NONLINEAR ALFVEN WAVES AND SOLITONS IN COLD PLASMA

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A finite-zone sector for the derivative nonlinear Schrodinger equation, which describes nonlinear small-amplitude Alfven waves in the long-wave approximation, is investigated. The formulas for periodic one-phase and one-soliton solutions and the general formula for an n-soliton solution envelope are obtained.

Introduction

Nonlinear Alfven waves and soliton effects in magnetized plasma are studied for several decades till now [1–10]. Recently, the active interest in those problems has been stimulated by new observations of oscillations of the high-amplitude magnetic fields in the solar wind and the interstellar medium [11] and by investigations of the nonlinear processes and large-scale turbulence in dusty plasma [12, 13].

In the long-wave approximation, nonlinear transverse oscillations of a magnetic field propagating along the force lines are described by a derivative nonlinear Schrodinger (DNLS) equation with a cube nonlinearity. This equation was obtained for the first time in paper [1] within a microscopic kinetic model of "mean field" (the Maxwell—Vlasov system of equations) with the following averaging of fast processes by the Krylov—Bogolyubov method. In papers [3, 4], the DNLS equation was obtained from a complete system of equations of the two-liquid magnetohydrodynamics at zero temperature by the methods of reductive perturbation theory [2].

In dimensionless units, the DNLS equation takes the form [1, 3, 4, 8]

$$i\frac{\partial \mathbf{b}}{\partial t} = \pm\mu\frac{\partial^2 \mathbf{b}}{\partial x^2} - i\frac{\partial}{\partial x}\left\{\mathbf{b} + \frac{1}{4(1-\beta)}\mathbf{b}^2\mathbf{b}^*\right\}, \qquad (1)$$

where $\mathbf{b}(t,x) = \frac{1}{B_0}(B_2 + iB_3)$, B_2, B_3 — magnetic field transverse components, $\mathbf{B}_0 = B_0\mathbf{e}_1$ — magnetic field in the equilibrium state, $\beta = 4\pi p_0/B_0^2$ — parameter characterizing the ratio of equilibrium thermodynamic pressure p_0 and magnetic pressure due to the Lorentz force; $\mu = \frac{1}{2}\left[1/R_i - 1/R_e\right]$, R_i, R_e — ion and electron Reynolds numbers (the signs "±" of the coefficient μ correspond to different polarizations of Alfven waves). With the units chosen, the Alfven velocity $V_A = \frac{B_0}{\sqrt{4\pi n_0(m_i+m_e)}}$ is equal to unity.

Let us note that, in the microscopic study of Eq. (1) with regard for kinetics, like it is done in [1], the coefficient of the nonlinearity depends on temperature and a magnetic field in a more complicated way and coincides with the expression $1/4(1-\beta)$ only at small values of the parameter β.

Though the model equation (1) for nonlinear Alfven waves was proposed rather long ago and was found to be completely integrable in the sense of the inverse scattering problem [5], there is no new classification of all its solutions with the physical content till now. Unfortunately, such a situation is typical of many equations involved in the general scheme of solving the inverse scattering problem. As to Eq. (1), many partial solutions for it were obtained by various methods [4–10]. Among the works sited, the most "advanced" from the viewpoint of mathematics are works [7, 8], where the general formulas for complex finite-phase solutions in terms of a multivariable theta-function were obtained. Though, the specification of those formulas according to the initial conditions chosen and the establishment of conditions for them to be real within the standard methods of algebraic geometry are extremely difficult problems.

In the present paper, we develop the new approach to the analysis of Eq. (1) that was initiated in [15–17] and is based on revealing the orbit Hamiltonian structure of a finite-phase sector of the phase space of Eq. (1). Within this method, one can get the explicit formulas for the N-soliton solutions (a hyperbolic soliton for odd

N) and describe the periodic one-phase solutions of Eq. (1) quite exactly for $\beta < 1$ (the case $\beta > 1$ is within the general scheme as well). We hope the method we apply is to be efficient also in the analysis of multiphase periodic solutions which are rather important in the investigation of the kinetics of Alfven waves.

1. Orbit-scheme for Interpretation of the Finite-zone Sector of the DNLS Equation

As noted in Introduction, the complete integrability of Eq. (1) within the method of inverse scattering problem was found in [5]. Integrability of the dynamical (Hamilton) system (1) results in the existence of finite-dimensional invariant subspaces in its phase space corresponding to the fast decaying (soliton) or special quasi-periodic (periodic) initial conditions. By the terminology commonly used in the inverse problem, such subspaces are called the finite-zone ones, and the respective solutions — finite-zone or finite-phase solutions [14].

In works [15, 16], it was shown that the finite-zone subspaces of many integrable equations can be realized as the orbits of co-adjoint action of the algebra of formal Laurent series with coefficients in classical simple Lie algebras. In [17], this construction was applied to the DNLS equation. Let us recall its main elements.

Let $\mathfrak{g} \simeq \mathrm{sl}(2)$ be the Lie algebra of complex 2×2 matrices with zero trace. In it, we fix the standard basis

$$X_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which satisfies the following commutation relations:

$$[X_0, X_1] = X_1, \quad [X_0, X_2] = -X_2, \quad [X_1, X_2] = 2X_0.$$

Consider the algebra $\mathrm{sl}(2) \otimes \mathcal{P}(\lambda, \lambda^{-1})$, $\lambda \in \mathbb{C}$, of Laurent polynomials with coefficients in the simple complex Lie algebra sl(2), and, within it, a subalgebra $\tilde{\mathfrak{g}}$ of matrix-valued polynomials, for which the coefficients of the parameter λ to even and odd orders are, respectively, diagonal and antidiagonal matrices:

$$\tilde{\mathfrak{g}} = \left\{ A(\lambda) = \sum_l \begin{pmatrix} \alpha_{2l} & 0 \\ 0 & -\alpha_{2l} \end{pmatrix} \lambda^{2l} + \right.$$

$$\left. + \sum_l \begin{pmatrix} 0 & \beta_{2l+1} \\ \gamma_{2l+1} & 0 \end{pmatrix} \lambda^{2l+1} \right\}.$$

Obviousely, $\tilde{\mathfrak{g}}$ is closed with respect to the commutation operation for the matrix-valued Laurent series and thus is a Lie subalgebra in $\mathrm{sl}(2) \otimes \mathcal{P}(\lambda, \lambda^{-1})$.

Let us divide the $\tilde{\mathfrak{g}}$ into a sum of two subalgebras: $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_+ + \tilde{\mathfrak{g}}_-$,

$$\tilde{\mathfrak{g}}_+ = \left\{ \sum_{i \ge 0} A_i \lambda^i \right\}, \quad \tilde{\mathfrak{g}}_- = \left\{ \sum_{i \le 0} B_i \lambda^{+i} \right\}.$$

In the subalgebra $\tilde{\mathfrak{g}}_+$, we choose the finite-dimensional linear subspace

$$M^N = \left\{ \hat{\mu}(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}; \right.$$

$$\alpha(\lambda) = \sum_{l=0}^{N+1} \alpha_{2l} \lambda^{2l}, \quad \beta(\lambda) = \sum_{l=0}^{N} \beta_{2l+1} \lambda^{2l+1},$$

$$\left. \gamma(\lambda) = \sum_{l=0}^{N} \gamma_{2l+1} \lambda^{2l+1} \right\}. \qquad (2)$$

The idea of the method of orbits in the theory of nonlinear integrable equations consists in the embedding of the finite-zone phase space of an integrable system to the linear space M^N in the form of an orbit of co-adjoint action of the subalgebra $\tilde{\mathfrak{g}}_+$ (or $\tilde{\mathfrak{g}}_-$). On the orbit, the space and time variables x and t are separated in the sense that the nonlinear partial differential equation is represented as the condition of compatibility of two integrable Hamilton systems: trajectories of one of them and those of the other are parametrized, respectively, by the variables x and t. To realize this structure, the manifold M^N should be endowed by the Poisson structure (the Poisson bracket is to be defined). As is often observed for integrable systems, there exist several mutually consistent Lie — Poisson brackets in the M^N space. We are interested in two of them.

Let f_1 and f_2 be smooth (differentiable) functions on the manifold M^N. Their Poisson bracket can be defined in two ways:

$$\{f_1, f_2\}_1 = \langle \hat{\mu}(\lambda), [\nabla^{(1)} f_1, \nabla^{(1)} f_2] \rangle_1, \qquad (3)$$

$$\{f_1, f_2\}_2 = \langle\hat{\mu}(\lambda), [\nabla^{(2)} f_1, \nabla^{(2)} f_2]\rangle_2, \quad (4)$$

where

$$\langle\hat{\mu}(\lambda), A(\lambda)\rangle_1 = \mathrm{res}\lambda^{-1}\mathrm{Tr}\hat{\mu}(\lambda)A(\lambda),$$
$$\langle\hat{\mu}(\lambda), A(\lambda)\rangle_2 = \mathrm{res}\lambda^{-2(N+1)-1}\mathrm{Tr}\hat{\mu}(\lambda)A(\lambda)$$

are two ad-invariant couplings in the algebra $\tilde{\mathfrak{g}}$ and

$$\nabla^{(1)} f = \sum_{l=0}^{N}\left(\frac{\partial f}{\partial\alpha_{2l}}X_0\lambda^{-2l} + \frac{\partial f}{\partial\gamma_{2l+1}}X_1\lambda^{-(2l+1)} + \right.$$

$$\left. + \frac{\partial f}{\partial\beta_{2l+1}}X_2\lambda^{-(2l+1)}\right), \quad (5)$$

$$\nabla^{(2)} f = \sum_{l=0}^{N}\left(\frac{\partial f}{\partial\alpha_{2l}}X_0\lambda^{-2l} + \frac{\partial f}{\partial\gamma_{2l+1}}X_1\lambda^{-(2l+1)} + \right.$$

$$\left. + \frac{\partial f}{\partial\beta_{2l+1}}X_2\lambda^{-(2l+1)}\right), \quad (6)$$

are matrix-valued gradients with values in the subalgebras $\tilde{\mathfrak{g}}_-$ and $\tilde{\mathfrak{g}}_+$, respectively.

By the *Poisson manifold*, we call the space M^N endowed with Poisson structure. In our case, it is complex, and the coordinates on it are the variables α_0, β_1, γ_1, α_2, β_3, γ_3, ..., α_{2N}, β_{2N+1}, γ_{2N+1}, α_{2N+1}, being matrix elements of the matrix $\hat{\mu}(\lambda)$. In the case where a function h is chosen as a Hamiltonian, the Hamilton equations take the form

$$\frac{d\mu_i(\lambda)}{d\tau} = \{\mu_i(\lambda), h\}_{1,2} \quad (7)$$

where $\mu_i(\lambda) = \langle\hat{\mu}(\lambda), X_i\rangle = \mathrm{Tr}\hat{\mu}(\lambda)X_i$, $i = 0, 1, 2$.

It is obvious that Eq. (7) is noncanonical, since the Poisson brackets (3) and (4) are noncanonical and degenerate as well. According to the Darboux theorem, a canonical structure can be found under narrowing Eq. (7) to the respective symplectic sheet, where the Poisson bracket becomes nondegenerate. It is obvious from the structure of the Lie—Poisson brackets (3) and (4) that the symplectic sheet of bracket (3) is an *orbit of the algebra* $\tilde{\mathfrak{g}}_-$; the symplectic sheet of bracket (4) is, respectively, an *orbit of the algebra* $\tilde{\mathfrak{g}}_+$.

Let us consider the ad-invariant function

$$I(\hat{\mu}(\lambda)) = \frac{1}{2}\mathrm{Tr}\hat{\mu}^2(\lambda) = \alpha^2(\lambda) + \beta(\lambda)\gamma(\lambda), \quad (8)$$

that is, obviously, a polynomial in the complex variable λ:

$$I(\hat{\mu}(\lambda)) = h_0 + \lambda^2 h_2 + \ldots + \lambda^{4N+4}h_{4N+4}.$$

The coefficient functions h_α can be easily calculated as

$$h_0 = \alpha_0^2,$$

$$h_2 = 2\alpha_0\alpha_2 + \beta_1\gamma_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$h_{4N} = 2\alpha_{2N-2}\alpha_{2N+2} + \alpha_{2N}^2 + \beta_{2N-1}\gamma_{2N+1} + \beta_{2N+1}\gamma_{2N-1},$$

$$h_{4N+2} = 2\alpha_{2N}\alpha_{2N+2} + \beta_{2N+1}\gamma_{2N+1},$$

$$h_{4N+4} = \alpha_{2N+2}^2. \quad (9)$$

For the function h_α, the following statements are true:

Proposition 1. *The functions h_α of collection (9) commute pairwise with respect to both Lie —Poisson brackets (3) and (4).*

Proposition 2. *The functions h_0, h_2, ..., h_{2N+2} are the annihilators (Casimir functions) of the second Lie—Poisson bracket, i.e., for any function f,*

$$\{f, h_\alpha\}_2 = 0, \quad \alpha = 0, 2, \ldots, 2N+2.$$

The functions h_{2N+2}, h_{2N+4}, ..., h_{4N+2} are the annihilators for the first Lie—Poisson bracket, i.e.,

$$\{f, h_\beta\}_1 = 0, \quad \beta = 2N+2, \ldots, 4N+2, 4N+4,$$

for any function $f \in C^\infty(M^N)$.

It is easy to prove those propositions through direct calculations, by using the explicit form of the function h_α and formulas (3) and (4).

In particular, it follows from Proposition 2 that the algebraic equations

$$h_0 = c_0, \quad h_2 = c_2, \quad \ldots, \quad h_{2N} = c_{2N},$$

c_α =const, play the role of constraints for the dynamical systems (7) with the Lie—Poisson bracket (4). They fix the orbit $\mathcal{O}_2^{\mathbb{C}} \subset M^N$ with respect to co-adjoint action of the group $G_+ = \exp \tilde{\mathfrak{g}}_+$. The algebraic equations

$$h_{2N+2} = c_{2N+2}, \quad h_{2N+4} = c_{2N+4}, \quad \dots,$$
$$h_{4N+4} = c_{4N+4},$$

c_α =const play a similar role. They fix the orbit $\mathcal{O}_1^{\mathbb{C}}$ generated by co-adjoint action of the group $G_- = \exp \tilde{\mathfrak{g}}_-$. Since the functions h_α, $\alpha = 0, 2, \dots, 4N+4$ are functionally independent, one has

$$\dim \mathcal{O}_1^{\mathbb{C}} = \dim \mathcal{O}_2^{\mathbb{C}} = 2(N+1).$$

It is obvious that complex dimensionality is meant here, since $\mathcal{O}_1^{\mathbb{C}}$ and $\mathcal{O}_2^{\mathbb{C}}$ are complex algebraic manifolds.

To construct the real Hamiltonian formalism, it is necessary to consider real orbits in the M^N space. In this space, two real ad^*-invariant Poisson structures related to two real subalgebras in $\mathrm{sl}(2,\mathbb{C})$ are possible. They are the subalgebras $\mathrm{su}(2)$ and $\mathrm{su}(1,1) \simeq \mathrm{sl}(2,\mathbb{R})$. While contracting, we must set

$$\alpha_{2l} = i a_{2l}, \quad a_{2l} \in \mathbb{R},$$

and

$$\gamma_{2l+1} = -\bar{\beta}_{2l+1}, \quad \text{(in the case su(2))}, \tag{10}$$
$$\gamma_{2l+1} = \bar{\beta}_{2l+1}, \quad \text{(in the case su(1,1))}. \tag{11}$$

Hereinafter, we continue to use the variables α_{2l}, γ_{2l+1}, β_{2l+1}, though we will have in mind that these variables satisfy one of the reality conditions, (10) or (11).

We fix the function h_{4N+2} as a Hamiltonian and write down the respective system of Hamilton equations:

$$\frac{\partial \mu_i(\lambda)}{\partial \tau_{4N+2}} = \{\mu_i(\lambda), h_{4N+2}\}_2. \tag{12}$$

Let us set $\tau_{4N+2} \equiv x$. In the coordinates α_{2l}, γ_{2l}, β_{2l}, system (12) takes the form

$$\frac{\partial \alpha_{2l}}{\partial x} = \beta_{2l-1}\gamma_{2N+1} - \gamma_{2l-1}\beta_{2N+1},$$
$$\frac{\partial \beta_{2l+1}}{\partial x} = 2(\alpha_{2l}\beta_{2N+1} - \beta_{2l-1}\alpha_{2N+2}),$$
$$\frac{\partial \gamma_{2l+1}}{\partial x} = -2(\alpha_{2l}\gamma_{2N+1} - \gamma_{2l-1}\alpha_{2N+2}). \tag{13}$$

Equations (13) yield that α_0 and α_{2N+2} are constant values. We fix them and exclude from the set of dynamical variables.

Equation (12) can be written in the matrix form (as the Euler —Arnold equation [16]):

$$\frac{\partial \hat{\mu}(\lambda)}{\partial x} = [\hat{\mu}(\lambda), \nabla_{(2)} h_{4N+2}],$$

where

$$\nabla_{(2)} h_{4N+2} = \begin{bmatrix} \alpha_{2N+2}\lambda^2 & \beta_{2N+1}\lambda \\ \gamma_{2N+1}\lambda & -\alpha_{2N+2}\lambda^2 \end{bmatrix}. \tag{14}$$

Let us consider the contraction of Eqs. (13) to the orbit $\mathcal{O}_1^{\mathbb{C}}$. To make this, it is necessary to choose local coordinates on it. It follows from the explicit form of the algebraic equations assigning the orbit that the variables β_{2l+1} and γ_{2l+1}, $l = 0, 1, \dots, N$ can serve as local coordinates and the other variables α_{2l} can be expressed through them. For instance:

$$\alpha_{2N} = \frac{1}{2\alpha_{2N+2}}(c_{4N+2} - \beta_{2N+1}\gamma_{2N+1}),$$
$$\alpha_{2N-2} = \frac{1}{2\alpha_{2N+2}}(c_{2N} - \alpha_{2N}^2 -$$
$$- \beta_{2N+1}\gamma_{2N-1} - \beta_{2N-1}\gamma_{2N+1}) \quad \text{and so on.} \tag{15}$$

In addition, the structure of Eqs. (13) makes it possible to express the variables β_{2l+1}, γ_{2l+1}, $l = 0, 1, \dots, N-1$, in terms of the basic variables β_{2N+1}, γ_{2N+1} and their derivatives of order $(n-l)$. For instance,

$$\beta_{2N-1} = \frac{1}{2\alpha_{2N+2}}\left(2\alpha_{2N}\beta_{2N+1} - \frac{\partial}{\partial x}\beta_{2N+1}\right). \tag{16}$$

A similar formula holds for the variable γ_{2N+1}. Upon such reductions, the system of equations (13) goes into two equations of order $(N+1)$ for the dynamical variables β_{2N-1} and γ_{2N-1}. As shown in [16], the transition from Eqs. (13) to equations with higher derivatives means the admissibility of the Lagrange formalism for Hamilton systems on the orbit of the algebras $\widetilde{\mathfrak{g}}_+$ and $\widetilde{\mathfrak{g}}_-$.

In M^N, we consider one more Hamilton system generated by the Hamiltonian h_{4N}:

$$\frac{\partial \mu_i(\lambda)}{\partial \tau} = \{\mu_i(\lambda), h_{4N}\}_2. \tag{17}$$

Similarly to the previous case, system (16) can be represented in the form of the Euler—Arnold matrix equation

$$\frac{\partial \hat{\mu}(\lambda)}{\partial \tau} = [\hat{\mu}(\lambda), \nabla_2 h_{4N}], \tag{18}$$

where

$$\nabla_2 h_{4N} = \begin{bmatrix} \lambda^4\alpha_{2N+2} + \lambda^2\alpha_{2N} & \lambda^3\beta_{2N+1} + \lambda\beta_{2N-1} \\ \lambda^3\gamma_{2N+1} + \lambda\gamma_{2N-1} & -(\lambda^4\alpha_{2N+2} + \lambda^2\alpha_{2N}) \end{bmatrix}.$$

Due to commutativity of the Hamiltonians h_{4N+2} and h_{4N}, evolution (17) can be defined on trajectories of the Hamilton system (13). Since, on its trajectories, all the variables α_{2l}, β_{2l+1}, γ_{2l+1} are given through β_{2N+1}, γ_{2N+1} and their derivatives, it is enough to trace only the variables β_{2N+1} and γ_{2N+1} in Eq. (17):

$$\frac{\partial\beta_{2N+1}}{\partial\tau} = 2(\alpha_{2N-2}\beta_{2N+1} - \alpha_{2N+2}\beta_{2N-3}),$$
$$\frac{\partial\gamma_{2N+1}}{\partial\tau} = -2(\alpha_{2N-2}\gamma_{2N+1} - \alpha_{2N+2}\gamma_{2N-3}).$$

Though, (13) yields:

$$\frac{\partial\beta_{2N-1}}{\partial x} = 2(\alpha_{2N-2}\beta_{2N+1} - \alpha_{2N+2}\beta_{2N-3}),$$
$$\frac{\partial\gamma_{2N-1}}{\partial x} = -2(\alpha_{2N-2}\gamma_{2N+1} - \alpha_{2N+2}\gamma_{2N-3}),$$

therefore,

$$\frac{\partial\beta_{2N+1}}{\partial\tau} = \frac{\partial}{\partial x}\beta_{2N-1}; \quad \frac{\partial\gamma_{2N+1}}{\partial\tau} = \frac{\partial}{\partial x}\gamma_{2N-1}.$$

With regard for formulas (15) and (16), we get the partial differential equation

$$\frac{\partial\beta_{2N+1}}{\partial\tau} = -\frac{1}{2\alpha_{2N+2}}\times$$

$$\times\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\beta_{2N+1} + \frac{\beta_{2N+1}}{\alpha_{2N+2}}(\beta_{2N+1}\gamma_{2N+1} - c_{4N+2})\right]. \tag{19}$$

A similar equation holds for the variable γ_{2N+1} as well. By restricting Eq. (19) to a real orbit of the algebra $su(2)\otimes\mathcal{P}(\lambda,\lambda^{-1})$, we set

$$\alpha_{2N+2} = ia, \quad \gamma_{2N+1} = -\bar{\beta}_{2N+1}.$$

Then we get

$$\frac{\partial\beta_{2N+1}}{\partial\tau} = -\frac{1}{2a}\frac{\partial^2\beta_{2N+1}}{\partial x^2} -$$

$$-\frac{i}{2a^2}\frac{\partial}{\partial x}(|\beta_{2N+1}|^2\beta_{2N+1} + c_{4N+2}\beta_{2N+1}). \tag{20}$$

It is obvious that Eq. (20) coincides with Eq. (1) [in the case $(1-\beta)>0$] after the scale transformation $x \to d\cdot x$ (this is equivalent to choosing the "Hamiltonian" of the stationary problem as dh_{4N+2}) and setting $\beta_{2k+1}(t,x) = \mathbf{b}(t,x)$, $c_{4N+2} = 2a^2 d$,

$$d = \frac{1}{2}\left(\frac{1}{\mu^2(1-\beta)}\right)^{1/3}, \quad \frac{1}{2ad^2} = \pm\mu.$$

In the case where $(1-\beta)<0$, we perform another reduction in Eq. (19). In particular, we set $\gamma_{2N+1} = \bar{\beta}_{2N+1}$. Such a reduction means the contraction of a complex Hamilton system to a real orbit of the algebra $su(1,1)\otimes\mathcal{P}(\lambda,\lambda^{-1})$. The resulting equation has another sign of the nonlinear term and describes the wave propagation in plasma with $\beta>1$.

2. Finite-zone Integration of the DNLS Equation

The above-presented scheme of construction for the nonlinear DNLS equation gives, at the same time, a method for its integration. For this purpose, one has to solve the system of Hamilton equations (13) and to assign the evolution on its trajectories according to Eqs. (17). Both systems are integrable by Liouville on both the orbit $\mathcal{O}_1$ and the orbit $\mathcal{O}_2$ as well. The intersection of the complex orbits $\mathcal{O}_1^{\mathbb{C}}$ and $\mathcal{O}_2^{\mathbb{C}}$ is a manifold fixed by the collection of algebraic equations

$$h_\alpha = c_\alpha, \quad \alpha = 2m,\ m = 0, 1, \ldots, 2N+1,$$

(the constant α_{2N+2} is fixed from the very beginning), and is a complex Liouville torus for the integrable Hamilton systems (13) and (17).

To integrate Eqs. (13), we apply the method, being well developed in the theory of finite-zone integration of soliton-type equations and applied to the DNLS equations in [7]. In Eqs. (13), we substitute the variables $\beta_1, \beta_2, \ldots, \beta_{2N+1}$ by the variables s_j, $j = 1, 2, \ldots, N$ being the squares of zeroes of the polynomial $\beta(\lambda) = \sum_{l=0}^N \beta_{2l+1}\lambda^{2l+1}$:

$$\beta(\lambda) = \lambda\beta_{2N+1}\prod_{j=1}^N(\lambda^2 - s_j). \tag{21}$$

By the Vieta theorem, all the values $\beta_{2l+1}/\beta_{2N+1}$ are expressed through symmetric functions of the variables s_j, in particular,

$$\frac{\beta_{2N-1}}{\beta_{2N+1}} = -\sum_{j=1}^N s_j, \quad \frac{\beta_1}{\beta_{2N+1}} = (-1)^N\prod_j s_j. \tag{22}$$

As follows from representation (14), Eq. (13) can be given in a compact form:

$$\frac{\partial\alpha(\lambda)}{\partial x} = \lambda(\beta(\lambda)\gamma_{2N+1} - \gamma(\lambda)\beta_{2N+1}), \tag{23}$$
$$\frac{\partial\beta(\lambda)}{\partial x} = 2(\lambda\alpha(\lambda)\beta_{2N+1} - \lambda^2\beta(\lambda)\alpha_{2N+1}),$$

From the last equation, we get

$$\frac{\partial}{\partial x}\ln\beta(\lambda) = \frac{2[\lambda\alpha(\lambda)\beta_{2N+1} - \lambda^2\beta(\lambda)\alpha_{2N+2}]}{\beta(\lambda)} =$$

$$= \frac{2\alpha(\lambda)}{\prod_j(\lambda^2 - s_j)} - 2\lambda^2\alpha_{2N+2}. \tag{24}$$

On the other side, due to (21), the left-hand side of Eq. (24) can be given as

$$\frac{\partial}{\partial x}\ln\beta(\lambda) = \frac{\partial}{\partial x}\ln\beta_{2N+1} - \sum_j \frac{\frac{\partial s_j}{\partial x}}{\lambda^2 - s_j}. \tag{25}$$

By multiplying (24) and (25) by the factor $\lambda^2 - s_k$ and by directing λ^2 to s_k, we get

$$\frac{\partial s_k}{\partial x} = -\frac{2\alpha(\sqrt{s_k})}{\prod_{j\neq k}(s_k - s_j)}. \tag{26}$$

The time dependence of the variables s_k is determined by Eq. (18). For the polynomial $\beta(\lambda)$, we have the evolutionary equation

$$\frac{\partial\beta(\lambda)}{\partial\tau} = 2\alpha(\lambda)(\lambda^3\beta_{2N+1} + \lambda\beta_{2N-1}) -$$

$$-2\beta(\lambda)(\lambda^4\alpha_{2N+2} + \lambda^2\alpha_{2N}),$$

whence it is not difficult to get the equation for its zeroes (i.e., the variables s_k):

$$\frac{\partial s_k}{\partial\tau} = \frac{2\alpha(\sqrt{s_k})}{\prod_{j\neq k}(s_k - s_j)}\left(\sum_{j=1}^{N} s_j - s_k\right). \tag{27}$$

Since $\beta(\sqrt{s_k}) = 0$, we get

$$\alpha^2(\sqrt{s_k}) = h_0 + s_k h_2 + \ldots + s^{2N+2}h_{4N+4} \equiv$$

$$\equiv h_{4N+4}P_{2N+2}(s_k), \tag{28}$$

where h_α are the integrals of motion for both Hamilton systems, (14) and (18). Upon the reductions we are interested in ($\alpha_{2l} = ia_{2l}\ \gamma_{2l+1} = \mp\beta_{2l+1}$), coefficients of the polynomial $P_{2N+2}(s)$ become real numbers. Therefore, its roots are real or mutually complex conjugate.

The general principles for the integration of equations of the type (26) and (27) are well known. The respective dynamical systems are studied on the Riemannian surface associated to the algebraic curve $\mathcal{R}$:

$$\mathcal{R} = \{s, w;\ w^2 = P_{2N+2}(s), \quad s = \lambda^2\}. \tag{29}$$

In the case where all roots of the polynomial P_{2N+2} are different, one has a hyperelliptic curve of the type $g = N$. In the case of twice degenerate roots, the hyperbolic curve is degenerated to a rational one. This case corresponds to soliton solutions. Let

$$\omega_1 = \frac{ds}{w(s)}, \quad \omega_2 = \frac{sds}{w(s)}, \quad \ldots, \quad \omega_N = \frac{s^{N-1}ds}{w(s)} \tag{30}$$

are basic holomorphic differentials on $\mathcal{R}$ (Abel differentials of the first kind). Then the Abel mapping

$$\{s_1, s_2, \ldots, s_N\} \to\ A_1(s_1, s_2, \ldots, s_N),$$

$$A_2(s_1, s_2, \ldots, s_N), \ldots, A_N(s_1, s_2, \ldots, s_N)$$

where

$$A_k(s_1, s_2, \ldots, s_N) = \sum_{j=1}^{N}\int_{s_j(0)}^{s_j(x,\tau)}\omega_k, \tag{31}$$

linearizes systems (26) and (27). Indeed, we make sure through direct calculations that

$$\frac{\partial A_k}{\partial x} = 0,\ k = 1, 2, \ldots, N-1; \frac{\partial A_N}{\partial x} = 2i,$$
$$\frac{\partial A_k}{\partial\tau} = 0,\ k = 1, 2, \ldots, N-2, N; \frac{\partial A_{N-1}}{\partial\tau} = 2i.$$

Thus, in terms of the variables $A_k(x,\tau)$, the Hamilton flows corresponding to the Hamiltonians h_{4N} and h_{4N+2} (and to the other Hamiltonians h_α) are linear. From the viewpoint of the integrable systems (14) and (18), the variables A_k coincide with the complex angular variables on a complexified Liouville torus to within numerical coefficients.

The most difficult stage in integrating systems (26) and (27) is concerned with the problem of inversion of the integral formulas (31), i.e., with finding the values $s_j(x,\tau)$ through the angular variables $A_k(x,t)$. This is

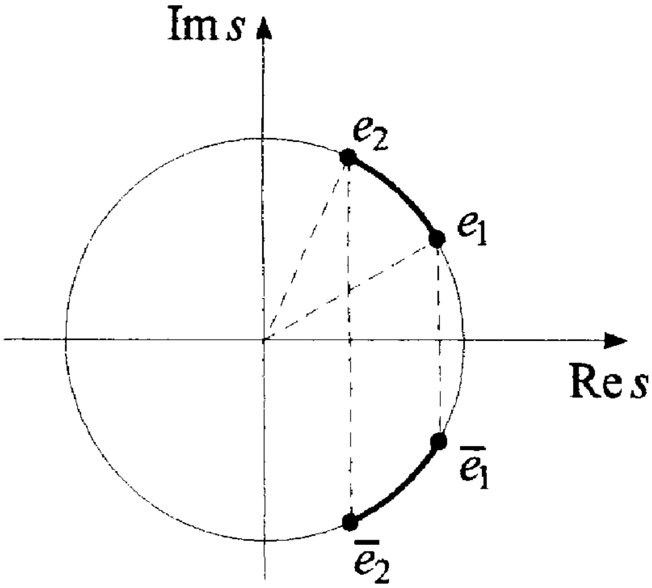

Fig. 1. $e_1 = Re^{i\theta_1}$, $e_2 = Re^{i\theta_2}$

the well-known problem of classical algebraic geometry [18], and its solution is given in terms of multivariable θ-functions. Unfortunately, the formulas [7,8] obtained by such a scheme work rather poorly. First, in θ-functional formulas, there are many undefined parameters which require the calculation of complicated integrals to find them (in the case of an arbitrary distribution of zeroes of the polynomial $P_{2N+2}(s)$, this problem looks to be extremely difficult). Secondly, complex solutions are obtained, because the Abel mapping and its inversion are complex-analytic. Therefore, to obtain the contractions to the real parts of respective complex manifolds, one has to perform an additional study. We do not go to these problems in detail and propose to turn to papers [19, 20], where the problem of making θ-functional formulas to be efficient is solved in several cases interesting for applications.

If the problem of inversion of Abel integrals is solved, the symmetric combinations of values $s_k(x,\tau)$ are expressed through the variables $A_k(x,t)$ by explicit formulas. In such a case, the function $\beta_{2N+1}(x,\tau) = |\beta_{2N+1}(x,\tau)|e^{i\phi(x,\tau)}$ can be found via the integration of the following relations:

$$\frac{\partial}{\partial x}\ln|\beta_{2N+1}| = -2a_{2N+2}\mathrm{Im}\sum_j s_j, \qquad (32)$$

$$\frac{\partial\phi}{\partial x} = -\frac{h_{4N+2}+|\beta_{2N+1}|^2}{a_{2N+2}} + 2a_{2N+2}\mathrm{Re}\sum_j s_j. \qquad (33)$$

3. One-phase and N-soliton Solutions of the DNLS Equation

Here, we describe the one-phase (one-soliton, rational, and periodic) solutions of Eq. (20) and give a compact formula for the N-soliton hyperbolic solutions for $N = 3$. As is shown in [23], hyperbolic solitons are stable formations and are important for kinetic processes.

Let us set $a \equiv a_{2N+2} = 1$ та $\beta_{2N+1} = \mathbf{b}$. Then Eq. (20) takes the form

$$2i\frac{\partial\mathbf{b}}{\partial t} = -\frac{\partial^2\mathbf{b}}{\partial x^2} - i\frac{\partial}{\partial x}(|\mathbf{b}|^2\mathbf{b} + c_{4N+2}\mathbf{b}). \qquad (34)$$

The simplest solution of this equation is a harmonic wave with circular polarization [20]

$$\mathbf{b}(x,t) = |\mathbf{b}|e^{i(kx-\omega t)},\ |\mathbf{b}| = \mathrm{const}, \qquad (35)$$

and with the dispersion law

$$2\omega(k) = k^2 + k(|\mathbf{b}|^2 + c_{4N+2}).$$

From the viewpoint of the orbit scheme, solution (35) corresponds to the situation $N = 0$. In this case, the phase space of a stationary Hamilton system is two-dimensional; it is a complex surface of the variable $\beta_1 = b$; and the respective Hamilton equation is linear:

$$\frac{\partial\beta_1}{\partial x} = 2ia_0\beta_1.$$

For the case $N = 1$, the characteristic distribution of zeroes of the polynomial $P_{2N+2}(s) = h_{2N+2}P(s)$,

$$P(s) = (s-e_1)(s-\bar{e}_1)(s-e_2)(s-\bar{e}_2)$$

is shown in Fig. 1. The polynomial $\beta(\lambda) = \lambda(\beta_1 + \lambda^2\beta_3)$ has only one nonzero root $s_1 = -\beta_1/\beta_3$. Equation (26) for the variable $s_1(x)$ takes the form:

$$\frac{\partial s_1}{\partial x} = 2ia_4\sqrt{(s_1^2 - 2s_1R\cos\theta_1 + R^2)}\times$$

$$\times\sqrt{(s_1^2 - 2s_1R\cos\theta_2 + R^2)}. \qquad (36)$$

From this equation, one can see that the choice of a_4 is the choice of the variable x scale. Thus, hereinafter we set $a_4 = 1$. For the given choice of zeroes e_1 and e_2, the coefficients h_α of the polynomial $P_{2N+2}(s)$ are as

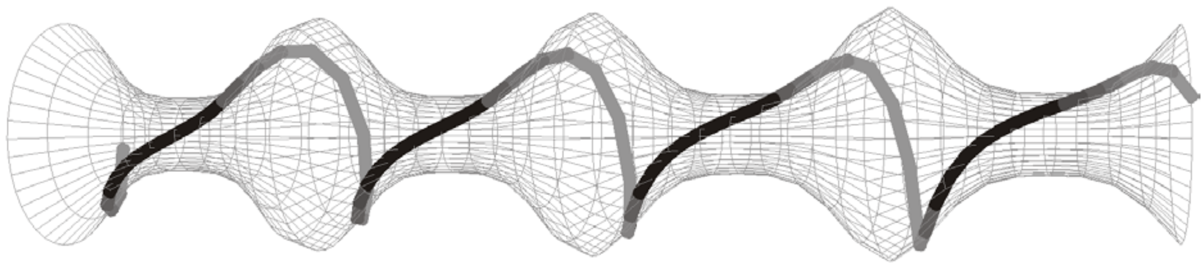

Fig. 2.

follows:

$$h_0 = -a_0^2 = -R^4,$$

$$h_2 = -2a_0a_2 - |s_1|^2|\beta_3|^2 = 2R^3(\cos\theta_1 + \cos\theta_2),$$

$$h_4 = -a_2^2 - 2a_0 + (s_1 + \bar{s}_1)|\beta_3|^2 =$$

$$= -2R^2(1 + 2\cos\theta_1\cos\theta_2),$$

$$h_6 = -2a_2 - |\beta_3|^2 = 2R(\cos\theta_1 + \cos\theta_2),$$

$$h_8 = -1. \tag{37}$$

From those equations, one can easily get the reality condition for the variable $s_1(x)$:

$$|s_1(x)|^2 = R^2.$$

This condition matches the initial data for Eq. (36) to the choice of zeros, $e_1 = Re^{i\theta_1}$ and $e_2 = Re^{i\theta_2}$.

The substitution

$$s_1 = R\frac{1 + iz}{1 - iz}$$

brings Eq. (36) to the standard equation for a Jacobi elliptic function:

$$z = \mathrm{tg}\,\tfrac{\theta_1}{2}\mathrm{sn}(Ax;k), \quad A = 4R\cos\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2}, \quad k^2 = \frac{\mathrm{tg}^2\frac{\theta_1}{2}}{\mathrm{tg}^2\frac{\theta_2}{2}}.$$

If $s_1(0) = R$, then

$$s_1 = R\frac{1 + i\,\mathrm{tg}\,\frac{\theta_1}{2}\mathrm{sn}(Ax;k)}{1 - i\,\mathrm{tg}\,\frac{\theta_1}{2}\mathrm{sn}(Ax;k)}.$$

If $s_1(0) = -R$, then

$$\widetilde{s}_1 = -R\frac{1 - i\,\mathrm{ctg}\,\frac{\theta_2}{2}\mathrm{sn}(Ax;k)}{1 + i\,\mathrm{ctg}\,\frac{\theta_2}{2}\mathrm{sn}(Ax;k)}.$$

We find the function $|\beta_3|^2$ from the algebraic equation (37):

$$|\beta_3|^2 = 4R\frac{\left(\sin\frac{\theta_1}{2}\mathrm{cn}(Ax;k) \pm \sin\frac{\theta_2}{2}dn(Ax;k)\right)^2}{1 + \mathrm{tg}^2\,\frac{\theta_1}{2}\mathrm{sn}^2(Ax;k)}. \tag{38}$$

The phase multiplier for the function $\beta_3(x) = |\beta_3(x)|e^{i\phi(x)}$ is obtained by the additional integration of Eq. (33) and has the form

$$e^{i\phi(x)} = \sqrt{\frac{\sigma(4R(x - ix_0))}{\sigma(4R(x + ix_0))}} \times \left[\frac{1 + i\,\mathrm{tg}\,\frac{\theta_1}{2}\mathrm{sn}(Ax;k)}{1 - i\,\mathrm{tg}\,\frac{\theta_1}{2}\mathrm{sn}(Ax;k)}\right]^{\pm 1} \times$$

$$\times e^{2Rx(2\zeta[4Rix_0] - i)}, \tag{39}$$

where σ is the Weierstrass “sigma”-function, ζ is the Weierstrass “zeta”-function, and x_0 is a constant that is obtained from the relation $\mathrm{sn}^2(Ax_0;k') = \cos^2\frac{\theta_1}{2}$. The behavior of the elliptic one-zone solution $\beta_3(x)$ is shown in Fig.2.

The time dependence of the function is given by Eq. (18). For $N = 1$, it takes the form

$$\frac{\partial\beta_3}{\partial t} = 2ia_0\beta_3.$$

This is a linear equation, therefore the time dependence in the reference system moving with the Alfven velocity, besides the argument $x - vt$, is included to the phase multiplier as well:

$$\mathbf{b}(x,t) = |\beta_3(x - vt)|e^{i\phi(x - vt) + 2ia_0t}$$

where $v = -h_6 = -2R(\cos\theta_1 + \cos\theta_2)$ is the velocity of a nonlinear wave with respect to the Alfven wave.

Let us consider several cases of a special arrangement of the zeroes e_1 and e_2.

1. Nonlinear wave with harmonic modulation (Fig.3). Let $\theta_2 = \pi$, $\pi > \theta_1 > 0$. Then $k = 0$, $A = 4R\cos\frac{\theta_1}{2}$,

$$s_1 = R\left(\frac{1-\mathrm{tg}^2\frac{\theta_1}{2}\sin^2(Ax)}{1+\mathrm{tg}^2\frac{\theta_1}{2}\sin^2(Ax)} + \frac{2i\,\mathrm{tg}\frac{\theta_1}{2}\sin(Ax)}{1+\mathrm{tg}^2\frac{\theta_1}{2}\sin^2(Ax)}\right),$$

$$|\beta_3|^2 = 4R\frac{\left(\sin\frac{\theta_1}{2}\cos Ax \pm 1\right)^2}{1+\mathrm{tg}^2\frac{\theta_1}{2}\sin^2 Ax},$$

$$e^{i\phi} = \sqrt{\frac{1-i\,\mathrm{tg}\,Ax/\cos\frac{\theta_1}{2}}{1+i\,\mathrm{tg}\,Ax/\cos\frac{\theta_1}{2}}} \times \left[\frac{1+i\,\mathrm{tg}\frac{\theta_1}{2}\sin Ax}{1-i\,\mathrm{tg}\frac{\theta_1}{2}\sin Ax}\right]^{\mp 1} \times$$

$$\times e^{2a_4 Rix}. \qquad (40)$$

The time dependence of the function β_3 is included similarly to the previous case.

2. Hyperbolic soliton (Fig 4). Let $\theta_1 = \theta_2 = \theta$. Then $k = 1$, $A = 2R\sin\theta$,

$$s_1 = R\left(\frac{1-\mathrm{tg}^2\frac{\theta}{2}\,\mathrm{th}^2 Ax}{1+\mathrm{tg}^2\frac{\theta}{2}\,\mathrm{th}^2 Ax} + \frac{2i\,\mathrm{tg}\frac{\theta}{2}\,\mathrm{th}\,Ax}{1+\mathrm{tg}^2\frac{\theta}{2}\,\mathrm{th}^2 Ax}\right),$$

$$|\beta_3(x)|^2 = 16R\sin^2\frac{\theta}{2}\frac{1-\mathrm{th}^2 Ax}{1+\mathrm{tg}^2\frac{\theta_1}{2}\,\mathrm{th}^2 Ax}$$

or

$$|\beta_3(x)|^2 = \frac{16R\sin^2\frac{\theta}{2}}{\mathrm{ch}^2 Ax + \mathrm{tg}^2\frac{\theta}{2}\,\mathrm{sh}^2 Ax},$$

$$e^{i\phi} = \left(\frac{1-i\,\mathrm{tg}\frac{\theta}{2}\,\mathrm{th}\,Ax}{1+i\,\mathrm{tg}\frac{\theta}{2}\,\mathrm{th}\,Ax}\right)^{3/2} e^{-2Rix\cos\theta}.$$

The final expression for a hyperbolic soliton is

$$\mathbf{b}(x,t) = \frac{4\sqrt{R}\sin\frac{\theta}{2}}{\cosh u(x,t)}\frac{[1-\mathrm{tg}\frac{\theta}{2}\,\mathrm{th}\,u(x,t)]}{[1+\mathrm{tg}\frac{\theta}{2}\,\mathrm{th}\,u(x,t)]^2} \times$$

$$\times e^{-2iR\cos\theta u(x,t)+2iR^2t}, \qquad (41)$$

where $u(x,t) = 2R\sin\theta(x - x_0 - vt)$, $v = -4R\cos\theta$. The configuration described by formula (41) is called a hyperbolic soliton. Such a solution is given in [5] as well.

3. Rational one-phase solution. If $\theta \to \pi$, then solution (41) degenerates to a rational one,

$$b(x,t) = \frac{4\sqrt{R}}{[1+16R^2(x-x_0-vt)]^{1/2}}\, e^{2iR(x-vt)+2iR^2t}. \qquad (42)$$

4. Hyperbolic solitons for $N = 3$. Consider the case of four twice degenerate roots of the polynomial $P_{2N+2}(s)$ for $N = 3$. The "reality" conditions give restriction for the variables s_i, namely,

$$|s_i| = R, \quad i = 1,2,3.$$

Since the "turn points" or "interruption points" (where $\frac{\partial s_i}{\partial x} = \frac{\partial s_i}{\partial t} = 0$) have, according to Eqs. (26) and (27), to coincide with zeroes of the polynomial $P_{2N+2}(s)$, the latter are positioned on a circle of radius R. Let $e_1 = Re^{i\theta_1}$, $e_2 = Re^{i\theta_2}$, $\theta_2 > \theta_1$, $e_3 = \bar{e}_1$, $e_4 = \bar{e}_2$ be twice degenerate roots. Then

$$P_8(s) = (s-e_1)^2(s-\bar{e}_1)^2(s-e_2)^2(s-\bar{e}_2)^2.$$

In this case, the Abel integrals (31) degenerate to integrals of rational functions; one can easily calculate those and find the Abel mapping in the explicit form:

$$A_1(s_1,s_2,s_3) = \ln\left\{C_1\prod_{j=1}^{3}(s_j-e_1)^{q(1)} \times \prod_{j=1}^{3}(s_j-\bar{e}_1)^{q(\bar{1})}\prod_{j=1}^{3}(s_j-e_2)^{q(2)}\prod_{j=1}^{3}(s_j-\bar{e}_2)^{q(\bar{2})}\right\},$$

$$A_2(s_1,s_2,s_3) = \ln\left\{C_2\prod_{j=1}^{3}(s_j-e_1)^{e_1q(1)} \times \prod_{j=1}^{3}(s_j-\bar{e}_1)^{\bar{e}_1q(\bar{1})}\prod_{j=1}^{3}(s_j-e_2)^{e_2q(2)}\prod_{j=1}^{3}(s_j-\bar{e}_2)^{\bar{e}_2q(\bar{2})}\right\},$$

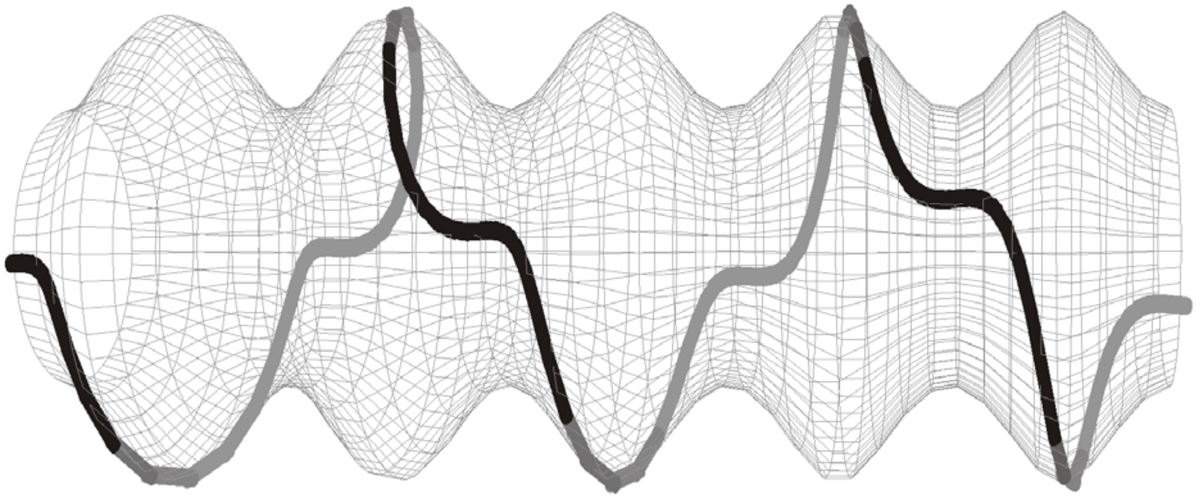

Fig. 3.

$$A_3(s_1,s_2,s_3)=\ln\left\{C_3\prod_{j=1}^{3}(s_j-e_1)^{e_1^2q(1)}\times\prod_{j=1}^{3}(s_j-\bar e_1)^{\bar e_1^2q(\bar 1)}\prod_{j=1}^{3}(s_j-e_2)^{e_2^2q(2)}\prod_{j=1}^{3}(s_j-\bar e_2)^{\bar e_2^2q(\bar 2)}\right\}. \quad (43)$$

In the formulas given, the following notations are used:

$$q^{-1}(1)=(e_1-\bar e_1)(e_1-e_2)(e_1-\bar e_2),$$

$$q^{-1}(\bar 1)=(\bar e_1-e_1)(\bar e_1-e_2)(\bar e_1-\bar e_2),$$

$$q^{-1}(2)=(e_2-e_1)(e_2-\bar e_1)(e_2-\bar e_2),$$

$$q^{-1}(\bar 2)=(\bar e_2-e_1)(\bar e_2-\bar e_1)(\bar e_2-e_2),$$

C_k, $k=1,\,2,\,3$ are the integration constants defined by initial conditions for the variables s_k.

As pointed out in the previous item, the variables A_1, A_2, and A_3 are angular variables on a 3-dimensional torus. Their space-time evolution according to Eqs. (12) and (13) is linear, in particular,

$$A_1=\text{const},\quad A_2=2i(\tau-\tau_0),\quad A_3=2i(x-x_0).$$

Hereafter, to simplify the formulas, we set $t_0=x_0=0$, and the choice of initial conditions is related to the choice of constants C_k, $k=1,\,2,\,3$. Equations (43) yield the relations

$$\prod_{j=1}^{3}(s_j-e_1)=C_1e^{2i[x(e_1-\bar e_1)+\tau(e_1^2-\bar e_1^2)]}\prod_{j=1}^{3}(s_j-\bar e_1),$$

$$\prod_{j=1}^{3}(s_j-e_2)=C_2e^{2i[x(e_2-\bar e_2)+\tau(e_2^2-\bar e_2^2)]}\prod_{j=1}^{3}(s_j-\bar e_2),$$

$$\prod_{j=1}^{3}(s_j-e_1)=C_3e^{2i[x(e_1-\bar e_2)+\tau(e_1^2-\bar e_2^2)]}\prod_{j=1}^{3}(s_j-\bar e_2). \quad (44)$$

After opening the products on the right- and left-hand sides, these relations are reduced to a system of linear inhomogeneous algebraic equations for three symmetric functions:

$$z_1=s_1s_2s_3,$$

$$z_2=s_1s_2+s_1s_3+s_2s_3,$$

$$z_3=s_1+s_2+s_3.$$

They can be found through solving the system of equations (44) by Cramer's rule. We use the notations

$$f_1=e^{-2i(xe_1+\tau e_1^2)}+C_1e^{-2i(x\bar e_1+\tau\bar e_1^2)},$$

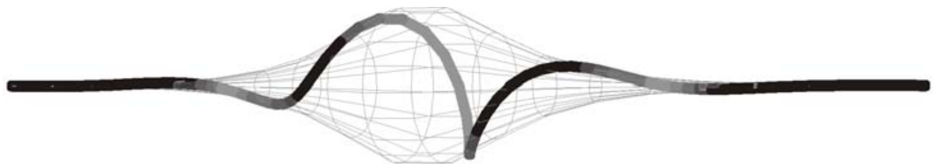

Fig. 4.

$$f_2 = e^{-2i(xe_2+\tau e_2^2)} + C_2 e^{-2i(x\bar{e}_2+\tau\bar{e}_2^2)},$$

$$f_3 = e^{-2i(xe_1+\tau e_1^2)} + C_3 e^{-2i(x\bar{e}_2+\tau\bar{e}_2^2)},$$

and set $-2ix = y$. Then, the determinant of system (44) can be written as a Wronskian

$$W(f_1, f_2, f_3) = \det\begin{pmatrix} f_1 & \frac{\partial}{\partial y} f_1 & \frac{\partial^2}{\partial y^2} f_1 \\ f_2 & \frac{\partial}{\partial y} f_2 & \frac{\partial^2}{\partial y^2} f_2 \\ f_3 & \frac{\partial}{\partial y} f_3 & \frac{\partial^2}{\partial y^2} f_3 \end{pmatrix},$$

and compact formulas can be obtained for the functions z_1, z_2, z_3. In particular, one can easily see that

$$z_3 = s_1 + s_2 + s_3 = -\frac{1}{2i}\frac{\partial}{\partial x}\ln W(f_1, f_2, f_3). \qquad (45)$$

With regard for the equation

$$\frac{\partial \beta_N}{\partial x} = -2i\beta_{N-1} = 2iz_3\beta_N$$

and the real form of the function z_3, we get the formula

$$|\beta_N|^2 = \frac{C}{WW^*}. \qquad (46)$$

The constant C and constants C_1, C_2 , C_3 determined by the initial conditions can be calculated from the algebraic relations

$$\alpha(s_k) = \sqrt{P_8(s_k)}.$$

Formulas (43), (46) can be easily generalized to arbitrary odd $N = 2n - 1$, $n = 1, 2, \ldots$ In this case, the number n corresponds to the number of solitons. That is, in asymptotics as $\tau \to \pm\infty$, the function $\beta_N(x, \tau)$ can be approximated as a sum of n terms, where each term has the form (41). For arbitrary $N = 2n - 1$, formula (46) preserves the general form. The velocity of the i-th soliton and its amplitude are determined, respectively, by the real and imaginary parts of the root e_i. Similarly to the case of solitons for the nonlinear Schrodinger equation, the scattering of solitons is accompanied by a shift of both the soliton center coordinate and its phase. The general shift for the center and phase of the i-th soliton is represented additively through two-particle shifts. In the case $n = 2$, we get

$$\Delta x_{01} = \frac{2}{\mathrm{Im} e_1}\ln\left|\frac{e_1 - \bar{e}_2}{e_1 - e_2}\right|,$$

$$\Delta x_{02} = -\frac{2}{\mathrm{Im} e_2}\ln\left|\frac{e_1 - \bar{e}_2}{e_1 - e_2}\right|.$$

Scattering processes for n-soliton configurations will be studied in detail in another paper.

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НЕЛІНІЙНІ АЛЬФВЕНІВСЬКІ ХВИЛІ ТА СОЛІТОНИ У ХОЛОДНІЙ ПЛАЗМІ

П.І. Голод, Ю.М. Бернацька

Р е з ю м е

Досліджено скінченнозонний сектор нелінійного рівняння Шредінгера з похідною, яке описує нелінійні альфвенівські хвилі малої амплітуди в довгохвильовому наближенні. Отримано формули для періодичних однофазних та односолітонних розв'язків, а також загальну формулу для огинаючої n-солітонного розв'язку.