ROTATION OF ARBITRARY OPTICAL IMAGE AND THE ROTATIONAL DOPPLER EFFECT

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We present a new insight into the understanding of an image rotation by an optical system, based on the rotational Doppler effect (RDE). Any image can be decomposed into a series of Laguerre — Gaussian modes which acquire concerted phase shifts due to the RDE. The interference of the phase-shifted modes exactly produces the whole image rotation. Examples of a specific manifestation of the proposed interpretation and possible applications for the image structure analysis are discussed.

Introduction

A specific frequency shift appearing due to relative rotation of the observer and a light beam with helical wave front, known as the rotational Doppler effect (RDE), attracted the increasing attention during the last years [1—6]. The RDE origination is connected to the peculiar spatial structure of such a beam due to which the beam rotation around the z-axis of propagation is completely equivalent to its translation along the z direction. The radiation field of a monochromatic helical light beam can be mathematically expressed as [4, 7]

$$E \propto F\left(r,z\right)\cos\left(l\varphi + kz - \omega t\right), \tag{1}$$

where k is the radiation wave number, ω is its optical frequency, r and φ denotes the polar coordinates within the beam cross-section, and $F(r,\,z)$ is a slowly varying function of r and z. The integer l determines the pitch $2\pi|l|$ and the winding direction of the helicoidal wave front so that a right helicoid corresponds to negative l (see Fig. 1).

The beam rotation (achieved, e.g. by the source rotation) around the z-axis with angular velocity Ω and translation along z with velocity v can be expressed in Eq. (1) by the transformation $\varphi = \varphi' - \Omega t$, $z = z' - vt$. Then, with respect to the fixed coordinates φ', z' the beam field acquires the representation $E \propto F\left(r,z\right)\cos\left(l\varphi' + kz' - \omega' t\right)$ [the difference between $F(r,\,z)$ and $F(r,\,z')$ is neglected], where

$$\omega' = \omega + kv + l\Omega \tag{2}$$

is the light frequency "seen" by a fixed observer. The second term on the right-hand side of Eq. (2) accounts for the usual translational Doppler effect while the last one just describes the RDE. Note that, in contrast to the translational case, the RDE frequency shift does not depend on the beam wavelength [1—3].

Additional physical meaning of the Doppler effect is revealed in the processes of energy exchange between a light wave and optical elements [3, 8, 9]. For example, when the wave is reflected from a mirror, the electromagnetic momentum of the light field is changed and the ponderomotive force acts upon the mirror. The mirror being moved along or against the wave propagation, the mechanical work is done by (or on) the wave, thus changing its energy. Since the number of photons involved remains the same, this energy change is distributed between them so that every photon of the beam gets (positive or negative) energy variation $\Delta\varepsilon$. The corresponding frequency shift

$$\Delta\omega = \frac{\Delta\varepsilon}{\hbar} \tag{3}$$

expresses the "interactive aspect" of the Doppler effect [9].

Likewise, a helical light beam carries the orbital angular momentum L with respect to the z-axis, which equals $l\hbar$ per photon or $L = lW/\omega$ for the total beam energy W [7, 10]. In other words, such a beam is analogous to an "intrinsically" spinning mechanical body [7]. To set this "body" into "extrinsic" rotation, an angular velocity Ω must be algebraically added to this "intrinsic" motion, for which a certain torque should be applied or experienced by corresponding optical elements. In this process, the beam energy changes by $L\Omega$ or, per photon, by $\Delta\varepsilon = l\hbar\Omega$. Hence, via Eq. (3), the frequency shift follows that exactly coincides with the last term of Eq. (2).

1. General Consideration

Usual schemes for RDE observation employ the interference between a rotating signal beam and a stationary reference one. Mutual coherence is ensured by the specific way of preparing the two beams, which are obtained by splitting a single laser beam into two arms, one of which contains a rotating element causing the beam rotation (for example, an astigmatic lens system [1], a Dove prism [2, 3] or a spiral zone plate [4]). However, the mechanical motion of this element appears to be a source of incontrollable disturbances and instabilities, which makes observation of the RDE in optical experiments a very difficult technical problem.

To avoid these difficulties, a scheme was proposed where the RDE is visualized due to the interference of two beams passing the same optical elements [5]. Each beam experiences identical perturbations, which thus do not affect the interference pattern. Obviously, this scheme can be used for quantitative measurements if one of the beams knowingly demonstrates no RDE, for example, because of its smooth wave front.

In the simplest case, the signal Laguerre-Gaussian (LG_{0l}) and reference Gaussian beams were used [5, 6]. A coaxial superposition of these beams is characterized by the complex amplitude distribution is

$$u\left(r,\varphi\right) \propto \left[1 + A\exp\left(i\delta\right)\left(\frac{r}{b}\right)^{|l|}\exp\left(il\varphi\right)\right]\exp\left(-\frac{r^2}{2b^2}\right), \tag{4}$$

where b is the Gaussian envelope radius, A and δ stand for the relative magnitude and phase shift of the "vortex" LG component with respect to the Gaussian one. The corresponding intensity distribution

$$|u(r,\varphi)|^2 \propto \left[1 + 2A\left(\frac{r}{b}\right)^{|l|}\cos(\delta + l\varphi) + A^2\left(\frac{r}{b}\right)^{2|l|}\right]\exp\left(-\frac{r^2}{b^2}\right). \tag{5}$$

Let now the superposition beam (4) rotate around the z-axis with constant angular velocity Ω. In this situation, the Gaussian component is not changed, while the vortex one obtains the RDE frequency shift $\Delta\omega = l\Omega$ [1—3]. Consequently, the phase difference δ in Eqs. (4), (5) acquires the dependence on time:

$$\delta = \delta_0 - l\Omega t. \tag{6}$$

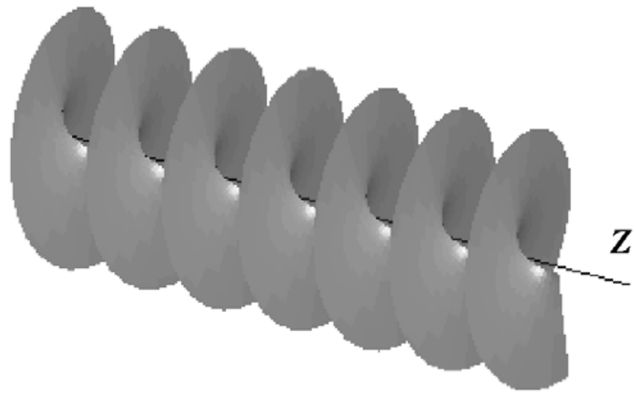

Fig. 1. Example of a helical wavefront structure (the $LG_{0,-1}$ mode)

The resulting intensity distribution (5) also becomes time-dependent. The signal $S(t)$ of a photodetector with small aperture placed at a point with coordinates r, φ will contain an alternating component

$$S_{\sim} \propto \left(\frac{r}{b}\right)^{|l|}\exp\left(\frac{r^2}{b^2}\right)\cos\left(\delta_0 + l\varphi - l\Omega t\right). \tag{7}$$

Now we stress the essential point: Exactly as in two-arm methods of the RDE observation [2—4] and typically for the interference approaches to frequency difference measurements, a beating signal arises, whose frequency coincides with the measured frequency shift.

However, the situation with the coaxial superposition can be viewed quite differently. If any beam rotates around the z-axis with constant angular velocity Ω, its transverse profile $u\left(r,\varphi\right)$ is modified according to the substitution

$$\varphi \to \varphi - \Omega t. \tag{8}$$

Being applied to Eqs. (4), (5), this rule immediately gives $u\left(r,\varphi\right) \to u\left(r,\varphi,t\right)$ where

$$u(r,\varphi,t) \propto \left\{1 + A\exp(i\delta)\left(\frac{r}{b}\right)^{|l|}\exp[il(\varphi - \Omega t)]\right\}\times \exp\left(-\frac{r^2}{2b^2}\right) = \left\{1 + A\exp[i(\delta - l\Omega t)]\left(\frac{r}{b}\right)^{|l|}\times \exp(il\varphi)\right\}\exp\left(-\frac{r^2}{2b^2}\right). \tag{9}$$

The expression in the first line of the above equation describes the rotating superposition (4), while the second line represents a superposition of Gaussian and frequency-shifted LG_{0l} beams. Two cases which seem very different physically, appear to have identical mathematical representations. Correspondingly, the beating signal (7) can be ascribed to the purely geometric effect of the inhomogeneous beam spot motion with respect to an off-axis aperture. As a result, the superposition beam rotation itself can serve as the indication of a frequency shift of its vortex component.

In application to the RDE, this looks as a tautology: rotation of the beam pattern causes the frequency shift which, in turn, causes the same rotation. This conclusion is a direct consequence of the specific transformational property of beams with helicoidal structure (shared with circularly polarized waves), due to which any turn around the beam axis is equivalent to a certain additive modification of the beam phase [8, 11]. Properly speaking, it seems meaningless to discuss whether the rotation of such a beam is the cause of its frequency shift or, vice versa, the frequency shift leads to the beam rotation: in a very deep physical sense, both things are the same. On the other hand, this observation leads to the interesting and far-reaching conclusions relating a profound connection between the rotation and the frequency spectrum of generic optical fields: Any beam rotation around its own axis is equivalent to certain concerted frequency shifts of its components. In other words, in any image rotation within its own plane, the RDE is present implicitly.

Really, in a fixed cross section of an arbitrary monochromatic light beam with cyclic frequency ω, the wave field can be written as a superposition

$$E\left(r,\varphi,t\right)=\sum_{p,l}a_{pl}\psi_{p}^{l}\left(r,\varphi,t\right) \tag{10}$$

[Eq. (4) represents its very special case]. In this sum, a_{pl} are complex numbers, and the basic functions of expansion (10) have the general form

$$\psi_{p}^{l}\left(r,\varphi,t\right)=u_{p}^{l}(r,\varphi)\exp\left(-i\omega t\right). \tag{11}$$

Here

$$u_{p}^{l}\left(r,\varphi\right)=F_{p}^{|l|}\left(r\right)\exp\left(il\varphi\right) \tag{12}$$

are Laguerre — Gaussian modes which form a complete orthogonal set [3, 12]. Expansion (10) may be called the LG spectrum of the beam [13, 14]. If the beam rotates with angular velocity Ω, the transformation law (8) yields, for arbitrary indices p and l,

$$\psi_{p}^{l}\left(r,\varphi,t\right)\to\psi_{p}^{\prime l}\left(r,\varphi,t\right)=$$

$$=u_{p}^{l}\left(r,\varphi\right)\exp\left[-i\left(\omega+l\Omega\right)t\right]. \tag{13}$$

Therefore, rotation of a monochromatic light beam makes it polychromatic [1]; sidebands differ from the base frequency by $l\Omega$ where l takes values of the azimuthal indices of all LG components of the beam. The genetic connection between the sidebands and RDE is clear; obviously, their very existence can be considered as an evidence of RDE. Essentially, it is no matter which beam to rotate and, anyway, the beam is not obliged to be helicoidal. Since the RDE shift is wavelength-independent, this important conclusion is fully applicable to polychromatic images.

Positions and the intensity of sidebands demonstrate relations between the LG spectrum of the beam and the Fourier spectrum of the signal produced by the rotating beam. They can be readily observed in various interference experiments and used for the LG spectrum retrieval [13, 14]. For example, the results of mutual interference of the sidebands are seen in the local intensity variations. Indeed, the intensity distribution corresponding to Eqs. (10) — (12) is, in general, inhomogeneous and behaves in accord with the immediate generalization of Eq. (5)

$$I\left(r,\varphi,t\right)=\left|\sum_{p,l}a_{pl}u_{p}^{l}(r)\exp\left(-il\Omega t\right)\right|^{2}. \tag{14}$$

The corresponding signal spectrum will contain frequencies $(l-l')\Omega$ where l and l' are arbitrary values of the azimuthal index present in sum (10). The more complicated procedure, where another coherent beam is mixed with the observed one, can give an additional information on its LG spectrum [13]. However, since the sidebands do not depend on the radial index p, the whole information they contain is, generally, insufficient for reconstructing the complete LG spectrum of an arbitrary beam. Simple superpositions of LG modes [like the Gaussian — LG_{0l} mode mixture (4) or the three-mode combination used in [13]], for which the beating spectrum directly shows RDE shifts of its vortex components and gives possibility to reveal their weights, are rather exclusions.

Of course, the whole picture is reversible: if we impart concerted frequency shifts to the beam LG components, it will result in the beam rotation. Relevant conditions are realized sometimes in deformed optical fibers, which leads to the optical Magnus effect [15].

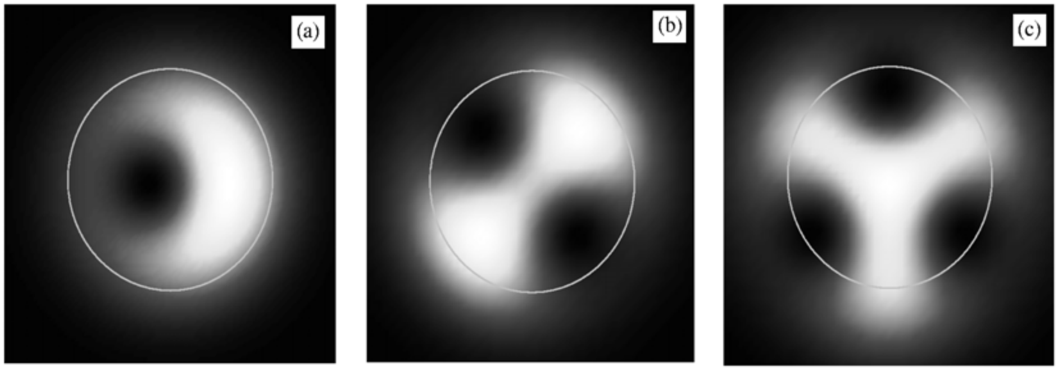

Fig. 2. Transverse patterns of light beams composed by the coaxial superposition of the Gaussian and LG_{01} (*1*), Gaussian and LG_{02} (*2*), and Gaussian and LG_{03} (*3*) modes. Circumferences show the traces of the detector aperture center in the course of the beams' rotation

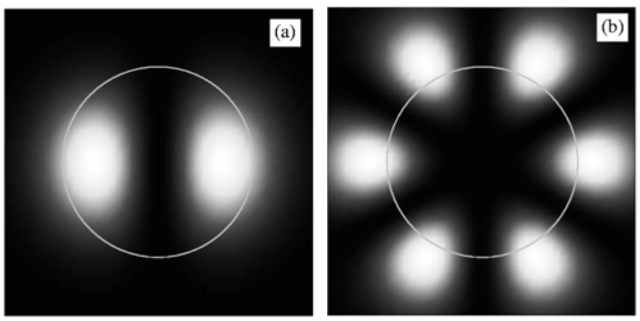

Fig. 3. The same as in Fig. 1 for coaxial superpositions of the $LG_{0,-1}$ and LG_{01} modes (*1*); $LG_{0,-3}$ and LG_{03} modes (*2*)

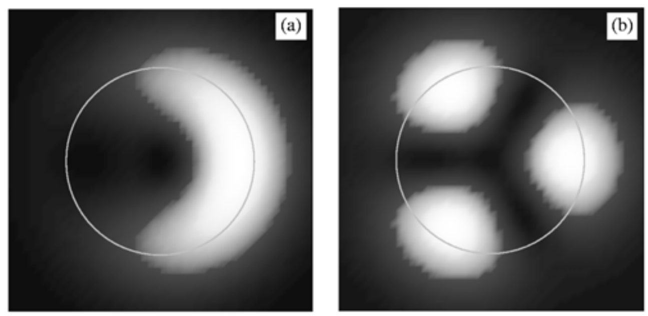

Fig. 4. The same as in Fig. 1 for coaxial superpositions of the LG_{01} and LG_{02} modes (*1*); $LG_{0,-1}$ and LG_{02} modes (*2*)

2. Numerical Examples and Discussion

To illustrate the outlined concept, we analyzed several calculated beam patterns (see Figs. 2—4) with usual image processing tools. This procedure allowed us to model, to a certain degree, beam distortions and noise appearing in real situations.

The first example was already employed — it is the simplest superposition (4) of Gaussian and vortex modes. The transverse profile of this beam presented in Fig. 2 contains a set of $|l|$ off-axis "holes" situated at the azimuthal positions [5]

$$\varphi_n = \frac{2n-1}{|l|}\pi - \frac{\delta}{|l|} \; (n = 1, 2, \ldots |l|), \tag{15}$$

so that the intensity performs $|l|$ oscillations on any circular round-trip about the beam axis [see Fig. 2 and Eq. (7)]. Now, let the light power be collected by an off-axis stationary detector with a small aperture. The beam rotating with angular velocity Ω, the aperture center moves along the circumference contours presented in Fig. 2,*a*—*c*. This causes modulations of the detector signal with frequency $|l|\Omega$ to which a distinct peak in the power spectrum corresponds (see curves *1* and *3* in Fig. 5). This geometric picture is remarkably clear and seems to have no need in employing the frequency shifts. Nevertheless, the signal modulation represents the modal beatings which prove the reality of RDE in this situation.

A combination of the LG_{0l} and $LG_{0,-l}$ modes with equal magnitudes represents another instructive example (Fig. 3). In this case,

$$u(r,\varphi) = F_0^{|l|}(r)[\exp(il\varphi) +$$

$$+\exp(-il\varphi)] \propto \left(\frac{r}{b}\right)^{|l|} \exp\left(-\frac{r^2}{2b^2}\right) \cos(il\varphi), \tag{16}$$

the resulting beam possesses no vortex, and its wavefront is not helical. The intensity distribution demonstrates $2|l|$ oscillations on a round trip. The geometric picture shows that rotation of this beam will cause modulation of the detector signal with frequency $2|l|\Omega$. Again, this can be interpreted in the RDE spirit: the LG_{0l} and $LG_{0,-l}$ components experience frequency shifts $+l\Omega$ and $-l\Omega$

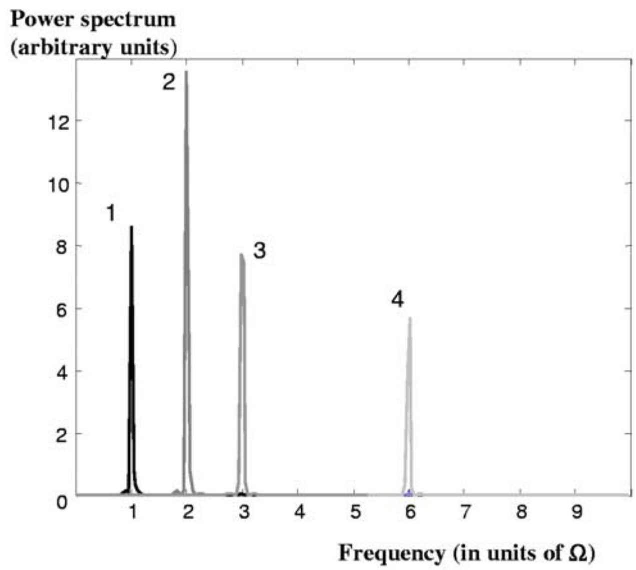

Fig. 5. Periodograms of the signals produced by composed beams rotating with respect to a small off-axis aperture: LG_{01}+Gaussian and LG_{01}+ LG_{02} (*1*); LG_{01}+ $LG_{0,-1}$ (*2*); LG_{03} + Gaussian and LG_{01}+ $LG_{0,-2}$ (*3*); LG_{03}+ $LG_{0,-3}$ (*4*)

and produce a beating signal with the difference frequency. Fig. 5 represents the corresponding peaks in the signal power spectrum for $|l| = 1$ and $|l| = 3$ (curves *2* and *4*).

Fig. 4 shows a situation where the combined beam is formed by vortex components with different non-zero $|l|$. For $l = 1$, $l' = 2$ (Fig. 4,*a*), the combined beam contains axial and off-axial vortices, but the resulting spot symmetry is similar to that of the Gaussian-LG_{01} combination (Fig. 2,*a*). Hence, the beam rotation leads to the detector signal modulation with frequency Ω (see also Fig. 5, curve *1*); one may recognize here the beatings between RDE-shifted beam components: $|l - l'|\Omega = \Omega$.

In an alternative situation, when the vortices being combined are of opposite signs, three additional off-axial vortices appear (Fig. 4,*b*). The whole pattern acquires the triangle symmetry due to which the beam rotation generates the signal modulation with the triple frequency (the power spectrum peak is presented by curve *3* in Fig. 5). Again, this exactly corresponds to the beating signal produced by the interference of the RDE-shifted $LG_{0,-1}$ and LG_{02} modes.

The examples presented give enough arguments supporting the idea of possibility to treat any image rotation in terms of the RDE language, but they are of rather special character far from most of the real images occurring in practice. Nevertheless, they clearly show

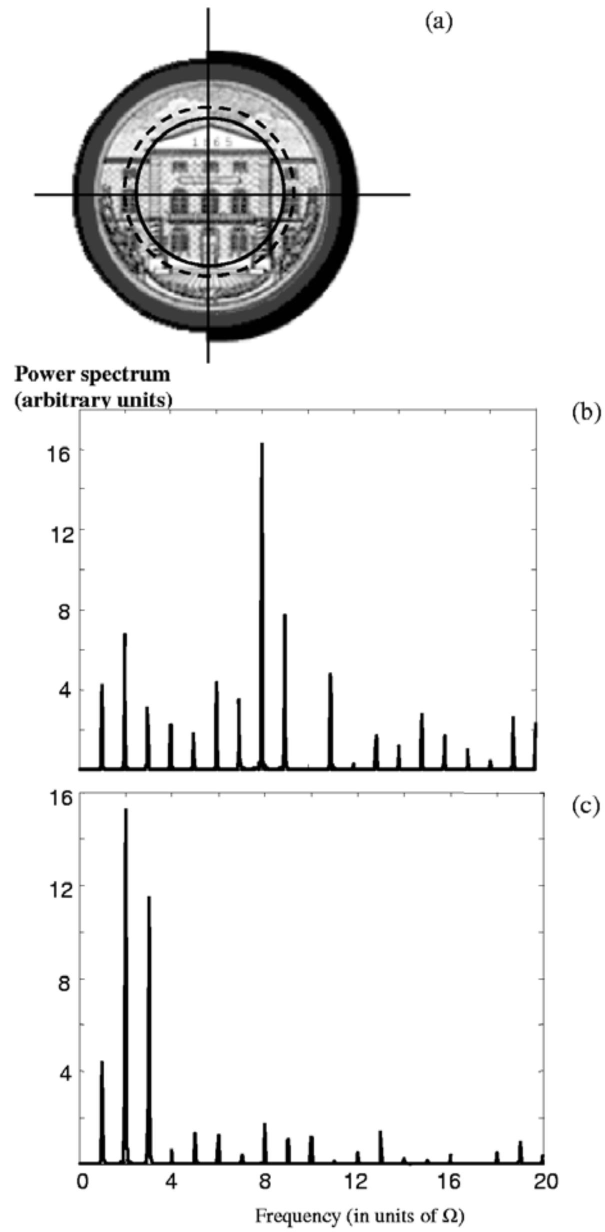

Fig. 6. *a* — the test image (Odesa University's logotype), *b*, *c* — power spectra of the beating signal arising upon the beam rotation. *b* — the detector aperture trace coincides with the solid contour in (*a*), *c* — dashed contour in (*a*)

some fundamental restrictions of the approaches to the LG spectrum retrieval based on the RDE techniques. Since the sideband positions depend only on $|l - l'|$, completely different rotating beams can demonstrate quite similar beating spectra (see Fig. 5). To handle such situations, the method of additional reference beam with known l and controllable frequency deviation may be helpful [13].

An analysis of more "natural" images exhibits further difficulties. For example, if the detector aperture moves along the contour denoted by the solid circumference in Fig. 6,*a*, the ac signal power spectrum has the form presented in Fig. 6,*b*. Its complicated structure is not extraordinary; only the outstanding role of the eighth harmonic seems rather surprising. It is likely to testify the considerable weight of the LG components with $l = \pm 4$ in the corresponding expansion (10), which cannot be seen in the image pattern directly. But, after a slight displacement of the aperture (dashed contour), the power spectrum changes radically and other peaks become predominant (Fig. 6,*c*). This is connected, most probably, with the role of LG_{pl} components with different $p \neq 0$ that cannot be resolved by the RDE approach.

Conclusion

We have proven and illustrated that the rotation of an optical image is completely equivalent to the RDE frequency shift of its LG components. In this view, modulations of the local light intensity, appearing in the course of the image rotation, are nothing else than the beating signal formed by the interference of those components. In principle, this beating signal can be used for the retrieval of the image LG spectrum, but, except for some trivial cases, this procedure is rather ambiguous. Nevertheless, the explicit manifestation of the LG spectra of usual images, provided by beating signals, seems impressive and promising.

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ОБЕРТАННЯ ДОВІЛЬНОГО ОПТИЧНОГО ЗОБРАЖЕННЯ ТА ОБЕРТАЛЬНИЙ ЕФЕКТ ДОППЛЕРА

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Р е з ю м е

Ми пропонуємо новий підхід для розуміння явища обертання зображення оптичною системою, що грунтується на обертальному ефекті Допплера. Довільне зображення може бути представлене у вигляді суперпозиції мод Лагерра—Гаусса, які набувають фазового зсуву завдяки обертальному ефекту Допплера. Результат інтерференції мод із фазовим зсувом в точності відтворює обертання зображення як цілого. Наведено деякі приклади, що ілюструють запропоновану інтерпретацію, а також обговорюються можливі прикладні аспекти для аналізу структури зображення.