LANDAU—LIFSHITS FLUCTUATING FORCES AND NONEQUILIBRIUM IN HYDRODYNAMICS

V.P. LESNIKOV

Odesa National Polytechnic University
(1, Shevchenko Ave., Odesa 65044, Ukraine)

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The theory of hydrodynamic fluctuations for steady states of a continuous medium has been developed on the basis of its local equilibrium. It is proved that the Landau—Lifshits fluctuating forces (LLFF) are described by the equilibrium fluctuation-dissipation theorem (FDT) and the nonequilibrium FDT has the same form as the equilibrium one, but the matrices which determine the one-time correlation functions and regression of fluctuations correspond to a nonequilibrium state under study. Different possible formulations of the FDT are shown to be equivalent, and its use in the simple cases of nonequilibrium states is considered.

Introduction

The theory of equilibrium hydrodynamic fluctuations was developed by Einstein and Onsager and completed by Landau and Lifshits [1], who found the formulas for the intensity of fluctuating forces (Langevin sources) for hydrodynamic fluctuations in an equilibrium unbounded liquid.

Then the intense studies were focused on nonequilibrium fluctuations in the presence of the gradients of macroscopic fields such as velocity, temperature, etc. Of significant interest is the fact that these nonequilibrium states may reveal instability.

The idea of using LLFF near the vicinity of a stability threshold belongs to Uhlenbeck [2], who expected an enhancement of hydrodynamic fluctuations similarly to a second-order phase transition in thermodynamics. Zaitsev and Shliomis [3] were the first who analyzed the Rayleigh—Benard instability with LLFF and confirmed the idea of the unbounded growth of fluctuations. Since the fluctuations, which are spoken about, are internal or natural, the most important conclusion that follows from Zaitsev and Shliomis's work consists in that any external factors cannot stop the growth of fluctuations when approaching the instability [4]. After that, the application of LLFF was continued in the subsequent works on nonequilibrium hydrodynamic states [5—9]. Although Lesnikov and Fisher [10], proceeding from a solution of the initial problem for the correlation functions of fluctuations, have received absolutely another results for the same system as in [3], no attention was paid to their results. In 1978, Keizer [11] postulated that fluctuations in a nonequilibrium hydrodynamic system must be described by LLFF with thermodynamic quantities replaced by their local values. This conclusion was also drawn, using microscopic kinetics, in works [12—14]. The direction in statistical physics based on local LLFF was called fluctuating hydrodynamics. It was developed intensively during many years [15—19]. However, a contradiction existed always within the frame of fluctuating hydrodynamics between the description of fluctuations in terms of local LLFF and the solution of the initial problem for fluctuations. Therefore, the last method was thought to be inadequate for the description of nonequilibrium states (see, e.g., [20]).

The active experimental study of nonequilibrium hydrodynamic fluctuations is carried out now (see, e.g., [21, 22]). All these works interpret the results of researches by proceeding from fluctuating hydrodynamics. But is it possible to conclude that they confirm fluctuating hydrodynamics if the origin and the place of LLFF in statistical physics are not clarified till now?

The present work is devoted to the answer to this question. Below, we show that LLFF are described by the equilibrium FDT and thus can be used for equilibrium states only. In the general case of nonequilibrium states, fluctuating forces differ from the Landau—Lifshits ones even if the latter are local.

1. The Fluctuation-Dissipation Theorem in Hydrodynamics

The first and basic point, from which we proceed, is that the hydrodynamic equations describe a gas or a liquid as a continuous medium. For a continuous medium, there is no problem with the definition of temperature as for a rarefied nonequilibrium gas or a nonequilibrium plasma: temperature and other physical quantities are set at every space point. Their definiteness means that the continuous medium is in a local equilibrium and, due to this fact, we know the local equilibrium distribution function of fluctuations x_i:

$$f(\vec{x}) \propto \exp(-\frac{1}{2}\beta_{ij}x_ix_j). \tag{1}$$

Thus, a nonequilibrium in the continuous medium is reduced to a local equilibrium. Indeed, the time of relaxation to a local equilibrium distribution for ordinary liquids is of order 10^{-13} s (see, e.g., [23]). Therefore, a local equilibrium in hydrodynamics always takes place except only such phenomena as turbulence or shock waves. The misunderstanding of the fact that hydrodynamics is valid in the approximation of local equilibrium has resulted in the appearance of many works which used kinetic methods in the study of nonequilibrium hydrodynamic systems. Such an approach was started by work [24]. Kinetic methods are very complicated and go beyond the limits of usual hydrodynamics, requiring, for example, the introduction of the greater number of coefficients as compared with those in the hydrodynamic equations. Results got within those methods are heavily controlled. For example, work [25] contains no answer to the question why works [3] and [10] gave different results for the same problem. But it was said repeatedly without any grounds that both works were incorrect.

Our second starting point is the Onsager hypothesis, according to which the regression of fluctuations in the continuous medium is determined by the linearized hydrodynamic equations near the state under consideration:

$$x_i = -\lambda_{ij}x_j. \tag{2}$$

Further, we will restrict ourselves by the analysis of the ordinary medium, for which the matrix λ is determined by the Navier-Stokes equations.

The problem of fluctuations in the appropriate system will be solved if the spectral density $(x_ix_j)_\omega$ of two-time correlation functions of fluctuations $\langle x_i(t)x_j(0)\rangle$ will be found. This can be done in several ways. According to the first way, which can be named as the Einstein approach or Einstein—Onsager one, one should solve the homogeneous system of equations (2) and perform the averaging of initial conditions with the distribution function (1):

$$(x_ix_j)_\omega = \int_{-\infty}^{\infty} \langle x_i(t)x_j(0)\rangle e^{i\omega t}dt. \tag{3}$$

In the second way (the Langevin approach), one should add fluctuating forces y_i to the right-hand side of (2) with the following correlation functions:

$$\langle y_i(\tau)y_j(0)\rangle = Q_{ij}\delta(\tau). \tag{4}$$

Then the spectral density of fluctuations can be found as a solution of the inhomogeneous system (2) averaged over (4):

$$(x_ix_j)_\omega = \Lambda_{im}(\omega)\Lambda_{jn}(-\omega)Q_{mn}, \tag{5}$$

where the matrix Λ is determined by the equation

$$\Lambda_{ik}(\omega)(-i\omega\delta_{kj} + \lambda_{kj}) = \delta_{ij}. \tag{6}$$

For the state of equilibrium, the matrix Q_{ij} was found by Landau and Lifshits.

From the mathematical point of view, a stochastic process with the initial distribution function (1) and regression equations (2) is the Ornstein—Uhlenbeck process. As has been already noted, using LLFF as Q_{ij} in nonequilibrium states yielded different results for fluctuations in the above approaches. In fact, using LLFF in nonequilibrium problems has no sense, because the sole formula for the intensity of fluctuating forces for the Ornstein—Uhlenbeck process is

$$Q_{ij} = \gamma_{ij} + \gamma_{ji}, \quad \gamma_{ij} = \lambda_{ik}\beta^{-1}_{kj}, \quad \beta^{-1}_{kj} = \langle x_kx_j\rangle. \tag{7}$$

There exist several proofs of formula (7), which can be found in textbooks. It is important that, in this case, the change of the time sign is not used in any way and consequently no restrictions are imposed on the symmetry of kinetic coefficients γ_{ij}. On a choice of the intensity of the fluctuation sources by formulas (7), any contradiction between the Einstein and Langevin approaches for any hydrodynamic state cannot be found.

In addition to formulas (3) and (5), the spectral density of fluctuations can also be expressed through the generalized susceptibility or the response function α. The corresponding formula was obtained by Callen and Welton and has the following form in the classical limit:

$$(x_i x_j)_\omega = \frac{iT}{\omega}(\alpha^*_{ji}(\omega) - \alpha_{ij}(\omega)). \tag{8}$$

Taking into account that the definition of susceptibility yields its connection with the matrices λ and β [26]:

$$\alpha_{ij}(\omega) = \frac{1}{T}(\beta_{ij} - i\omega\beta_{im}\lambda^{-1}_{mj})^{-1}, \tag{9}$$

it is easy to reduce (8) to (3) and (5) or, on the contrary, to get (8) from (3).

Thus, formulas (3), (5)—(7), and (8) are equivalent and give the possibility to calculate the spectral density of fluctuations in three different ways. Because formula (8) is named the Callen—Welton FDT, it is expedient to call (3) as the Einstein—Onsager FDT and (5)—(7) as the Langevin FDT. Let's emphasize that the Einstein—Onzager FDT is the most straight method to determine the spectral density of fluctuations.

All formulations of FDT in an identical manner require to know the matrices λ and β. Above we did not specify anywhere a state of the continuous medium under study. Therefore, the written down formulas express the equilibrium FDT, if the matrices λ and β are determined by an equilibrium state, and the same formulas express the nonequilibrium FDT if λ and β correspond to a nonequilibrum. From this position, we consider equilibrium states in the following section and further the most simple nonequilibrium states in the unbounded liquid with gradients of temperature and velocity.

2. Landau—Lifshits Fluctuating Forces

The existing method of derivation of LLFF [1] is based on the use of the expression for the entropy production and the stress tensor

$$\sigma_{ik} = \eta\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i} - \frac{2}{3}\delta_{ik}\operatorname{div}\vec{v}\right) + \zeta\delta_{ik}\operatorname{div}\vec{v} + s_{ik} \tag{10}$$

and the heat flow

$$\vec{q} = -\kappa\vec{\nabla}T + \vec{g}, \tag{11}$$

as Eqs. (2) with fluctuating forces s_{ik} and $\vec{g}$and makes them rather enigmatic. We note that their nature has been never discussed in the works on nonequilibrium fluctuations, where they were used as the sources of fluctuations, since work [2]. In (10) and (11), the customary notations are used.

Let us prove that the Landau—Lifshits formulas

$$\langle s_{ik}\left(\vec{r},t\right) s_{lm}\left(\vec{r}',t'\right)\rangle =$$

$$= 2T\left[\eta\left(\delta_{il}\delta_{mk} + \delta_{im}\delta_{kl}\right) + \left(\zeta - \frac{2}{3}\eta\right)\delta_{ik}\delta_{lm}\right]\times$$

$$\times\delta\left(\vec{r} - \vec{r}'\right)\delta\left(t - t'\right), \tag{12}$$

$$\langle g_i\left(\vec{r},t\right) g_j\left(\vec{r}',t'\right)\rangle =$$

$$= 2T^2\kappa\delta_{ij}\delta\left(\vec{r} - \vec{r}'\right)\delta\left(t - t'\right), \tag{13}$$

$$\langle s_{ik}\left(\vec{r},t\right) g_l\left(\vec{r}',t'\right)\rangle = 0 \tag{14}$$

represent the equilibrium FDT (7) and can be found in a more formal way.

The matrix λ is given by the Navier—Stokes hydrodynamic equations for the velocity $\vec{v}$, pressure p, entropy s, and temperature θ fluctuations in an equilibrium liquid, which we write down with fluctuating forces

$$\rho\frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial r_i} + \frac{\partial}{\partial r_k}\left[\eta\left(\frac{\partial v_i}{\partial r_k} + \frac{\partial v_k}{\partial r_i} - \right.\right.$$

$$\left.\left. -\frac{2}{3}\delta_{ik}\operatorname{div}\vec{v}\right) + \zeta\delta_{ik}\operatorname{div}\vec{v}\right] + \frac{\partial s_{ik}}{\partial r_k} \tag{15}$$

$$\rho T\frac{\partial s}{\partial t} = -\kappa\triangle\theta - \operatorname{div}\vec{g}, \tag{16}$$

The 1-time correlation functions and thus the matrix β^{-1} are also well known in equilibrium.

The fluctuating force in Eq. (16) is $-\frac{1}{\rho T}\mathrm{div}\vec{g}$. In accordance with FDT (7), we have

$$\frac{1}{\rho^2 T^2}\langle \mathrm{div}\vec{g}\left(\vec{r},t\right)\mathrm{div}\vec{g}\left(\vec{r'},t'\right)\rangle =$$

$$= \left[-\frac{\kappa}{\rho T}\langle\triangle\theta s'\rangle - \frac{\kappa}{\rho T}\langle\triangle'\theta' s\rangle\right]\delta\left(t-t'\right), \tag{17}$$

where the prime means that the corresponding value is determined for $\vec{r'}$. Using the thermodynamic formula

$$\theta = \frac{T}{c_p}s + \left(\frac{\partial T}{\partial p}\right)_s p \tag{18}$$

and the equilibrium relations

$$\langle ps'\rangle = 0, \quad \langle ss'\rangle = \frac{c_p}{\rho}\delta\left(\vec{r}-\vec{r'}\right), \tag{19}$$

we obtain

$$\langle \mathrm{div}\vec{g}\left(\vec{r},t\right)\mathrm{div}\prime\vec{g}\left(\vec{r'},t'\right)\rangle =$$

$$= \left[-\kappa T^2\triangle\delta\left(\vec{r}-\vec{r'}\right) - \kappa T^2\triangle'\delta\left(\vec{r}-\vec{r'}\right)\right]\times$$

$$\times\delta\left(t-t'\right). \tag{20}$$

Multiplying both sides of this equation by $r_i r'_j$ and integrating by parts, we find:

$$\int\int\langle g_i\left(\vec{r},t\right)g_j\left(\vec{r'},t'\right)\rangle d\vec{r}\,d\vec{r'} =$$

$$= \kappa T^2\int\int r'_j\frac{\partial}{\partial r_i}\delta\left(\vec{r}-\vec{r'}\right)d\vec{r}\,d\vec{r'}+$$

$$+\kappa T^2\int\int r_i\frac{\partial}{\partial r'_j}\delta\left(\vec{r}-\vec{r'}\right)d\vec{r}\,d\vec{r'}. \tag{21}$$

Taking into account that $\frac{\partial}{\partial r_i}\delta\left(\vec{r}-\vec{r'}\right) = -\frac{\partial}{\partial r'_i}\delta\left(\vec{r}-\vec{r'}\right)$ and integrating by parts once more, we arrive at (13).

Similarly, the fluctuating force in Eq. (15) is $\frac{1}{\rho}\frac{\partial s_{ik}}{\partial r_k}$. Then, using the equilibrium formulas

$$\langle v_i v'_j\rangle = \frac{T}{\rho}\delta_{ij}\delta\left(\vec{r}-\vec{r'}\right), \langle p\vec{v'}\rangle = 0, \tag{22}$$

and integrating twice as above, we obtain (12). In the same manner, keeping in mind the equilibrium formulas

$$\langle ps'\rangle = \langle\vec{v}s'\rangle = \langle\theta\vec{v'}\rangle = 0, \tag{23}$$

we derive (14). The above derivation reveals that smallskip LLFF correspond to the equilibrium FDT (7) and are valid for this reason only for equilibrium liquids.

3. Nonequilibrium Langevin FDT

Fluctuation perturbations in any hydrodynamic problem can be expanded in a series in eigenfunctions (spatial modes which are determined by solving the corresponding boundary-value problem). It is obvious that the matrix λ for these modes in the presence of the macroscopic fields of physical quantities will be distinct from the equilibrium one. In addition, only conjugated modes correlate with one another in equilibrium due to a spatial homogeneity. But, in nonequilibrium states, correlations among nonconjugated modes of the same quantity or different physical quantities come into existence. These correlations are caused by the dispersion of parameters of the local-equilibrium distribution function. Let us demonstrate this by a simple example.

Let us consider fluctuations of a certain scalar quantity n occurring in the unbounded liquid, whose temperature varies according to the linear law

$$T\left(\vec{r}\right) = T_0 + \vec{r}\vec{\nabla}T. \tag{24}$$

The local-equilibrium 1-time correlation function be

$$\langle n\left(\vec{r}\right)n\left(\vec{r'}\right)\rangle = \alpha_n T\left(\vec{r}\right)\delta\left(\vec{r}-\vec{r'}\right), \tag{25}$$

where α_n depends on the specific quantity n.

Expanding n in a Fourier series

$$n\left(\vec{r}\right) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} n_{\vec{k}}\exp i\vec{k}\vec{r}, \tag{26}$$

and using (25), we find

$$\langle n_{\vec{k}}n_{\vec{k'}}\rangle = \alpha_n T_0 V^{-1}\int_V d\vec{r}\left(1+\vec{q}\vec{r}\right)e^{-i\left(\vec{k}+\vec{k'}\right)\vec{r}}d\vec{r}, \tag{27}$$

where

$$\vec{q} = \vec{\nabla} T / T_0 \tag{28}$$

The quantity $\vec{q}\,\vec{r}$ is small in the hydrodynamic limit. Then, approximately, $\vec{q}\,\vec{r} \cong \sin \vec{q}\,\vec{r}$, and the integral in (27) can be evaluated. Then we get

$$\langle n_{\vec{k}} n_{\vec{k}'} \rangle = \alpha_n T_0 \triangle_{\vec{k},\vec{k}'} \tag{29}$$

where

$$\triangle_{\vec{k},\vec{k}'} = \delta_{\vec{k},-\vec{k}'} + \frac{i}{2}\delta_{\vec{k}+\vec{q},-\vec{k}'} - \frac{i}{2}\delta_{\vec{k}-\vec{q},-\vec{k}'} \tag{30}$$

So, there are the correlations between modes $\vec{k}$ and $-\vec{k}$, which take place in equilibrium and the nonequilibrium correlations between modes with $\vec{k}$ and $-\vec{k} \mp \vec{q}$. Expanding in modes increases the number of variables. Each variable is identified by two subscripts i and $\vec{k}$. The generalization of the Langevin FDT (7) for these variables is given by

$$\langle y_{i,\vec{k}}(\tau)\, y_{j,\vec{k}'}(0) \rangle = Q_{i,\vec{k};j,\vec{k}'} \delta(\tau), \tag{31}$$

where

$$Q_{i,\vec{k};j,\vec{k}'} = \lambda_{im,\vec{k}} \langle x_{m,\vec{k}} x_{j,\vec{k}'} \rangle + \lambda_{jm,\vec{k}'} \langle x_{m,\vec{k}'} x_{i,\vec{k}} \rangle, \tag{32}$$

Under nonequilibrium conditions, the intensities $Q_{i,\vec{k};j,\vec{k}'}$ are different from the intensities of LLFF, which is caused by two factors: 1-time correlation functions in nonequilibrium states are different from equilibrium ones, and the matrix λ is not such as in equilibrium. For the first time, the nonequilibrium Langevin FDT, which has removed the contradictions in the Rayleigh-Benard problem, was formulated in [27].

4. Temperature Gradient in the Unbounded Liquid

Nonequilibrium states with gradients of temperature and velocity (a Couette or shear flow) in the unbounded liquid are the simplest examples of nonequilibrium. Therefore, they were intensively studied by both the gas-kinetic or microscopic methods and the Langevin method with local LLFF as sources of fluctuations. The main result obtained by these methods is the determination of long-range corrections to the spectra of equilibrium fluctuations. In particular, in the case of the nonequilibrium in the presence of the temperature gradient, the spectrum of fluctuations of density turn out to be asymmetric with respect to the frequency. This fact was initially established by the first group of the methods [12—14]. However, because of their complexity and discrepancy in numerical results, using the Langevin approach with LLFF seemed to be more attractive. Below, these problems will be reconsidered on the basis of the theory advanced above .

Let's calculate the spatio-temporal Fourier transforms of two-time correlation functions of density fluctuations in the unbounded liquid with temperature gradient, which is most interesting from the point of view of molecular light scattering,

$$S_{n,n}\left(\vec{k},\omega\right) = \frac{1}{TV}\int_V d\vec{r} \int_V d\vec{r}' \int_{-T/2}^{T/2} dt \times$$

$$\times \int_{-T/2}^{T/2} dt' \langle n\left(\vec{r},t\right) n\left(\vec{r}',t'\right)\rangle \times$$

$$\times e^{i\omega(t-t') - i\vec{k}\left(\vec{r}-\vec{r}'\right)}, \tag{33}$$

where the duration of the experiment T is assumed to be long and the volume V to be big. The condition of stationarity of fluctuations allows us to represent (33) as

$$S_{n,n}\left(\vec{k},\omega\right) = \frac{1}{V}\int_V d\vec{r} \int_V d\vec{r}' (n_{\vec{r}} n_{\vec{r}'})_\omega e^{-i\vec{k}\left(\vec{r}-\vec{r}'\right)}. \tag{34}$$

The 1-time correlation function of fluctuations in the presence of the temperature gradient in the unbounded liquid is given by formulas (29),(30). Let us use the Einstein approach in the calculation of the spectra of fluctuations. Let the solution of the corresponding initial problem written down with the help of the unilateral Fourier transformation with respect to time and the spatial expansion (26) be

$$n_{\vec{k},\omega} = \Lambda^{n,n}_{\vec{k},\omega} n_{\vec{k}}. \tag{35}$$

Taking into consideration the condition of stationarity, relation (34) yields

$$S_{n,n}\left(\vec{k},\omega\right) = \frac{1}{V^2}\int_V d\vec{r} \int_V d\vec{r}' e^{-i\vec{k}\left(\vec{r}-\vec{r}'\right)} \times$$

$$\times\sum_{\vec{p},\vec{p}'}\left(\Lambda^{n,n}_{\vec{p},\omega}+\Lambda^{n,n}_{\vec{p}',-\omega}\right)\langle n_{\vec{p}}n_{\vec{p}'}\rangle e^{i\vec{p}\,\vec{r}}e^{i\vec{p}'\vec{r}'}. \quad (36)$$

Substituting (29) and carrying out the summation over p', we have

$$S_{n,n}\left(\vec{k},\omega\right)=\alpha_n T_0\sum_{\vec{p}}\left(\Lambda^{n,n}_{\vec{p},\omega}+\Lambda^{n,n}_{-\vec{p},-\omega}\right)\times$$

$$\times I\left(\vec{k},\vec{p},0\right)+\frac{i}{2}\left(\Lambda^{n,n}_{\vec{p},\omega}+\Lambda^{n,n}_{-\vec{p}-\vec{q},-\omega}\right)\times$$

$$\times I\left(\vec{k},\vec{p},\vec{q}\right)-\frac{i}{2}\left(\Lambda^{n,n}_{\vec{p},\omega}+\Lambda^{n,n}_{-\vec{p}+\vec{q},-\omega}\right)\times$$

$$\times I\left(\vec{k},\vec{p},-\vec{q}\right), \quad (37)$$

where $\vec{q}$ is defined by (28) and

$$I\left(\vec{k},\vec{p},\vec{q}\right)=\frac{1}{V^2}\int\limits_V d\vec{r}\times$$

$$\times\int d\vec{r}'e^{-i\vec{k}\left(\vec{r}-\vec{r}'\right)}e^{i\vec{p}\left(\vec{r}-\vec{r}'\right)}e^{-i\vec{q}\,\vec{r}'}. \quad (38)$$

Introducing the new variables by

$$\vec{x}=\vec{r}-\vec{r}',\vec{R}=\left(\vec{r}+\vec{r}'\right)/2 \quad (39)$$

and taking into account that, for small $\vec{q}$ $\int\limits_V\exp\left(i\vec{q}\vec{R}\right)d\vec{R}\cong V$, we find

$$I\left(\vec{k},\vec{p},\pm\vec{q}\right)=\delta_{\vec{k}\mp\frac{\vec{q}}{2},\vec{p}}. \quad (40)$$

The summation over $\vec{p}$ yields:

$$S_{n,n}\left(\vec{k},\omega\right)=\alpha_n T_0\Bigg[2\mathrm{Re}\Lambda^{n,n}_{\vec{k},\omega}+$$

$$+\frac{i}{2}\left(\Lambda^{n,n}_{\vec{k}-\frac{\vec{q}}{2},\omega}+\Lambda^{n,n}_{-\vec{k}-\frac{\vec{q}}{2},-\omega}\right)-$$

$$-\frac{i}{2}\left(\Lambda^{n,n}_{\vec{k}+\frac{\vec{q}}{2},\omega}+\Lambda^{n,n}_{-\vec{k}+\frac{\vec{q}}{2},-\omega}\right)\Bigg]. \quad (41)$$

Retaining the terms linear in $\vec{q}=\vec{\nabla}T/T_0$, we finally find:

$$S_{n,n}\left(\vec{k},\omega\right)=\alpha_n T_0\left(2\mathrm{Re}+\vec{q}\vec{\nabla}_{\vec{k}}\mathrm{Im}\right)\Lambda^{n,n}_{\vec{k},\omega}. \quad (42)$$

The second term in the above expression is a long-range correction to the equilibrium result. The correction is caused by a spatial dispersion of the temperature field, and thus it has statistical origin. The function $\Lambda^{n,n}_{\vec{k},\omega}$ in this correction term must be calculated in the zero-order in $\vec{\nabla}T$. That's why, it is the equilibrium one. At the same time, the corresponding function in the first term must be calculated in the first order in $\vec{\nabla}T$. Therefore, the first term in (42) contains an extra term with the temperature gradient caused by the difference of the hydrodynamic equations being used from the equilibrium equations.

Below, we consider the example of a process, for which the temperature gradient causes no changes in the dynamical equations.

The temperature gradient is not present in the dynamical equations describing a liquid, in which the pressure depends only on the density $p=p(\rho)$, and the propagation of sound fluctuations is isothermal. Such a situation takes place for water at a temperature of $4°C$. The appropriate problem with local LLFF was examined in [15, 16]. In that case, fluctuations of both the density ρ and the velocity potential φ obey the system of equations

$$\frac{\partial\rho}{\partial t}+\rho_0\triangle\varphi=0,\quad \frac{\partial\varphi}{\partial t}-D\triangle\varphi+\frac{c^2}{\rho_0}\rho=0, \quad (43)$$

where $D=\frac{4}{3}\nu+\xi$; $\nu=\eta/\rho_0$ and $\xi=\varsigma/\rho_0$ represent the shear and bulk kinematic viscosity coefficients, respectively, and c is the isothermal speed of sound. It follows from (43) that

$$\Lambda^{\rho,\rho}_{\vec{k},\omega}=\frac{-i\omega+Dk^2}{-\omega^2+c^2k^2-i\omega Dk^2} \quad (44)$$

Formula (42) gives

$$S_{\rho,\rho}\left(\vec{k},\omega\right)=2T_0\rho_0\left\{\frac{Dk^4}{\left(\omega^2-c^2k^2\right)^2+\omega^2D^2k^4}+\right.$$

$$+\frac{\omega\vec{k}\vec{q}\left[\left(\omega^2-c^2k^2\right)^2-\omega^2D^2k^4\right]}{\left[\left(\omega^2-c^2k^2\right)^2+\omega^2D^2k^4\right]^2}\Bigg\}, \quad (45)$$

where the relation $\alpha_\rho=\rho_0/c^2$ was used.

Let us find the fluctuating forces for the above example. Making the Fourier transformation of (43) with respect to the spatial variables, we obtain the following expression for the matrix λ :

$$\lambda_{ij,\vec{k}}=\begin{pmatrix}0 & -\rho_0k^2\\ c^2/\rho_0 & Dk^2\end{pmatrix} \quad (46)$$

The 1-time correlation functions of the fluctuations of ρ and $\vec{v}$ can be found in a way similar to the finding of (29):

$$\left\langle\rho_{\vec{k}}\rho_{\vec{k}'}\right\rangle=\frac{T_0\rho_0}{c^2}\Delta_{\vec{k},\vec{k}'},$$

$$\left\langle v_{\alpha,\vec{k}}v_{\beta,\vec{k}'}\right\rangle=\frac{T_0}{\rho_0}\delta_{\alpha\beta}\Delta_{\vec{k},\vec{k}'}. \quad (47)$$

For the longitudinal part of velocity from the second equation in (47) we have

$$\left\langle\varphi_{\vec{k}}\varphi_{\vec{k}'}\right\rangle=-\frac{\vec{k}\vec{k}'}{k^2k'^2}\frac{T_0}{\rho_0}\Delta_{\vec{k},\vec{k}'}. \quad (48)$$

According to the nonequilibrium FDT (32), the fluctuating forces are given by

$$\langle y_{i,\vec{k}}(\tau)\,y_{j,\vec{k}'}(0)\rangle=$$

$$=\begin{pmatrix}0 & 1+\frac{\vec{k}\vec{k}'}{k'^2}\\ 1+\frac{\vec{k}\vec{k}'}{k^2} & -\frac{D\vec{k}\vec{k}'}{\rho_0k^{*2}}\end{pmatrix}T_0\Delta_{\vec{k},\vec{k}'}\delta(\tau), \quad (49)$$

where

$$k^{*-2}=k^{-2}+k'^{-2}. \quad (50)$$

It follows from (43) that

$$\rho_{\vec{k},\omega}=\frac{\left(-i\omega+Dk^2\right)y_{1,\vec{k},\omega}+\rho_0k^2y_{2,\vec{k},\omega}}{-\omega^2+c^2k^2-i\omega Dk^2}. \quad (51)$$

The substitution of (51) in (33) results in

$$S_{\rho,\rho}\left(\vec{k},\omega\right)=\frac{1}{V^2}\int_V d\vec{r}\int_V d\vec{r}'e^{-i\vec{k}\left(\vec{r}-\vec{r}'\right)}\times$$

$$\times\sum_{\vec{p},\vec{p}'}\frac{\langle\left[\left(-i\omega+Dp^2\right)y_{1,\vec{p},\omega}+\rho_0p^2y_{2,\vec{p},\omega}\right]}{\left(-\omega^2+c^2p^2-i\omega Dp^2\right)}\times$$

$$\times\frac{\left[\left(i\omega+Dp'^2\right)y_{1,\vec{p}',-\omega}+\rho_0p'^2y_{2,\vec{p}',-\omega}\right]\rangle}{\left(-\omega^2+c^2p^2+i\omega Dp'^2\right)}\times$$

$$\times e^{i\vec{p}\vec{r}}e^{i\vec{p}'\vec{r}'}. \quad (52)$$

Averaging over forces (49), integrating, and then carrying out the summation, we obtain (45). Thus, the nonequilibrium Langevin FDT gives the same result as the Einstein approach.

The local LLFF correspond to only the nonzero-term with D in matrix (49). If we restrict ourselves only to it like in [15, 16], the expression in the square brackets in the numerator of the second term in (45) will contain $-2\omega^2D^2k^4$, rather than $\left(\omega^2-c^2k^2\right)^2-\omega^2D^2k^4$. Therefore, the results obtained within the Einstein and Langevin approaches will contradict to one another in principle.

As seen from (45), the spatial dispersion of temperature and the nonequilibrium correlations caused by it contribute to the spectrum of fluctuations asymmetrically in frequency. For instance, the height of one of the Mandelstam—Brillouin satellites increases comparing to the equilibrium result, whereas the height of the other one decreases by a factor $\left(1\pm\frac{c\vec{k}\vec{q}}{Dk^3}\right)$. We note that the local LLFF give the factor $\left(1\pm\frac{2c\vec{k}\vec{q}}{Dk^3}\right)$, i.e. a twice larger correction at the frequency of maxima.

5. Shear Flow

The stationary profile of velocity in a shear flow is

$$\vec{U}=\vec{x}^{\circ}\Gamma y, \quad (53)$$

and the temperature and density of a liquid are independent of spatial coordinates.

Let's consider fluctuations for the same liquid as above. The corresponding problem with Landau—Lifshits sources was considered in [15]. Again, we begin to analyze this problem by using the Einstein method.

The local-equilibrium distribution function is determined by the kinetic energy and the potential energy depending on the density fluctuations. Let's assume that the velocity is $\vec{U} + \vec{v}$ and the density is $\rho_0 + \rho$. By retaining the terms linear in Γ and keeping quadratic fluctuations, we obtain

$$w \propto \exp\left[-\frac{1}{2T_0}\int\left(\rho_0\vec{v}^{\,2} + 2\Gamma y\rho\vec{v}\vec{x}^{\circ} + \frac{c^2}{\rho_0}\rho^2\right)d\vec{r}\right] \quad (54)$$

We denote

$$\Gamma y \equiv \frac{\Gamma}{q}qy = \epsilon\sin\vec{q}\,\vec{r}, \quad (55)$$

where $\epsilon = \Gamma/q, \vec{q} = \vec{y}^{\circ}q$. The variable $\vec{q}$ plays an auxiliary role and will not appear in the final result.

Formula (54) yields the following distribution for the Fourier components (26) of fluctuations:

$$w \propto \exp\left\{-\frac{1}{2T_0}\sum_{\vec{k}}\left[\rho_0\vec{v}_{\vec{k}}\vec{v}_{-\vec{k}} - i\epsilon\rho_{\vec{k}}\times\right.\right.$$

$$\left.\left.\times\left(\vec{v}_{-\vec{k}-\vec{q}} - \vec{v}_{-\vec{k}+\vec{q}}\right)\vec{x}^{\circ} + \frac{c^2}{\rho_0}\rho_{\vec{k}}\rho_{-\vec{k}}\right]\right\}. \quad (56)$$

It follows from this that

$$\left\langle\rho_{\vec{k}}\rho_{\vec{k}'}\right\rangle = \frac{T_0\rho_0}{c^2}\delta_{\vec{k},-\vec{k}'},$$

$$\left\langle v_{\alpha,\vec{k}}v_{\beta,\vec{k}'}\right\rangle = \frac{T_0}{\rho_0}\delta_{\alpha\beta}\delta_{\vec{k},-\vec{k}'},$$

$$\left\langle\rho_{\vec{k}}\vec{v}_{\vec{k}'}\right\rangle = \mp\vec{x}^{\circ}\frac{i\epsilon T_0}{2c^2}\delta_{\vec{k}\pm\vec{q},-\vec{k}'} \quad (57)$$

and

$$\left\langle\varphi_{\vec{k}}\varphi_{\vec{k}'}\right\rangle = \frac{T_0}{\rho_0 k^2}\delta_{\vec{k},-\vec{k}'},$$

$$\left\langle\rho_{\vec{k}}\varphi_{\vec{k}'}\right\rangle = \mp\frac{\epsilon k'_x T_0}{2c^2k'^2}\delta_{\vec{k}\pm\vec{q},-\vec{k}'}, \quad (58)$$

if we separate the longitudinal and transverse parts of velocity fluctuations.

Therefore, the shear flow causes the appearence of correlation between the fluctuations of density and velocity. The same time, the density-density and the velocity-velocity correlations remain to be equilibrium ones.

Then let's write down a system of equations for density fluctuations ρ and fluctuations of the velocity potential φ:

$$\frac{\partial\rho}{\partial t} + \Gamma y\frac{\partial\rho}{\partial x} + \rho_0\Delta\varphi = 0$$

$$\frac{\partial\varphi}{\partial t} + \Gamma y\frac{\partial\varphi}{\partial x} - D\Delta\varphi + \frac{c^2}{\rho_0}\rho = 0. \quad (59)$$

Expanding the fluctuations in a series in spatial coordinates (26) and retaining the terms linear in ϵ, we obtain:

$$\frac{\partial\rho_{\vec{k}}}{\partial t} - \rho_0k^2\varphi_{\vec{k}} = \frac{1}{2}\epsilon k_x\left(\rho_{\vec{k}+\vec{q}} - \rho_{\vec{k}-\vec{q}}\right)$$

$$\frac{\partial\varphi_{\vec{k}}}{\partial t} + Dk^2\varphi_{\vec{k}} + \frac{c^2}{\rho_0}\rho_{\vec{k}} = \frac{1}{2}\epsilon k_x\left(\varphi_{\vec{k}+\vec{q}} - \varphi_{\vec{k}-\vec{q}}\right). \quad (60)$$

In order to find the function $S_{\rho,\rho}\left(\vec{k},\omega\right)$ by the method of perturbations in ϵ, we make the unilateral Fourier transform with respect to time and write down the solution of the initial problem for $\rho_{\vec{k},\omega}$:

$$\rho_{\vec{k},\omega} = \sum_{\mu}\Lambda^{\rho,\mu}_{\vec{k},\omega}\left[\mu_{\vec{k}} + \frac{1}{2}\epsilon k_x\left(\mu_{\vec{k}+\vec{q},\omega} - \mu_{\vec{k}-\vec{q},\omega}\right)\right]. \quad (61)$$

Here, $\mu = \rho, \varphi$, and the quantities $\Lambda^{\nu,\mu}_{\vec{k},\omega}$ are given by the left part of (60), i.e. they correspond to the equilibrium state without a shear flow:

$$\Lambda^{\nu,\mu}_{\vec{k},\omega} = \begin{pmatrix}\Lambda^{\rho,\rho}_{\vec{k},\omega} & \Lambda^{\rho,\varphi}_{\vec{k},\omega}\\ \Lambda^{\varphi,\rho}_{\vec{k},\omega} & \Lambda^{\varphi,\varphi}_{\vec{k},\omega}\end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix}-i\omega + Dk^2 & \rho_0k^2\\ -c^2/\rho_0 & -i\omega\end{pmatrix},$$

$$\Delta = c^2k^2 - \omega^2 - i\omega Dk^2. \quad (62)$$

Estimations (57) and (58) reveal that, within the approach linear in ϵ, only the terms with $\rho_{\vec{k}}, \varphi_{\vec{k}}$ and also those parts of $\mu_{\vec{k}\pm\vec{q},\omega}$ which depend on the initial value of the density fluctuations contribute to $S_{\rho,\rho}\left(\vec{k},\omega\right)$. Therefore, we have:

$$\rho_{\vec{k},\omega}=\sum_{\mu}\Lambda^{\rho,\mu}_{\vec{k},\omega}\left[\mu_{\vec{k}}+\frac{1}{2}\epsilon k_x\times\right.$$

$$\left.\times\left(\Lambda^{\mu,\rho}_{\vec{k}+\vec{q},\omega}\rho_{\vec{k}+\vec{q}}-\Lambda^{\mu,\rho}_{\vec{k}-\vec{q},\omega}\rho_{\vec{k}-\vec{q}}\right)\right]. \tag{63}$$

Making manipulations similar to those used to derive (42) from (36), we obtain:

$$S_{\rho,\rho}\left(\vec{k},\omega\right)=\frac{\rho_0 T_0}{c^2}\left\{2Re\Lambda^{\rho,\rho}_{\vec{k},\omega}+\frac{1}{2}\epsilon k_x\times\right.$$

$$\times\left[\Lambda^{\rho,\varphi}_{\vec{k}-\frac{\vec{q}}{2},\omega}\Lambda^{\varphi,\rho}_{\vec{k}+\frac{\vec{q}}{2},\omega}-\Lambda^{\rho,\varphi}_{\vec{k}+\frac{\vec{q}}{2},\omega}\Lambda^{\varphi,\rho}_{\vec{k}-\frac{\vec{q}}{2},\omega}-\right.$$

$$-\Lambda^{\rho,\varphi}_{-\vec{k}-\frac{\vec{q}}{2},-\omega}\Lambda^{\varphi,\rho}_{-\vec{k}+\frac{\vec{q}}{2},-\omega}+\Lambda^{\rho,\varphi}_{-\vec{k}+\frac{\vec{q}}{2},-\omega}\Lambda^{\varphi,\rho-}_{-\vec{k}-\frac{\vec{q}}{2},-\omega}$$

$$-\frac{\Lambda^{\rho,\varphi}_{\vec{k}-\frac{\vec{q}}{2},\omega}}{\rho_0\left(\vec{k}-\frac{\vec{q}}{2}\right)^2}+\frac{\Lambda^{\rho,\varphi}_{\vec{k}+\frac{\vec{q}}{2},\omega}}{\rho_0\left(\vec{k}+\frac{\vec{q}}{2}\right)^2}-$$

$$\left.\left.-\frac{\Lambda^{\rho,\varphi}_{-\vec{k}-\frac{\vec{q}}{2},-\omega}}{\rho_0\left(-\vec{k}-\frac{\vec{q}}{2}\right)^2}+\frac{\Lambda^{\rho,\varphi}_{-\vec{k}+\frac{\vec{q}}{2},-\omega}}{\rho_0\left(-\vec{k}+\frac{\vec{q}}{2}\right)^2}\right]\right\}. \tag{64}$$

Substituting the explicit expressions for $\Lambda^{\nu,\mu}_{\vec{k},\omega}$ into (64) and taking into account the terms linear in $\vec{q}$, we finally obtain:

$$S_{\rho,\rho}\left(\vec{k},\omega\right)=\frac{2\rho_0 T_0 D k^4}{\left(\omega^2-c^2k^2\right)^2+\omega^2D^2k^4}\times$$

$$\times\left[1-\frac{2D\Gamma k_x k_y\omega^2\left(c^2k^2-\omega^2\right)}{c^2k^2\left(\left(\omega^2-c^2k^2\right)^2+\omega^2D^2k^4\right)}\right]. \tag{65}$$

This result essentially differs from that received in [15] by the presence of the factor $\left(c^2k^2-\omega^2\right)/c^2k^2$ in the nonequilibrium long-range correction. Thus, according to our calculations, the correction does not change the height of lines. The correction is symmetric with respect to the frequency and depends on the sign of $\Gamma k_x k_y$. This correction results from the fact that the 1-time correlation functions of fluctuations and the evolutionary hydrodynamic equations are nonequilibrium.

It is possible to reach the same result (65) if we use the nonequilibrium Langevin FDT (32). The matrix λ for the fluctuation variables is defined by (60), and the 1-time correlation functions by the first formula in (57) and formulas (58). Then, for the appropriate forces, we have

$$\left\langle y_{i,\vec{k}}(\tau)\,y_{j,\vec{k}'}(0)\right\rangle=\begin{pmatrix}0 & -\frac{\varepsilon k'_x}{2c^2}\left(\delta_{\vec{k}+\vec{q},-\vec{k}'}-\delta_{\vec{k}-\vec{q},-\vec{k}'}\right)\\ -\frac{\varepsilon k_x}{2c^2}\left(\delta_{\vec{k}+\vec{q},-\vec{k}'}-\delta_{\vec{k}-\vec{q},-\vec{k}'}\right) & \frac{2}{\rho_0}\delta_{\vec{k},-\vec{k}'}\end{pmatrix}DT_0\delta(\tau) \tag{66}$$

Like in the previous temperature gradient example, the nonequilibrium FDT gives rise to the forces which differ by nondiagonal elements from local Landau—Lifshits fluctuating forces which contain only one nonzero-term on the crossing of the second row and the second column.

Conclusion

In conclusion, we note that, for a continuous medium which is described by the Navier-Stokes equations, almost always the nonequilibrium states can be approximated as local-equilibrium ones. The assumption of local equilibrium and the Onsager regressive hypothesis allow one to construct a consistent theory of hydrodynamic fluctuations. Different formulations of FDT have been stated, and all they are proved to be equivalent. The LLFF correspond to the equilibrium FDT for fluctuation sources. Generally, for nonequilibrium steady states, the fluctuating forces are different from the Landau—Lifshits fluctuating forces. The reason for the appearance of long-range corrections to equilibrium spectra is the correlations between fluctuations which did not take place in equilibrium, and also the difference of the evolutionary matrix from an equilibrium one. The use of local LLFF for nonequilibrium states takes into account the spatial dispersion of macroscopic fields only partially, leaving the contradiction between various possible ways of the description of hydrodynamic fluctuations. The mistake of Zaitsev and Shliomis's work and of subsequent works on fluctuating hydrodynamics consists in the application of the equilibrium Langevin FDT to nonequilibrium hydrodynamic states, while one should use the above-formulated nonequilibrium Langevin FDT.

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СТОРОННІ СИЛИ ЛАНДАУ—ЛІФШИЦЯ ТА НЕРІВНОВАГА В ГІДРОДИНАМІЦІ

В.П. Леснiков

Р е з ю м е

На основі уявлень про суцільне середовище та його локальну рівновагу розвинуто теорію гідродинамічних флуктуацій. Доведено, що сторонні сили Ландау—Ліфшиця є рівноважною флуктуаційно-дисипативною теоремою (ФДТ), а нерівноважна ФДТ в гідродинаміці має таку саму форму, як і рівноважна, і відрізняється від останньої виразами матриць, що визначають одночасові кореляційні функції та регресію флуктуацій. Показано еквівалентність різних форм запису ФДТ та розглянуто приклади її застосування в простих випадках нерівноваги.