A GENERALIZATION OF THE ZIPOY—VOORHEES METRIC IN THE PRESENCE OF A CONFORMALLY INVARIANT SCALAR FIELD

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We find and investigate a generalization of the well-known Zipoy—Voorhees metric in the case where its central time-like singularity is also the source of a conformally invariant scalar field. The Zipoy—Voorhees metric known also as the γ-solution, plays an important role in the hierarchy of naked singularities, and its properties are characteristic, as a rule, of some more general classes of solutions. The analysis of the axial naked time-like singularity shows that it can be referred to one of 6 different types. Among them, there are surface singularities as well as linear and paradoxical singularities with negative mass, which was never obtained before. A possibility of forming such singularities by a collapse is discussed.

Introduction

The description of scalar fields in relativity theory is associated with some indeterminacy. Even considering no exotic variants with nontrivial Lagrangians, we arrive at the field equations

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{ik}\psi_{,k}\right)+m^2\psi-\xi R\psi=0 \qquad (1)$$

which include the parameter ξ called a coupling. This parameter appears in the energy-momentum tensor of the field deduced by Chernikov and Tagirov in [1] and then by Callan, Coleman, and Jackiw in [2]:

$$T_{ik}=\psi^*_{,i}\psi_{,k}+\psi_{,i}\psi^*_{,k}-g_{ik}\left[\psi^{*,l}\psi_{,l}-(m^2-\xi R)|\psi|^2\right]+$$
$$+2\xi\left[R_{ik}-\nabla_i\nabla_k+g_{ik}\nabla_i\nabla^i\right]\ |\psi|^2. \qquad (2)$$

Here, m is the mass of a quantum of the field which is considered to be complex in the general case. For a real field, formula (2) must be multiplied by $1/2$. In this work, we use the signature, signs of the curvature tensor, and designations from [3]. We take the Einstein gravity constant and the light velocity to be equal to unity.

Most frequently are used two values of the parameter ξ. At $\xi=0$, we get a scalar field with minimum coupling. If $\xi=1/6$ and, in addition, $m=0$, we deal with a conformally invariant massless scalar field [4] frequently used in quantum gravitation. Using (1) and (2), it is easy to prove that, for $m=0$ and $\xi=1/6$, we get $T=0$ and thus $R=0$. Just this fact confirms the conformal invariance of the field.

Distinctions between different variants of scalar fields increase with the space-time curvature and manifest themselves most clearly near singularities of the space-time. In the previous work [5], we analyzed Kasner singularities in the form of an infinite homogeneous thread as the source of a gravitational field and a massless scalar one. The analysis of the space-time showed that its structures are similar for any coupling parameter $\xi\neq 0$ and differ qualitatively from those in the case with minimum coupling $\xi=0$ which was considered earlier in [6].

However, a type of singularities cannot be determined more exactly due to their infinite length. An analogous problem arose also in the absence of scalar field [7]. To solve it, we needed to consider the space-time with a finite-length singularity possessing a constant linear density. This space-time was described by the Zipoy — Voorhees metric [8, 9]

$$ds^2=\mathrm{th}^{2\mu}\frac{v}{2}\times dt^2-\frac{L^2}{4}\mathrm{sh}^2v\times\mathrm{th}^{-2\mu}\frac{v}{2}\times$$

$$\times\left[\left(1+\frac{\cos^2 u}{\mathrm{sh}^2 v}\right)^{1-\mu^2}(du^2+dv^2)+\cos^2 u\times d\varphi^2\right]. \quad (3)$$

At $\mu = 0$, the coordinates v, u, and ϕ pass into those of a prolate spheroid with the distance between focuses L. The physical sense of this solution is studied in [10], where the method of determination of the types of time-like singularities was developed as well.

In the case where the singularity $v = 0$ is also the source of a massless field with minimum coupling, a generalization of (3) was derived and analyzed in work [11]. In the present work, we generalize solution (3) for a singularity, being the source of a conformally invariant field. The derived solutions are completely different from those for metrics with minimum coupling. The analysis of types of the singularity $v = 0$ showed that, in this case, we meet previously unknown variants such as linear or paradoxical singularities with negative mass as well as surface singularities.

1. Finite Massive Thread as the Source of a Conformally Invariant Scalar Field

We seek for a generalization of the Zipoy — Voorhees metric (3) to the case where a thread is also the source of a conformally invariant field as

$$ds^2 = e^{2\alpha(v)}\mathrm{th}^{2\mu}\frac{v}{2}dt^2 - \frac{L^2}{4}\mathrm{sh}^2 v\times\mathrm{th}^{-2\mu}\frac{v}{2}\times$$

$$\times\left[e^{2\beta(v)}\left(1+\frac{\cos^2 u}{\mathrm{sh}^2 v}\right)^{1-\mu^2}\times\right.$$

$$\left.\times(du^2+dv^2)+e^{2\gamma(v)}\cos^2 u\times d\varphi^2\right]. \quad (4)$$

We assume also that the field ψ depends only on the coordinate v. A similar functional dependence is also revealed by the metrics of both an electroststic field and a scalar field with minimum coupling [11].

With regard for $R = 0$ and $m = 0$, relation (1) yields that

$$\psi_{,v} = \frac{\Lambda e^{i\kappa}}{\mathrm{sh}v}e^{-\alpha-\gamma}, \quad (5)$$

where Λ and κ are real constants. The (12)-component of the Einstein equations takes the form

$$\left(1-\frac{1}{3}|\psi|^2\right)\left[(\gamma-\beta)_{,v}\mathrm{tg}\,u-(\alpha+\gamma)_{,v}(1-\mu^2)\times\right.$$

$$\left.\times\frac{\sin u\cos u}{\mathrm{sh}^2 v+\cos^2 u}\right] = \frac{\mu^2-1}{3}\frac{\sin u\cos u}{\mathrm{sh}^2 v+\cos^2 u}|\psi|^2_{,v}, \quad (6)$$

which is possible if the following two relations are satisfied:

$$\beta_{,v} = \gamma_{,v}; \quad (\alpha+\gamma)_{,v} = \frac{|\psi|^2_{,v}}{3-|\psi|^2}. \quad (7)$$

While integrating, we choose the constants so that, for $\psi = 0$, we would get $\alpha = \beta = \gamma = 0$, i.e., solution (3):

$$\beta = \gamma, \quad \alpha+\gamma = -\ln\left|1-\frac{1}{3}|\psi|^2\right|. \quad (8)$$

The first condition guarantees the absence of conic singularities on the axis $v \neq 0$, $\cos u = 0$. By substituting (8) into (5), we obtain

$$|\psi|_{,v} = \frac{\Lambda}{\mathrm{sh}v}\left|1-\frac{1}{3}|\psi|^2\right|. \quad (9)$$

This yields

$$\psi = \sqrt{3}\left|\frac{C-\mathrm{th}^p\frac{v}{2}}{C+\mathrm{th}^p\frac{v}{2}}\right|, \quad p = \frac{2\Lambda}{\sqrt{3}}, \quad C = \mathrm{const}. \quad (10)$$

From the (00)-component of the Einstein equations, we get

$$\alpha_{,vv}+\alpha_{,v}\,\mathrm{cth}v = \frac{\Lambda^2}{3\mathrm{sh}^2 v}\left|1-\frac{1}{3}|\psi|^2\right| =$$

$$= \frac{4\Lambda^2}{3\mathrm{sh}^2 v}\frac{\mathrm{cth}^p\frac{v}{2}}{\left(C+\mathrm{th}^p\frac{v}{2}\right)^2}. \quad (11)$$

A solution of this equation has the form

$$\alpha = \ln\left|C+\mathrm{th}^p\frac{v}{2}\right|+C_1\ln\,\mathrm{th}\frac{v}{2}+C_2;\ C_1,\ C_2 = \mathrm{const}. \quad (12)$$

From (8), (10), and (12), we obtain

$$\beta = \gamma = \ln\left|C+\mathrm{th}^p\frac{v}{2}\right|+C_3\ln\mathrm{th}\frac{v}{2}+C_4, \quad (13)$$

$$C_1+C_3 = -p, \quad C_2+C_4 = -\ln|4C|. \quad (14)$$

In this case, the (22)- and (33)-components of the Einstein equations are identically valid, and the (11)-component holds if the following two conditions are satisfied: $C = 1$ and

$$\mu\,(C_3-C_1) = C_1^2+C_3^2+C_1C_3. \quad (15)$$

From relations (14) and (15), we get

$$C_1 = -\frac{p}{2} - \mu \pm \sqrt{\mu^2 - \frac{3}{4}p^2},$$

$$C_3 = -\frac{p}{2} + \mu \mp \sqrt{\mu^2 - \frac{3}{4}p^2}. \qquad (16)$$

The signs of the roots and values of C_2 and C_4 are chosen in view of the condition $\alpha, \beta, \gamma \underset{p\to 0}{\longrightarrow} 0$. For $\mu>0$ or $\mu<0$, it is necessary to take, respectively, the upper or lower sign. It is seen from (16) that the scalar charge of a singularity is bounded from above: $\mu^2 \geq \frac{3}{4}p^2 = \Lambda^2$. Thus, a solution has the form (4) with

$$\alpha = \ln\left[\frac{1}{2}\left(1 + \mathrm{th}^p \frac{v}{2}\right)\right] +$$

$$+ \left(\pm\sqrt{\mu^2 - \frac{3}{4}p^2} - \mu - \frac{p}{2}\right) \ln \mathrm{th}\frac{v}{2}, \qquad (17)$$

$$\beta = \gamma = \ln\left[\frac{1}{2}\left(1 + \mathrm{th}^p \frac{v}{2}\right)\right] +$$

$$+ \left(\mu - \frac{p}{2} \mp \sqrt{\mu^2 - \frac{3}{4}p^2}\right) \ln \mathrm{th}\frac{v}{2}, \qquad (18)$$

and the field

$$\psi = \sqrt{3}\frac{1 - \mathrm{th}^p \frac{v}{2}}{1 + \mathrm{th}^p \frac{v}{2}} \qquad (19)$$

turns to $\sqrt{3}$ for $v = 0$ and to zero as $v\to \infty$. As $p \to 0$, the field ψ drops significantly for v of the order of $\exp(-p^{-1})$. It is easy to see that the sole singularity occurs at $v = 0$. Far from this point, the space-time becomes asymptotically flat, being written in prolate spherical coordinates. For $v \to \infty$, they are connected with the spherical transformation $r \to \frac{L}{4}e^v$. In view of the asymptotic form of the metric and the field,

$$g_{00} \to 1 \mp \frac{\sqrt{4\mu^2 - 3p^2}}{2r}L, \quad \psi \to \frac{\sqrt{3}Lp}{4r}, \qquad (20)$$

we found the masses of the singularity and the ambient scalar field,

$$M = \pm\frac{\sqrt{4\mu^2 - 3p^2}}{4}L, \qquad (21)$$

and its scalar charge

$$Q = -\frac{\sqrt{3}}{4}Lp = -\frac{\Lambda L}{2}. \qquad (22)$$

2. Analysis of the Singularity Types

Consider the space-time near the singularity $v = 0$. At a large distance from the singularity ends ($u = \pm\pi/2$), e.g., for $u \approx 0$, the solution acquires the asymptotic form

$$ds^2 \underset{v\to 0}{\longrightarrow} Av^{2q_1}dt^2 + Bv^{2q_3}\left(du^2 + dv^2\right) + Cv^{2q_2}d\varphi^2, \qquad (23)$$

where A, B, and C are constants and

$$2q_1 = \pm\sqrt{4\mu^2 - 3p^2} - p, \quad 2q_3 = 2\mu^2 - p \mp \sqrt{4\mu^2 - 3p^2},$$

$$2q_2 = 2 - p \mp \sqrt{4\mu^2 - 3p^2}. \qquad (24)$$

For any admissible values of the parameters p and μ, we have $q_3 > -1$, and the singularity is located at a finite distance from points with nonzero v, i.e., it is not a directional singularity. We pass now to the variable $x = v^{1+q_3}$ and change the scales on the axes. Then asymptote (23) has the Kasner form

$$ds^2 = x^{2p_1}dt^2 - dx^2 - x^{2p_2}dy^2 - x^{2p_3}dz^2. \qquad (25)$$

Its indices p_i equal

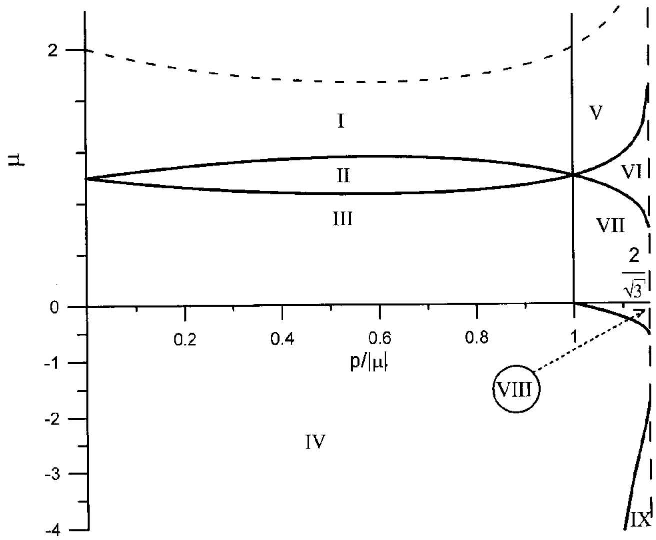
$$p_i = \frac{q_i}{1 + q_3} \qquad (26)$$

and, naturally, are not connected by the Kasner relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 \qquad (27)$$

which hold in the absence of a scalar field.

The signs of p_i coincide with those of q_i. According to (24), we can divide the entire region of values of μ and the ration $p/|\mu|$ (the latter belongs to the interval from 0 to $2/\sqrt{3}$) into 9 zones whose indices p_i have a certain sign (see. the figure). These signs define the form of the diagram introduced in [10], by which one can define the singularity type. In this case, considering the mass sign, we study the mass of the very singularity not adding the scalar field energy to it, like in (21). Zone I ($p_1 > 0$, $p_2 < 0$, $p_3 > 0$) corresponds to the paradoxical type of singularities with positive mass introduced in [10]. In zone II ($p_1 > 0$, $p_2 < 0$, $p_3 < 0$), the singularity $v = 0$ is a surface and has a positive mass. Zone III ($p_1 > 0$, $p_2 > 0$, $p_3 < 0$) covers the region of linear singularities with positive mass density. In the rest 6 zones, the singularity has a negative mass. In this case, the total mass of the singularity and the field (21) can be positive. Zones IV and VI ($p_1 < 0$, $p_2 > 0$, $p_3 > 0$) correspond to point-like singularities, zones V and IX



Regions with different types of naked singularities. The scales of both ordinate semiaxes are different

($p_1 < 0$, $p_2 < 0$, $p_3 > 0$) do to paradoxical ones, and zones VII and VIII ($p_1 < 0$, $p_2 > 0$, $p_3 < 0$) do to linear ones.

The obtained pattern differs considerably from the case with analogous metric (4) studied in [11], in which the singularity is the source of a massless scalar field with minimum coupling. A point-like singularity with positive mass appeared there is not observed in the case under study. However, for an arbitrary small scalar charge, a region of surface singularities appears near $\mu = 1$. This region separates the regions of linear and paradoxical types of singularities. If the limiting value $p = \mu$ is exceeded, singularities of new types, namely linear and paradoxical types with negative masses, become possible.

A special attention should be paid to the ends of singularities, to the points $v = 0$ and $u = \pm\pi/2$. Already without a scalar field for $\mu \geq 2$, they contain two directional singularities which correspond to infinitely removed points [10]. By moving to them along the axes $\cos u = 0$, we get the asymptotic form of space-time-(23) with the same q_1 and q_2 (24), but with

$$q_3 = 2 - p \mp \sqrt{4\mu^2 - 3p^2}. \tag{28}$$

Additional spatial infinities arise for $q_3 < -2$. In the figure, their limits are drawn with a dashed line. The minimum value of μ, at which the singularities with respect to direction arise, decreases to $\mu = \sqrt{3}$ for $p = 1$. In the region where $\mu < 0$, similar singularities are possible only in the presence of a conformally invariant scalar field.

The asymptotic form of the metric near a singularity, but far from its ends (23, 24) must coincide with a solution near an infinitely long singularity. Because the field ψ is finite, we may take solution (25) from [5]. Comparing two collections of indices p_i for weak scalar fields, we get the coupling

$$\Lambda \underset{\lambda\to 0}{\longrightarrow} \sqrt{\frac{2}{3q}}\left(1 + \mu^2 - \mu\right)\lambda. \tag{29}$$

Because the parameter q is arbitrary, a finite interval of variation in Λ corresponds to an infinite interval for λ.

Conclusion

We have derived an exact solution of the Einstein equations which generalize (3) to the case where the singularity $v = 0$ is the source of both a gravitational field a conformally invariant scalar one. The analysis of a singularity with finite length allows us to study the singularity type. Six possible variants contain the types unknown up to now. These are linear and paradoxical sources with negative mass and surface singularities. All they cannot appear in the case of a field with minimum coupling. Thus, the coupling of a scalar field influences significantly the type and properties of its singular source.

For $\mu = 1$, the singularity $v = 0$ becomes fictitious, and solution (3) can be transferred by a simple coordinate transformation to the space-time beyond the horizon of a Schwartzschild black hole (just it corresponds to $v = 0$). In the presence of a conformally invariant scalar field, the singularity $v = 0$ remains to be true even for $\mu = 1$. Therefore, the central source of a scalar field in the centrally symmetric space-time is a naked singularity rather than a black hole, which confirms the conclusions made in [12].

Solution (3) occupies the important place in the hierarchy of naked time-like singularities [10]. Two next levels in generality with the same type of singularities are occupied by Weyl's singularities and Israel's simple linear sources. Therefore, the qualitative conclusions on the types of singularities, which were obtained in the analysis of the specific solution (4), (17), and (18), can be used in more general situations.

While considering the validity of the Penrose's hypothesis of cosmic censorship, i.e. while studying the question about the possibility of the creation of naked singularities under a collapse, it is worth to discuss the influence of a hypothetical scalar charge on the collapse course. For a field with minimum coupling, an arbitrary small scalar charge leads to the qualitative change in the type of the most general solution near a singularity [6, 11]. This does not occur in the case of a field with nonminimum coupling. All the exotic types of singularities require the minimum limiting value of a scalar charge and thus cannot appear under a collapse.

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УЗАГАЛЬНЕННЯ
МЕТРИКИ ЗІПОЯ—ВУРХІЗА В ПРИСУТНОСТІ
КОНФОРМНО-ІНВАРІАНТНОГО СКАЛЯРНОГО ПОЛЯ

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Р е з ю м е

Знайдено і досліджено узагальнення відомої метрики Зіпоя—Вурхіза у випадку, коли її центральна часоподібна сингулярність є також джерелом конформно-інваріантного скалярного поля. Метрика Зіпоя—Вурхіза, відома також як γ-розв'язок, відіграє важливу роль в ієрархії голих особливостей, і її властивості, як правило, характерні для більш загальних класів розв'язків. Аналіз виникаючої на осі часоподібної голої сингулярності показує, що її тип може відноситися до шести різних видів. Серед них є особливості на поверхні, а також лінійні і парадоксальні сингулярності негативної маси, що не зустрічалися в раніше розглянутих випадках. Обговорюється можливість утворення подібних особливостей при колапсі.