{\lambda^4 e^{-4f} + \frac{q}{6} e^{-6f}}}. \tag{14}$$

We get

$$\alpha = f + q_1 \int e^{-3f} d\rho = f + q_1 \int \frac{e^{-3f} df}{\sqrt{\lambda^4 e^{-4f} + \frac{q}{6} e^{-6f}}} =$$

$$= f - q_1 \sqrt{\frac{6}{q}} \ln \left| e^{-f} + \sqrt{e^{-2f} + \frac{6\lambda^4}{q}} \right| + \text{const}, \tag{15}$$

$$\psi = \lambda \int e^{-f} d\rho = \lambda^{-3} \left(\sqrt{\lambda^4 e^{2f} + \frac{q}{6}} - \sqrt{\frac{q}{6}} \right) =$$

$$= \frac{\lambda e^{2f}}{\sqrt{\lambda^4 e^{2f} + \frac{q}{6}} + \sqrt{\frac{q}{6}}}. \tag{16}$$

We chose the integration constant in (16) from the condition $\psi \underset{\lambda \to 0}{\to} 0$.

To simplify the solution obtained, we change q_i by the collection of Kasner's indices

$$p_i = \frac{1}{3} + q_i \sqrt{\frac{2}{3q}}, \tag{17}$$

which satisfy conditions (2). We introduce now a new coordinate V and a constant η as

$$V = \left(\frac{3q}{2}\right)^{-\frac{1}{2}} e^{3f}, \quad \eta = \sqrt[3]{3}\sqrt[6]{2}\lambda q^{-\frac{1}{6}} \tag{18}$$

and obtain the solution

$$ds^2 = -\frac{dV^2}{1 + \eta^4 V^{\frac{2}{3}}} + F^2 \left[C_1 \left(\frac{V}{F^3}\right)^{2p_1} dt^2 - \right.$$

$$\left. -C_2 \left(\frac{V}{F^3}\right)^{2p_2} dy^2 - C_3 \left(\frac{V}{F^3}\right)^{2p_3} dz^2 \right],$$

$$F = \frac{1}{2}\left(1 + \sqrt{1 + \eta^4 V^{\frac{2}{3}}}\right), \quad C_1, C_2, C_3 = \text{const},$$

$$\psi = \frac{1}{2}\sqrt{3}\eta \frac{V^{\frac{2}{3}}}{F}. \tag{19}$$

2. Analysis of the Derived Metric

For small V, we have $F \underset{V\to 0}{\to} 1$, and (19) acquires the Kasner's form (1), (2) near the singularity $V = 0$. In this case, the field ψ tends to zero. For large V, we get $F \underset{V\to\infty}{\to} \frac{\eta^2}{2} V^{\frac{1}{3}}$, $\frac{V}{F^3} \underset{V\to\infty}{\to} \frac{8}{\eta^6}$, the metric tends to the form (1) with exponents $p_1 = p_2 = p_3 = 1/2$, and ψ grows infinitely. The singularities $V = 0$ and $V = \infty$ are located from each other at the infinite distance.

Solution (19) can be written in another form, by introducing the coordinate

$$U = \frac{V}{F^3} = \frac{8V}{\left(1+\sqrt{1+\eta^4 V^{\frac{2}{3}}}\right)^3} < \frac{8}{\eta^6} \quad , \tag{20}$$

$$ds^2 = -\frac{dU^2}{\left(1-\frac{\eta^4}{4}U^{\frac{2}{3}}\right)^6} + $$

$$+\frac{C_1 U^{2p_1} dt^2 - C_2 U^{2p_2} d\varphi^2 - C_3 U^{2p_3} dz^2}{\left(1-\frac{\eta^4}{4}U^{\frac{2}{3}}\right)^2},$$

$$\psi = \frac{2\sqrt{3}\eta}{4U^{-\frac{2}{3}} - \eta^4}. \tag{21}$$

Besides the region $U < 8\eta^{-6}$ which is equivalent to (19), it covers also the region $U > 8\eta^{-6}$. These regions are connected by the singularity $U = 8\eta^{-6}$, which corresponds to the spatial infinity. It may appear that we have obtained the analytic continuation of (19) to $U > 8\eta^{-6}$. But, upon the transformation $\xi = 64\eta^{-12}U^{-1}$, it pass, in fact, to solution (21) for $U < 8\eta^{-6}$ with a new collection of the Kasner's exponents $2/3 - p_1$, $2/3 - p_2$, $2/3 - p_3$ and with the field $\psi - 4\sqrt{3}\eta^{-3}$. In such a way, (21) describes two solutions of the type (19) which are connected by spatial infinities.

Conclusion

The conformally invariant scalar Korkina—Grigor'ev field does not change the form of the space-time near singularities and thus does not influence their properties. Its presence has no effect on the possibility of the creation of naked singularities via the collapse. At the same time, far from singularities, the space-time takes the asymptotic form (1) with $p_1 = p_2 = p_3 = 1/2$ irrespective of the parameters of a singularity. Such a behavior, according to the methods of determination of the type of singularities [7], corresponds to a singularity with positive mass. The finite length of any interval of a singularity (for finite V) tends to zero as $V \to \infty$, i.e., it becomes a point-like singularity. Only the infinite length of the whole singularity does not allow one to call a singularity at $V = \infty$ as point-like.

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ВПЛИВ СКАЛЯРНОГО ПОЛЯ КОРКІНОЇ—ГРИГОР'ЄВА НА ВЛАСТИВОСТІ ЧАСОПОДІБНИХ СИНГУЛЯРНОСТЕЙ У ЗАГАЛЬНІЙ ТЕОРІЇ ВІДНОСНОСТІ

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Р е з ю м е

Знайдено і проаналізовано точний розв'язок рівнянь Ейнштейна, що узагальнює метрику Казнера у випадку, коли сингулярність є також джерелом безмасового нелінійного конформно-інваріантного скалярного поля Коркіної—Григор'єва. Показано, що присутність такого поля не впливає на тип особливості.