SPATIAL DYNAMICS OF OPTICAL VORTICES WHEN GAUSS—LAGUERRE BEAM PROPAGATES IN THE KERR NONLINEAR MEDIUM

V.P. AKSENOV, I.V. IZMAILOV[1], B.N. POIZNER[1], O.V. TIKHOMIROVA

Institute of Atmospheric Optics, Siberian Branch of the Russian Academy of Sciences *(Tomsk, Russia),*

[1]**Tomsk State University** *(Tomsk, Russia)*

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The process of generation, evolution, and annihilation of optical vortices when a vortical laser beam interacts with a nonlinear medium is analyzed. The study is based on the parabolic wave equation allowing for the refractive index as a function of the light field intensity. Regularities of the vortex spatial evolution, intensity, phase, Umov—Poynting vector, and also the system of singular points for the phase gradient vector field in the beam cross section as functions of the longitudinal coordinate, medium parameters, and acting radiation have been discovered.

Introduction

Development of the controlled optical systems operating in the atmosphere, in particular, adaptive control systems, stimulates an exhaustive study of the fine structure of an optical field. A presence of the wave-front screw dislocations or optical vortices is the governing factor for the field structure.

The fundamental properties of the dislocation systems generated in light beams propagated through the vacuum, inhomogeneous, linear, and nonlinear media have been investigated in detail (see [1—7] and references therein). However, the development of adaptive optical systems with nonlinear feedback urges to study the regularities of interaction of a vortical optical beam with the nonlinear medium [8].

The present paper is devoted to a process of generation of the vortical structures in an optical wave propagating in the Kerr nonlinear medium [9]. The singular light field was investigated on the base of numerical solutions to the parabolic wave equation for the complex amplitude of a monochromatic wave field in the nonlinear medium. The theoretical models and approaches proposed before in [10, 11] were used for analyzing the field of directions of the energy flux density vector, constructing the energy stream-lines, and investigating the set of critical points of the corresponding dynamical system.

1. Basic Equations

The slow complex amplitude $U(\rho, z)$ of a monochromatic wave $V(\rho, z) = U(\rho, z)\exp(ikz)$ in a half-space $z \geq 0$ is assumed to obey the parabolic wave equation [7, 12—14]

$$2ik\langle n\rangle\frac{\partial U}{\partial z} + \Delta_\perp U + k^2\tilde{\varepsilon}(\rho, z)U(\rho, z) = 0, \tag{1}$$

where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, $\Delta_\perp = \nabla_\perp \cdot \nabla_\perp$, $\nabla_\perp = \mathbf{l}\frac{\partial}{\partial x} + \mathbf{m}\frac{\partial}{\partial y}$; $\rho = \{x, y\}$ is a vector in the pane perpendicuar to the radiation propagation direction, $\tilde{\varepsilon}(\rho, z) = [\varepsilon(\rho, z) - \varepsilon_0]$ is the field of medium permittivity inhomogeneities, $\varepsilon_0 = \langle n\rangle^2$, $\langle n\rangle$ is the refractive index mean value, $\varepsilon(\rho, z) = \varepsilon_0 + \varepsilon_2 I(\rho, z)$ is the permittivity for the nonlinear medium with the nonlinearity parameter ε_2 in the presence of an optical field with intensity $I = |U(\rho, z)|^2$.

If the complex wave amplitude is represented via the intensity $I(\rho, z)$ and phase $S(\rho, z)$ as $U(\rho, z) = \{I(\rho, z)\}^{1/2}\exp[iS(\rho, z)]$, a system of equations equivalent to Eq.(1) that consists of the eikonal equation

$$\frac{\partial S}{\partial z} + \frac{1}{2k}\{\nabla_\perp S\}^2 = k\tilde{\varepsilon}(\rho, z) + \frac{1}{4k}\left[\frac{\Delta_\perp I}{I} - \frac{1}{2}\frac{(\nabla_\perp I)^2}{I^2}\right] \tag{2}$$

and the transport equation

$$\nabla_\perp\{I(\rho, z)\nabla_\perp S\} = -k\frac{\partial I}{\partial z}, \tag{3}$$

can be written. In Eq.(3), a value

$$\mathbf{P}_\perp(\rho, z) = I(\rho, z)\nabla_\perp S(\rho, z) \tag{4}$$

is the transversal component of the Umov—Poynting vector $\mathbf{P}(\rho, z)$; and $P_z = kI$ is the longitudinal component of $\mathbf{P}(\rho, z)$. These vector components allow the energy stream-lines or diffraction rays [11, 14, 16] for the light field $U(\rho, z)$ to be constructed in the space. The diffraction rays are the integral curves of the differential equation [11,14,16] for the current transversal coordinate of the diffraction ray $\rho = \rho(z) = \{x(z), y(z)\}$:

$$\frac{d\rho}{dz} = \frac{\mathbf{P}_\perp(\rho, z)}{P_z(\rho, z)} = \frac{1}{k}\nabla_\perp S(\rho, z). \tag{5}$$

A family of rays presents a pattern of the spatial energy distribution in the optical field. The stream-lines coincide with the phase trajectories which are the tangent lines for the phase gradient [see Eq.(4)].

It is practical to rewrite the differential equation (5) as the system

$$\begin{cases} \dfrac{dx}{dz} = \dfrac{1}{k}\dfrac{\partial}{\partial x}S(x, y, z), \\ \dfrac{dy}{dz} = \dfrac{1}{k}\dfrac{\partial}{\partial y}S(x, y, z), \end{cases} \tag{6}$$

and to consider Eq.(6) as a model of the reduced dynamical system [11] with a single degree of freedom and two dynamic variables x and y. The coordinate z plays the role of an evolutional parameter. To study the vortical spatial dynamics for an optical field in the Kerr nonlinear medium, a structure of the phase space of system (6) should be investigated in detail.

2. Singular Wave Field Structure

The spatial dynamics of the singular field in a Gauss—Laguerre optical beam and a Gaussian beam passed through an astigmatic plate was considered in detail in [10, 11]. The investigation of the full set of critical points in the dynamic system corresponding to Eqs. (6) had shown that the energy redistribution in the beam and the appearance of optical vortices occurred in accordance with the transformation of singularities. The local focusing of a stream-line in the vicinity of a knot was a precursor of a generated pair of vortices, and the stream-line took on a helix shape near an unstable focus.

In the present paper, a laser operating in Gauss—Laguerre modes is considered as a source of the wave field [11]. In this case, a situation can be realized when three modes are excited and, as was shown in [11], the optical vortices appeared and disappeared in the beam at a certain distance from the radiating aperture in a homogeneous medium. Zero-intensity positions significantly depended on the longitudinal coordinate. The complex field amplitude had the form [11]

$$U(x, y, \mathcal{Z}) = (1 + \mathcal{Z}^2)^{-3}\times$$

$$\times \exp\frac{(x^2 + y^2)(i\mathcal{Z} - 1)}{2(1 + \mathcal{Z}^2)}(U_r + iU_i)(\mathcal{Z} + i)^3, \tag{7}$$

where $U_r = -3\mathcal{Z}^2 + 2\mathcal{Z} - 1 + 2(x - y)^2$, $U_i = 3\mathcal{Z}^2 + 2\mathcal{Z} + 1 - 2(x + y)^2$, $\mathcal{Z} = \frac{z}{ka^2}$ is the wave parameter [17], a is the effective beam radius.

Two pairs of the conjugate optical vortices are the peculiar properties of field (7) which can be observed at the distance $\mathcal{Z} = 0.11$ from the radiating aperture in a homogeneous medium [11]. When $\mathcal{Z} = 0.33$, these vortices annihilate and then appear again at $\mathcal{Z} = 1$.

Let now the Gauss—Laguerre beam (7) enters into the Kerr nonlinear (the parameter ε_2 in Eq.(1) takes nonzero values) medium at $\mathcal{Z} = 0.5$, where the field has no vortices but contains preconditions of dislocation generation when increasing in the longitudinal coordinate. In such a case, field (7) can be considered as the initial condition for a solution to the parabolic equation (1). Under these conditions, Eq.(1) and the ray differential equations (6) were solved numerically. To solve Eq.(1), the phase screen technique [18] as a version of the splitting methods [19 — 21] was used. Our computer programs provided for solving the stiff ordinary differential equation (1) in the approximation of medium steady-state response [18]. This allowed us to calculate the distributions of the real and imaginary parts of the optical field, its intensity, and phase. In order to study the spatial ray dynamics of the resulting light field, Eqs.(6) were solved by the Euler method with automatic step selection to determine the trajectory of a ray passing through an arbitrary given point (x_0, y_0, z_0) playing a role of the initial point.

Let us first study a structure of the phase space of system (6) by considering the spatial characteristic, which is described by Eq.(5) using the construction of the field of ray directions. We take into account the fact that the evolution of stream-lines for the diffracted laser beam has a unidirectional character. The vector field is mapped in Figs.1—3 by the segments of the tangents to phase trajectories with a point in the segment beginning. Such a field represents the propagation directions of light rays in the cross-sectional plane or the field of phase gradient projections on the xOy plane at a given value of the longitudinal coordinate z =const.

Analyzing the vector field of phase-gradient projections shows that the two-dimensional phase space of our dynamic system (the xOy plane) has a center of

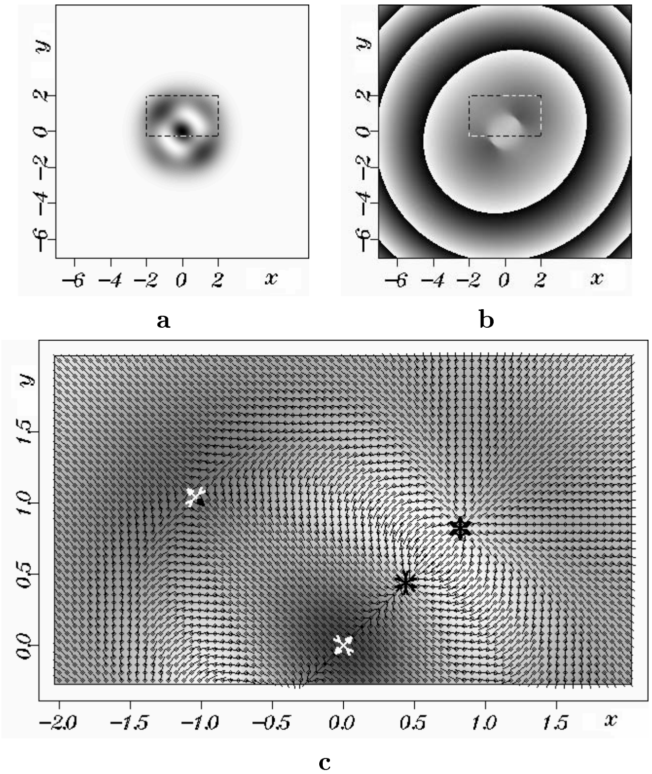
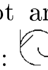
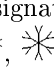
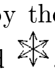


Fig. 1. Intensity (a), phase principal value (b), and field of the phase-gradient projections (c) for the Gauss—Laguerre beam at $\mathcal{Z} = 0.5$. The positions of some singular points are marked with symbols: x (saddle) and * (knot)

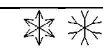
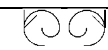
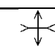
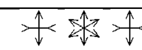
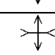
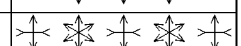
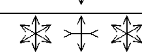
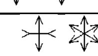
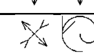
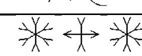
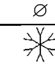
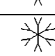
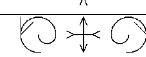
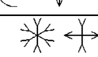
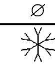
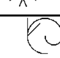
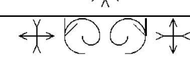
symmetry ($x = 0, y = 0$) and two symmetry planes running through the Oz axis and separatrices $x = y$ and $x = -y$. Therefore, it is sufficient to represent a fragment of the phase plane: $-2 < x < 2$ and $-0 < y < 2$. The intensity, phase, and structure of the vector field determined by the relative position of singular points, is presented in Fig. 1 for $\mathcal{Z} = 0.5$. For clarity, the field of directions is represented against the laser-beam intensity background: darker areas correspond to higher intensities. In the description of the propagation of an optical wave along the z-axis, the wave parameter $\mathcal{Z}$ was taken as an evolutional parameter, and it was assumed that $\lambda = 0.63\,\mu$ and $a = 0.05$ m.

It was established in [11] that the energy in the beam was spatially redistributed in accordance with the relative position of singular points of the field of phase-gradient projections. The appearance of optical vortices was associated with transformations of singular points, i.e., with bifurcations of the corresponding dynamical system. The key bifurcation was the generation of a pair of unstable focuses from the unstable and stable knots. This bifurcation was an event of generation of a pair of vortices, and the reverse bifurcation was an annihilation of vortices.

Under the conditions of light propagation through the Kerr nonlinear medium the scenarios of the generation, evolution, and disappearance of optical vortices change depending on the value ε_2.

To regulate and formalize a description of the singularities and bifurcations, we introduce the following designations: $\mathcal{K}$, $\mathcal{S}$, $\mathcal{F}$ are the knot, saddle, and focus, respectively; s, u are the stable and unstable singularities; B_i is the bifurcation transforming a group of singularities lc into a group $l'c'$, where l is the quantity of singularities, c is the singularity type; B_i^{-1} is the bifurcation reverse to B_i. Under the considered propagation conditions, the following bifurcations take place: B_1: $(u\mathcal{K}, s\mathcal{K}) \to 2\mathcal{F}$; B_2: $\mathcal{S} \to (2\mathcal{S}, u\mathcal{K})$; B_3: $\mathcal{S} \to (3\mathcal{S}, 2u\mathcal{K})$; B_4: $(\mathcal{S}, 2u\mathcal{K}) \to u\mathcal{K}$; B_5: $(\mathcal{S}, u\mathcal{K}) \to \varnothing$; B_6: $(\mathcal{S}, \mathcal{F}) \to \varnothing$; B_7: $(2s\mathcal{K}, \mathcal{S}) \to s\mathcal{K}$; B_8: $u\mathcal{K} \to (\mathcal{S}, 2\mathcal{F})$; B_9: $\varnothing \to (s\mathcal{K}, \mathcal{S})$; B_{10}: $s\mathcal{K} \to \mathcal{F}$; B_{11}: $(2\mathcal{S}, 2\mathcal{F}) \to \varnothing$. A symbol $\varnothing$ means the empty set of singularities. When $\varnothing$ appears as a result of bifurcation, it is an annihilation of singularities. When some singularities appear *ex nihil* (from nothing $\varnothing$), then it is a birth of singularities. The revealed bifurcations are presented in Table 1, where the focus, saddle, stable knot, and unstable knot are designated by the following symbols, respectively: , , , and .

Simulation shows that the scenario of the light field evolution can be described *pro forma* for the nonlinear

Bifurcation	Group of singularities	
	Before the bifurcation	After the bifurcation
B_1: $(u\mathcal{K}, s\mathcal{K}) \to 2\mathcal{F}$		
B_2: $\mathcal{S} \to (2\mathcal{S}, u\mathcal{K})$		
B_3: $\mathcal{S} \to (3\mathcal{S}, 2u\mathcal{K})$		
B_4: $(\mathcal{S}, 2u\mathcal{K}) \to u\mathcal{K}$		
B_5: $(\mathcal{S}, u\mathcal{K}) \to \varnothing$		$\varnothing$
B_6: $(\mathcal{S}, \mathcal{F}) \to \varnothing$		$\varnothing$
B_7: $(2s\mathcal{K}, \mathcal{S}) \to s\mathcal{K}$		
B_8: $u\mathcal{K} \to (\mathcal{S}, 2\mathcal{F})$		
B_9: $\varnothing \to (s\mathcal{K}, \mathcal{S})$	$\varnothing$	
B_{10}: $s\mathcal{K} \to \mathcal{F}$		
B_{11}: $(2\mathcal{S}, 2\mathcal{F}) \to \varnothing$		$\varnothing$

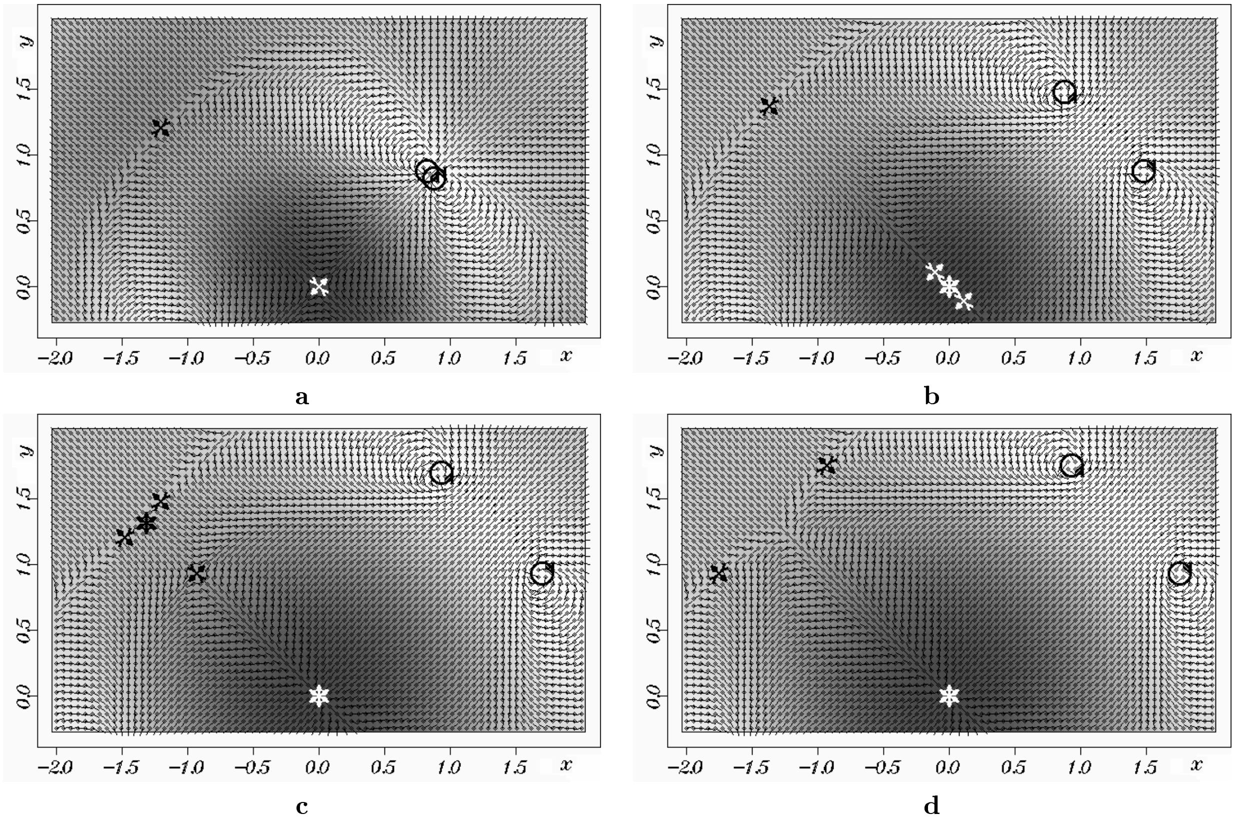


Fig. 2. Field of phase-gradient projections for the self-defocusing medium, $\varepsilon_2 = -5 \cdot 10^{-12}$. $\mathcal{Z}$: 0.787 (a), 1.109 (b), 1.258 (c), 1.309 (d). The positions of some singular points are marked with symbols: x (saddle), * (knot), and o (focus)

($\varepsilon_2 = 0$) and self-defocusing ($\varepsilon_2 = -5 \cdot 10^{-12}$) media as follows (Fig. 2): [$2B_1$; B_2; $2B_2$; $2B_5$; $4B_6$] and [$2B_1$; B_2; $2B_3$; $2B_4$; $2B_5$; $4B_6$], respectively. When $\varepsilon_2 = -7.5 \cdot 10^{-12}$, the second scenario transforms into [$2B_2$; $2B_4^{-1}$; $2B_1$; $2B_4$; B_2; $2B_5$; $4B_6$]. The appearance of vortices through the bifurcation B_1 (a pair of the stable and unstable knots $\rightarrow$ a pair of the unstable focuses) and the sequent disappearance of vortices through the bifurcation B_6 (annihilation of the saddle and focus) are the peculiarities of these scenarios.

The evolutional scenario changes essentially when the light propagates in the nonlinear medium, $\varepsilon_2 = 5 \cdot 10^{-12}$, (Fig. 3): [$B_7$; $2B_8$; ($4B_9$; B_7^{-1}; $2B_5^{-1}$); $4B_5^{-1}$; (B_2; $2B_7^{-1}$; $2B_8^{-1}$); ($2B_2$; $2B_4$); $4B_{10}$; $2B_9$; $2B_{11}$; $4B_5$; $4B_9^{-1}$; $2B_2^{-1}$; $2B_1$]. The bifurcations in the parentheses occur simultaneously. Along with the bifurcation B_1 revealed before [11], when the vortices appear from the stable and unstable knots, a situation is realized when a pair of the vortices appears because of the bifurcations B_8 (the unstable knot $\rightarrow$ the saddle and two focuses) and B_{10} (the stable knot $\rightarrow$ focus). At that, the bifurcations $2B_8$ occur instead of $2B_1$, since two stable knots taking part before in the bifurcation $2B_1$ are "indrawn" into the beam center due to the self-focusing effect and then create a stable knot together with a saddle after the bifurcation B_7. Note that the stable knot is a symbol of self-focusing.

The beam energy is concentrated in the areas corresponding with the dark spots in Fig. 3. One can see that the beam first is self-focused, then the stable knot transforms into the saddle and two stable knots (Fig. 3,c,d) and the dark area increases (Fig. 3,d,e,f). It is due to the nonlinear focus [17], where the beam cross-section is minimum. In the nonlinear medium, the bifurcation B_1 occurs in the same x and y coordinates of the beam cross-section as in vacuum [11], but subsequently far, at $\mathcal{Z} = 1.74$ instead of $\mathcal{Z} = 1$. Note that much more complex evolutional scenarios with the other bifurcations, for example, B_{11}^{-1}, are realized when $\varepsilon_2 > 5 \cdot 10^{-12}$.

Simulation shows that new evolutional scenarios are realized for the Gauss—Laguerre beam in the nonlinear

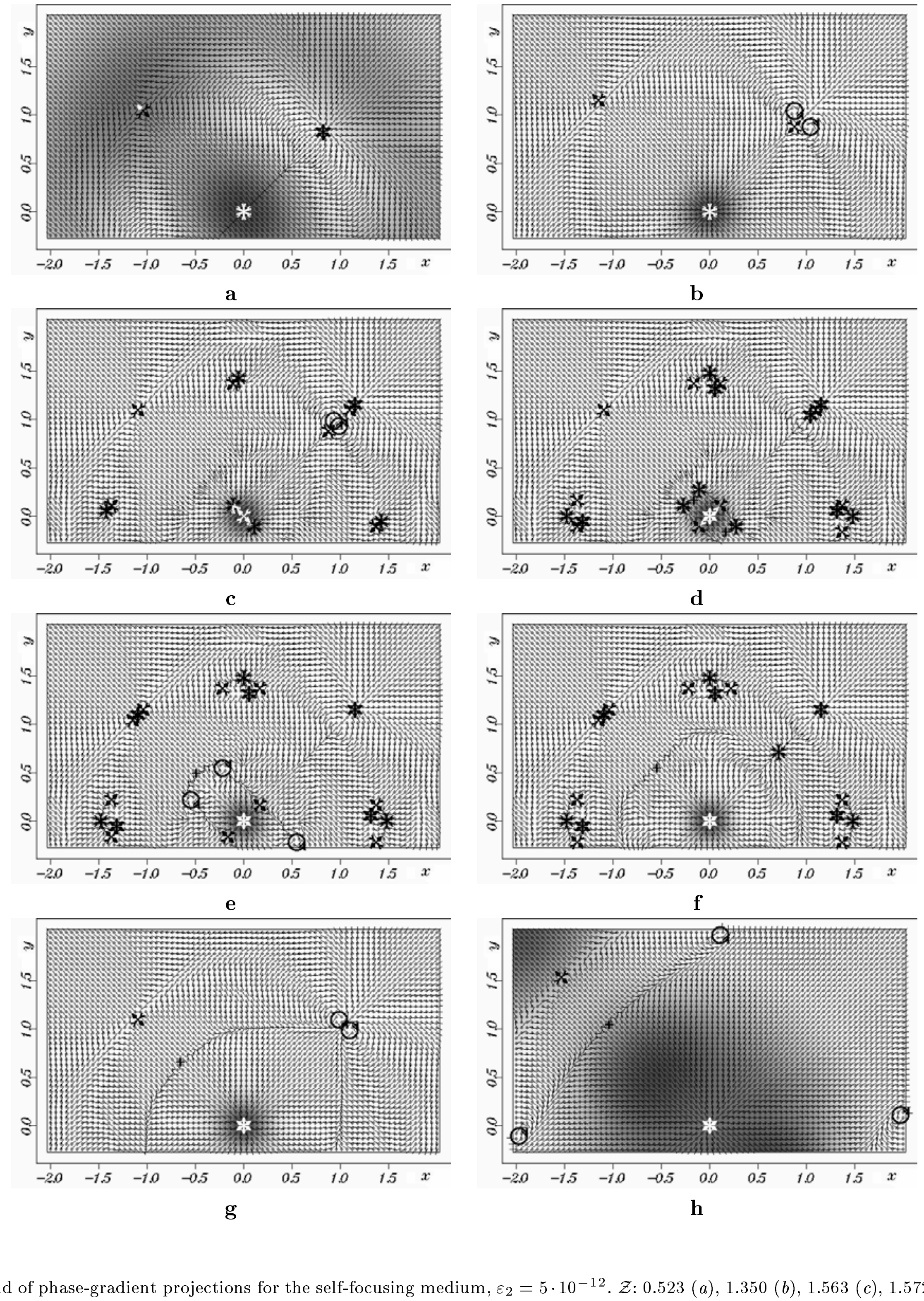


Fig. 3. Field of phase-gradient projections for the self-focusing medium, $\varepsilon_2 = 5\cdot10^{-12}$. $\mathcal{Z}$: 0.523 (*a*), 1.350 (*b*), 1.563 (*c*), 1.572 (*d*), 1.585 (*e*), 1.608 (*f*), 1.739 (*g*), 2.179 (*h*). The positions of some singular points are marked with symbols: x (saddle), * (knot), and o (focus)

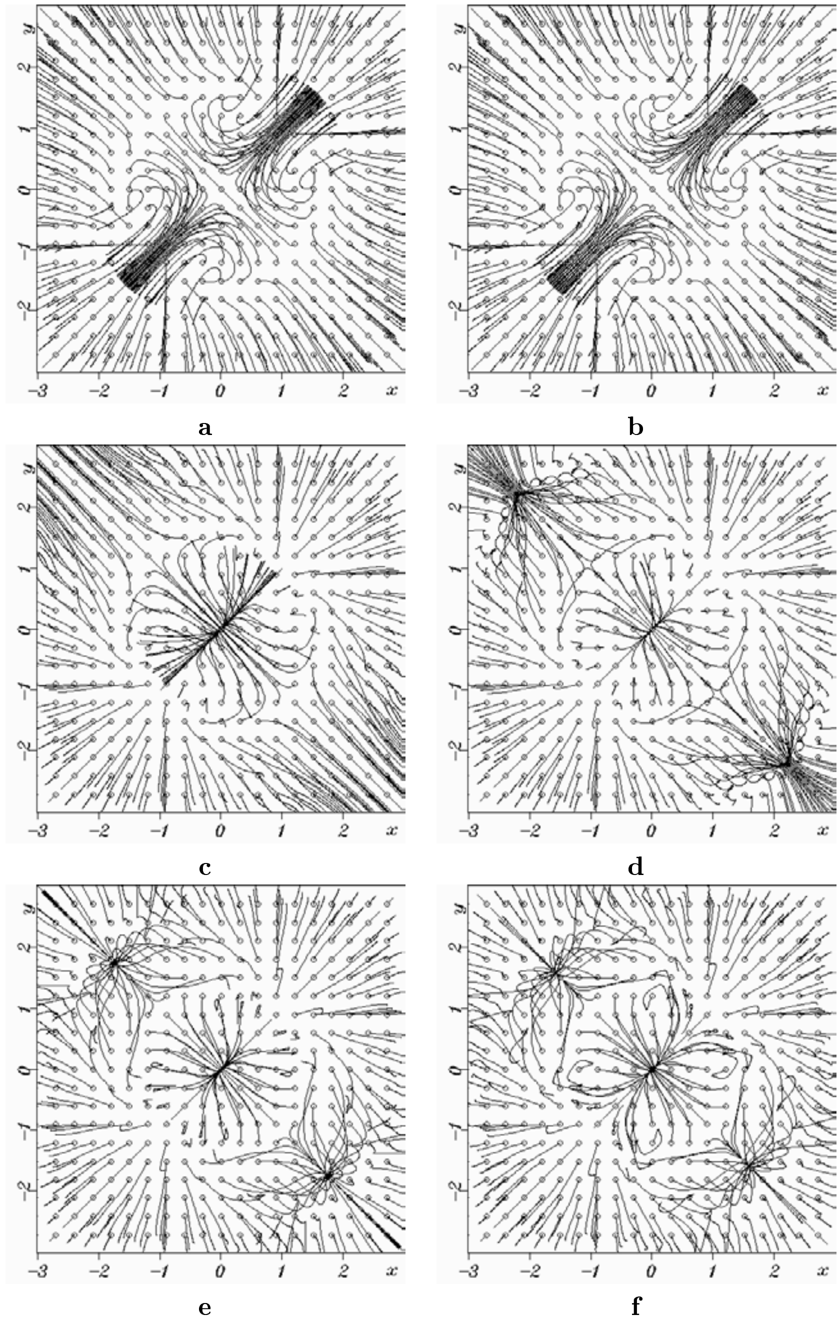


Fig. 4. Projections of the stream-lines for a light beam propagated in the Kerr self-(de)focusing medium. $0.5 < \mathcal{Z} < 203.3$. $\varepsilon_2 : -5\cdot 10^{-12}$ (*a*), $-7.5\cdot 10^{-12}$ (*b*), $5\cdot 10^{-12}$ (*c*), $7.5\cdot 10^{-12}$ (*d*), 10^{-11} (*e*), $1.25\cdot 10^{-11}$ (*f*)

medium with increase in the modulus of the nonlinear parameter ε_2. In this connection, the variation in ε_2 changes a scenario in a larger degree when $\varepsilon_2 > 0$. For the self- defocusing medium, $\varepsilon_2 < 0$, the vortex-generation scenario is changed on the whole, but the conditions for vortex appearance are the same as those in the linear medium [11]. Both in this case and in the linear medium ($\varepsilon_2 = 0$), there is no annihilation of the vortices in the beam far zone. The scenarios are changed sufficiently for the self-focusing medium ($\varepsilon_2 > 0$) due to the other bifurcations and the annihilation mentioned above is possible (B_8^{-1}; B_{10}).

The diffraction ray projections on the xOy plane for various propagation conditions over the region $0.5 < \mathcal{Z} < 203.3$ are presented in Fig. 4. The ray shapes are due to focusing the rays in the vicinities of stable knots and defocusing in the vicinities of unstable knots, approximately 90-degree turn near saddles, and curl along the helix trajectory around unstable focuses. At the same time, one ray can move along various trajectories by turns owing to a limited lifetime of the singularities. For example, the helix rays are focused at the stable knots after the vortices disappear and then these rays become rectilinear.

Conclusion

The wave and ray dynamics of a singular light field in the process of generation, evolution, and disappearance of optical vortices are studied with the use of the model of a laser-generated Gauss—Laguerre beam propagated in the nonlinear medium.

An analysis of the structure of the vector field of the phase gradient made it possible to expose the tendencies in the behavior of singular points of this field, which lead to the generation and annihilation of vortices, and to the determination of their spatial localization. The descriptive formalizm for the evolutional scenarios as a sequence of bifurcations is proposed. The bifurcations are the functions mapping one group of singularities into another group. The singular points discovered satisfy the theorem from [22] on the algebraic number of singular points of a vector field on a plane.

The projections of energy stream-lines onto the plane perpendicular to the beam propagation direction are plotted. An energy stream-line modifies its shape depending on the evolutional variable and the distance to singular points. In the vicinity of an unstable focus, this line takes form of a helix.

The results obtained indicate a diversity of the bifurcations of appearance and annihilation of optical vortices in dependence on a light beam model and medium properties. Problems of the bifurcation completeness and construction of the model for an optical field evolution can be formulated and solved on the base of these results. A conjecture can be frame on the creation of a system of local anti-vortical wave-front distortions by the facilities of adaptive optics. Such a structure is assumed to provide or promote the vortex annihilation at specified points.

1. *Berry M.V.*// Physics of Defects/Ed. by R. Balian, M. Kleman, J.-P. Poirier. — Amsterdam: North-Holland, 1981. — P. 453—543.
2. *Baranova N.B., Zel'dovich B.Ya.* //Zh. Eksp. Teor. Fiz. — 1981. — **80**, N5. — P.1789—1797 [Sov. Phys. JETP. — 1981. — **53**. — P. 925].
3. *Kreminskaya L.V., Soskin V.S., Khizhnyak A.I.* //Opt. Communs. — 1998. — **145**. — P. 377—384.
4. *Horizons* in World Physics. — Vol 228: Optical Vortices/ Ed. by M. Vasnetsov, K. Staliunas. — Huntington, New York: Nova Science, 1999.
5. *Weiss C.O., Vanpel M., Staliunas K. et al.* //Appl. Phys. B. — 1999. — **68**. — P. 151—168.
6. *Proc.* SPIE. — Vol 4403: Singular Optics (Optical Vortices) SO'2000/Ed. by M.Soskin.
7. *Deykoon A.V., Swartzlander G.A.* //J. Opt. Soc. Amer. B. — 2001. — **18**, N6. — P. 804—810.
8. *Vorontsov M.A., Firth W.J.* // Phys. Rev. A. — 1994. — **49**, N 4. — P. 2891—2906.
9. *Izmailov I.V., Aksenov V.P., Poizner B.N.* //J. for Problems of Open System Evolution. — 2003. — **1**, N5. — P. 112—119.
10. *Aksenov V.P. Tikhomirova O.V.* //J. Opt. Soc. Amer. A. — 2002. — **19**, N2. — P. 345—355.
11. *Aksenov V.P. , Izmailov I.V., Poizner B.N., Tikhomirova O.V.* //Opt. Spektr. — 2002. — **92**, N3. — P. 409—418.
12. *Rytov S.M., Kravtsov Yu.A., Tatarskii V.I.* Introduction to Statistical Radio Physics. — Moscow: Nauka, 1978. — Pt. 2. Random Fields (in Russian).
13. *Aksenov V.P., Banakh V.A. et al.* High-Power Laser Beams in the Random Inhomogeneous Atmosphere. — Novosibirsk: Publ. House of Siberian Branch of RAN, 1998.
14. *Aksenov V.P., Kolosov V.V., Tartakovskii V.A., Fortes B.V.* //Atmospheric and Ocean Optics. — 1999. — **12**, N10. — P. 952—958.
15. *Bespalov V.I., Litvak A.G., Talanov V.I.* Proc. of the II All-Union Symposium on Nonlinear Optics. — Novosibirsk: Nauka, 1968. — P. 428—463.
16. *Kolosov V.V.* //Atmospheric and Ocean Optics. — 1996. — **9**, N12. — P. 1631—1638.
17. *Vinogradova M.B., Rudenko O.V., Sukhorukov A.P.* Wave Theory.— Moscow: Nauka, 1990.
18. *Kandidov V.P.* //Uspekhi Fiz. Nauk. — 1996. — **166**, N12. — P. 1309—1338 [Phys. Usp. — 1996. — **39**. — P. 1243].
19. *Rikhtmaier R., Morton K.* Difference Methods to Solve the Boundary Problems. — Moscow: Mir, 1972 (in Russian).
20. *Godunov S.K., Ryaben'kii V.S.* Difference Schemes. — Moscow: Nauka, 1977 (in Russian).
21. *Samarskii V.* Introduction to Numerical Methods. — Moscow: Nauka, 1987 (in Russian).
22. *Krasnosel'skii M.A., Petrov A.I., Povolotskii A.I., Zabreiko P.P.* Vector Fields on Plane. — Moscow: Fizmatgiz, 1963 (in Russian).

ПРОСТОРОВА ДИНАМІКА ОПТИЧНИХ ВИХОРІВ ПРИ ПОШИРЕННІ ГАУСС-ЛАГЕРРІВСЬКОГО ПУЧКА В СЕРЕДОВИЩІ З КЕРРІВСЬКОЮ НЕЛІНІЙНІСТЮ

В.П.Аксьонов, І.В.Ізмайлов, Б.Н.Пойзнер, О.В.Тіхомірова

Р е з ю м е

Досліджено генерацію, перетворення та анігіляцію оптичних вихорів при поширенні лазерного пучка з вихорами крізь нелінійне середовище. Аналіз базується на розв'язанні параболічного хвильового рівняння, в якому показник заломлення залежить від потужності світла. Встановлено закономірності перетворення вихорів та систем сингулярних точок у поперечному перерізі пучка, а також інтенсивностей, фаз та вектора Умова—Пойнтінга при поширенні пучка для різних параметрів нелінійного середовища та інтенсивностей світла.