SCALING ORIGIN OF FLICKER NOISE

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Using the scaling approach, it is shown that the hierarchy of correlation times is the cause for the flicker noise appearance.

Introduction

The spectrum S(ω) and the correlation function $K(\tau)$ of some stationary stochastic process are connected by the well-known formula

$$S(\omega) = \int_{-\infty}^{+\infty} (-i\omega\tau) K(\tau) d\tau / 2\pi. \quad (1)$$

In some cases (see, e.g., [1]), the asymptotic behavior of the spectrum of a system is described by

$$S(\omega) \sim \omega^{-\beta} \ \ (\omega \to 0), \quad (2)$$

where $0 < \beta \leq 1$. The stochastic process with such a spectrum is called the flicker noise.

In the present work, we use the scaling approach for the description of the flicker noise. To the best of our knowledge, this approach was not applied earlier to solve the problem (see, e.g., [2, 3]), though the scaling concepts were considered in many publications (see, e.g., [4]).

1. The Hierarchy Model of a Fluctuating System

Let the behavior of a fluctuating system be described by the dependence of a parameter X on time t', and let the function $X(t')$ be a stationary stochastic process. Its correlation function is $K(\tau)$. By θ_2, we denote the correlation time of this process. We suppose that one more process runs in the system with the correlation time θ_1 that satisfies the condition

$$\theta_1 \ll \theta_2. \quad (3)$$

Such a condition is called usually a hierarchy. Accordingly, we call the model of a system, for which this condition is fulfilled, as the hierarchy model.

2. Independence of the Correlation Function on the Scale in the Hierarchy Model

We take θ_1 to be the measuring unit (the scale) of the time scale τ.

Divide this scale into the intervals $h = \lambda\theta_1$, where λ is the scale coefficient. We shall take h to be the measuring unit (the scale) of the new time scale

$$t = \lambda\tau. \quad (4)$$

Introduce the smoothed correlation function

$$H(t) = \int_{t}^{t+h} K(\tau) d\tau / h. \quad (5)$$

Let the times τ and t satisfy the conditions

$$\theta_1 \ll \tau \ll \theta_2, \quad (6)$$

$$\theta_1 \ll t \ll \theta_2, \quad (7)$$

We expand $K(\tau)$ in a series in τ. Owing to condition (6), one can limit oneself by the zero-degree term by writing

$$K(\tau) = K(0). \quad (8)$$

In this case, the equalities

$$H(t) = K(t), \quad (9)$$

$$K(t) = K(\tau) \quad (10)$$

are valid, i.e. the correlation function does not on the scale.

3. The Spectrum is a Homogeneous Function in the Hierarchy Model

The spectrum corresponding to function (5) has the form

$$Q(\omega) = \int_{-\infty}^{\infty} (-i\omega t)H(t)dt/2\pi. \quad (11)$$

Substituting (9) into (11), we get

$$Q(\omega) = \int_{-\infty}^{\infty} (-i\omega t)K(t)dt/2\pi. \quad (12)$$

Comparing (12) and (1) gives

$$S(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega t)K(t)dt/2\pi. \quad (13)$$

Substituting (4) into (13), we obtain

$$S(\omega) = \left\{ \int_{-\infty}^{\infty} (-i\omega\lambda\tau)K(\lambda\tau)\lambda d\tau) \right\} /2\pi. \quad (14)$$

Substituting (10) into (14), we obtain

$$S(\omega) = \int_{-\infty}^{\infty} (-i\omega\lambda\tau)K(\tau)\lambda d\tau)2\pi. \quad (15)$$

Comparing (15) and (1) leads to

$$S(\omega) = S(\omega\lambda)\lambda. \quad (16)$$

It follows from equality (16) that S(ω) must be a homogeneous function of degree -1:

$$S(\omega) \sim \omega^{-1}. \quad (17)$$

4. Spectrum Exponents in the Hierarchy Model

The exponent -1 was obtained because of formula (8). In this case, the equality

$$\int_{t}^{t+h} K(\tau)d\tau = K(0)h \quad (18)$$

is valid. But let us assume that the function $K(\tau)$ has a small ripple. Then, instead of (18), we obtain

$$\int_{t}^{t+h} K(\tau)d\tau \sim h^{\beta}, \quad (19)$$

where the exponent $\beta < 1$. Now, instead of (17), we get

$$S(\omega) \sim \omega^{-\beta}. \quad (20)$$

It is clear that formulae (17), (20) reveal the form of the flicker noise spectrum, $S(\omega)$.

5. The Example: the Diffusion Model of Flicker Noise is a Particular Case of the Hierarchy Model

According to the above considerations, the flicker noise arises when condition (3) is fulfilled. It is enough a general condition. Therefore, if it is valid, it must be fulfilled for various physical models of flicker noise. We will test whether condition (3) is fulfilled for the well-known diffusion model [3].

In this model, a stochastic process $X(t')$ can be presented in the form of a sum of independent stochastic values $X_J(t')$, namely

$$X(t') = \sum_{j} X_j(t'). \quad (21)$$

The value $X_J(t')$ is a certain diffusion mode with the wave vector $\mathbf{k}$ and the time relaxation $\theta_{\mathbf{k}} = (D\mathbf{k}^2)^{-1}$, where D is the diffusion coefficient. We also introduce into consideration the diffusion volume with the size $L = (D/\omega)^{1/2}$.

We assume that the condition

$$L \gg l, \quad (22)$$

is fulfilled, where l is the size of the system.

Then we introduce an ensemble, for which the values $\mathbf{k}$ are given statistically with the help of the distribution function

$$\Psi(\mathbf{k}) = (\mathbf{L^2/2\pi})^{3/2} \exp(-\mathbf{L^2k^2/2}). \quad (23)$$

Now the spectrum is described by the formula

$$S(\omega) = \int D\mathbf{k}^2(\omega^2 + D^2\mathbf{k}^4)^{-1}\Psi(\mathbf{k})d\mathbf{k}. \quad (24)$$

Substituting equality (23) into (24), we get relation (17).

As seen, the diffusion model of flicker noise is based on the assumption of the hierarchy for space scales (22). It is not difficult to pass to the hierarchy of time scales (3) from hierarchy (22). Thus, we see that the diffusion model is a particular case of the more general hierarchy model. It is not necessary to take the additional assumption (23) in the diffusion theories. Without this assumption, we must get (17), as it has been shown above.

Conclusions

Based on the obtained results, we conclude that conditions (3), (6), and (7) are sufficient for the appearance of the flicker noise in a fluctuating system. The thing is that, under these conditions, the correlation function of a stochastic process describing the behavior of the fluctuated system does not depend on the scale and, therefore, the spectrum of this process is a homogeneous function of frequency. Such a property of the spectrum leads to the spectrum form characteristic of the flicker noise.

APPENDIX

The Limiting Passage for Correlation Functions and Spectra

Some trouble concerning equality (8) may arise. Indeed, when we substitute (8) in (1), the corresponding integral does not exist. The answer consists in that we consider, in this case, the limiting form $K_0(\tau)$ of the function $K(\tau)$, which we obtain by passing to the limit $\tau \to \infty$, $\theta_2 \to \infty$ with condition (6)

$$K_0(\tau) = \lim_{\tau\to\infty,\,\theta_2\to\infty} K(\tau) \quad (\tau << \theta_2). \tag{25}$$

Accordingly, the limiting function $K_0(\tau)$ is determined on all the τ axis, and it is correct to write

$$K_0(\tau) = K(0)I(\tau), \tag{26}$$

instead of formula (8), where $I(\tau)$ is the function which is equal to 1 at every point of the τ axis.

For the scale transformation (4), some transformation of the correlation function, $K \to K'$, happens. We consider the functions K and K' as elements of some functional space S_K and denote the last transformation as $K' = AK$. We believe also that K_0 is an element of this space. It is obvious that K_0 is the *immovable point* of the transformation A. In fact, we assume that the functions like K and K' form the vicinity of this point.

For the correlation function (26), the spectrum is

$$S_0(\omega) = K(0)\delta(\omega), \tag{27}$$

where $\delta(\omega)$ is the delta-function.

It is obvious that function (27) possesses property (16). Accordingly, the Fourier transforms of elements of the space S_K form some functional space S_S . In the space S_S, the transform $S' = BS$ corresponds to the transform $K' = AK$, and the delta-function S_0 is the immovable point of the transform B. We accept that the vicinity of this point is formed by the functions satisfying the same condition (15) which is valid for the function S_0. What functions form this vicinity?

As known, the delta-function is the limit of some delta-sequence. Let the terms of this sequence be

$$S' = K(0)\{e(\omega + b) - e(\omega b)\}/2b, \tag{28}$$

where $e(\omega)$ is the unit function. One can consider S' as the approximate delta-function. These terms form the vicinity of the delta-function in the space S_S.

Excluding the Scale Dependence of the Spectrum

Formulae (26) and (27) are idealized. Indeed, function (8) is determined on some finite interval

$$-\eta < \tau \leq \eta. \tag{29}$$

Accordingly, the smallest value of the ω measuring unit is equal to π/η. This is the scale of the spectrum. Since $S(\omega)$ is an even function, the smallest value $\omega_{\rm m}$ is described by the formula

$$\omega_{\rm m} = \pi/2\eta. \tag{30}$$

In view of condition (28), it is impossible to approach the zero of ω nearer than at the distance $\omega_{\rm m}$. Therefore, one can find $2\omega_{\rm m}$as the width $2b$ of the approximate delta-function. From the known property of the delta-function, the height of the approximate delta-function is equal to $(2\omega_{\rm m})^{-1}$. Thus, each term of the sequence corresponds to a certain time η. In accordance with formula (30), the certain value of $\omega_{\rm m}$ corresponds to each value of η. Respectively, for each value of η, we have the certain value $(2\omega_{\rm m})^{-1}$ of the height of the approximate delta-function. Moreover, we can write formula (28) as

$$S' = K(0)\{e(\omega + \pi/2\eta) - e(\omega - \pi/2\eta)\}\eta/\pi. \tag{31}$$

It follows from formula (31) that S' is a function of two variables,

$$S' = S'(\omega, \eta). \tag{32}$$

But the spectrum cannot be a function of two variables. The quantity S' can depend only on ω. The dependence of S' on η means that S' depends on the scale. It is necessary to exclude the dependence S' on η. For this purpose, the certain function

$$\eta = \eta(\omega) \tag{33}$$

should be selected. The quantities ω and η cannot be changed independently.

The condition $\omega \to 0$ in formula (2) means that the behavior of the function is considered for the nonzero values of ω. But, for each η, only one value exists, for which we can confirm with confidence that this value differs from zero. This is $\omega = \omega_{\rm m}$. Therefore, we can use equality (30) writing

$$\omega = \pi/2\eta. \tag{34}$$

Such a form of function (33) leads to the formula for the height of the approximate delta-function (in fact, for the value of the function $S'(\omega)$) which agrees with formula (17).

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СКЕЙЛІНГОВЕ ПОХОДЖЕННЯ ФЛІКЕР-ШУМУ

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Р е з ю м е

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