CALCULATION OF CRITICAL PARAMETERS IN SU(2) GAUGE THEORY WITH THE KOUVEL—FISHER METHOD

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We calculate the critical coupling $4/g_c^2$ and critical exponent β for the order parameter in SU(2) lattice gauge theory by applying the finite-size scaling technique and the method proposed by Kouvel and Fisher for the analysis of experimental data. In contrast to the standard finite size scaling approach, this method allows us to determine simultaneously both $g_c^2/4$ and β as two parameters of the linear fit to the Monte Carlo (MC) data.

Lattice calculations by the MC method have so far been the only method for studying the temperature phase transition in gauge theories from the first principles. By their nature, MC calculations are carried out on finite lattices. This is a most serious drawback of computer simulations, because no finite system can exhibit a true phase transition. Nevertheless, finite systems remind us of phase transitions, and systematic studies of these pseudo-transitions as functions of a system size may reveal information about a phase transition in the thermodynamic limit. One way to do this particularly in order to evaluate the critical exponents in the theory is to use the finite-size scaling (FSS).

FSS was proposed in statistical physics [1, 2] and applied to the high precision MC data of the Ising [3, 4] and other models. The validity of this method for gauge theories has been demonstrated through the investigation of critical properties of SU(2) lattice gauge theories at finite temperatures [5–9].

Here, we propose a method of analysis of MC data which is a combination of the FSS technique and the so-called Kouvel—Fisher method for the analysis of experimental data in solid-state physics [11]. We consider MC data for the SU(2) order parameter on the lattices $8^3 \times 4$, $12^3 \times 4$ and $18^3 \times 4$. In contrast to the standard FSS approach, our method allows one to determine the critical coupling $4/g_c^2$ and critical exponent β as two parameters of a linear fit.

The partition function for SU(2) gauge theory on the $N_\sigma^3 \times N_\tau$ lattice is

$$Z = \int \prod_{(\mu,\nu)} dU_{\mu,\nu} e^{-S(U)}, \qquad (1)$$

where $U_{\mu,\nu}$ are the SU(2) matrices on the link (μ,ν), S is the standard Wilson action

$$S(U) = \frac{4}{g^2} \sum_p \left(1 - \frac{1}{2} Tr Up\right), \qquad (2)$$

and Up is a product of link matrices around a plaquette. The temperature T (volume V) is defined by a size of the lattice in the time-like (space-like) direction (a is a lattice spacing)

$$T = 1/(N_\tau a), \qquad V = (N_\sigma a)^3 . \qquad (3)$$

The order parameter of the deconfinement phase transition on an infinite volume lattice is the expectation value of the lattice average of a Polyakov loop

$$\langle L \rangle = \left\langle \frac{1}{N_\sigma^3} \sum_{\mathbf{n}} L_{\mathbf{n}} \right\rangle, \qquad (4)$$

where

$$L_{\mathbf{n}} = \frac{1}{2} Tr \prod_{\tau=1}^{N_\tau} U_{\tau,\mathbf{n};0} \qquad (5)$$

and $U_{\tau,\mathbf{n};0}$ is the SU(2) link matrix at the point $(\mathbf{n},\tau)$ in the time-like direction. On the finite lattice, one considers $\langle|L|\rangle$ as the order parameter, since $\langle L\rangle$ is always zero in this case due to flips of the system between the two ordered states.

According to the FSS theory, the scaling function for the order parameter on the lattice $N_\sigma^3 \times N_\tau$, N_τ is fixed, is given by [2]

$$N_\sigma^{\frac{\beta}{\nu}} \langle|L|\rangle = Q_L\left(x, g_1 N_\sigma^{-y_1}\right), \quad (6)$$

where β, ν are the critical exponents for order parameter and correlation length correspondingly, g_1 is the irrelevant scaling field, and $y_1 \approx 1$. The scaling variable x is defined as

$$x = t \cdot N_\sigma^{\frac{1}{\nu}}, \qquad t = \frac{g_c^2/4 - g^2/4}{g_c^2/4}. \quad (7)$$

Our definition of reduced temperature t differs from one used in [7—10],

$$t = \frac{4/g^2 - 4/g_c^2}{4/g_c^2}, \quad (8)$$

and seems to be more natural because the coupling constant $g^2/4$ plays just the role of a "temperature" of the effective theory which can be obtained from the partition function (1) by integration over all link variables except those for Polyakov loops. It is clear that this difference disappears in the limit $g^2/4 \to g_c^2/4$.

At large N_σ, the scaling function Q_L must behave as

$$Q_L \sim (x)^\beta . \quad (9)$$

This behavior may be observed only if x becomes "sufficiently large". Deviations from (9) are in a close vicinity of the critical point due to the finite size rounding and further away from $x = 0$ due to a correction to scaling. Expanding Q_L in (6) around $x = 0$ at large N_σ results in

$$N_\sigma^{\frac{\beta}{\nu}} \langle|L|\rangle = a + \frac{b}{N_\sigma^{y_1}} + \left(c + \frac{d}{N_\sigma^{y_1}}\right) x, \quad (10)$$

where a, b, c, and d are unknown constants. Dropping the term $O(N_\sigma^{-y_1})$, one can obtain

$$N_\sigma^{\frac{\beta}{\nu}} \langle|L|\rangle = a + cx, \quad (11)$$

what is exactly the approach of [7—10]. On the other hand, the x-dependence is drastically changed at x "sufficiently large". In [6], the scaling function Q_L has been calculated in the form

$$Q_L = Ax^\beta \left[1 + \frac{B}{N_\sigma^{y_1}} x^{y_1 \nu}\right] \quad (12)$$

by the data collapsing method. It has been shown that the correction to the scaling term in (12) is negligibly small as compared to the leading term on the interval $0.5 < x < 2$. In both cases, the SU(2) MC data for the order parameter have been used which were computed on the $N_\sigma^3 \times N_\tau$ lattices with N_σ=8, 12, 18, 26, and $N_\tau = 4$ and have already reported in [5—10]. The numerical value of the critical exponent β has been found which is compatible with the corresponding value $\beta = 0.327$ of the 3D Ising model [4].

These two procedures have two disadvantages: i) $\nu = 0.631$ has to be used as the 3D Ising model value; ii) a very accurate calculation of the critical coupling $g_c^2/4$ is necessary. The latter is determined by the fourth cumulant of $\langle|L|\rangle$ as in [5, 7] or by the χ^2-method as proposed in [8]. Even a small shift leads to the additional uncertainty in the determination of the critical exponent β, because it depends very sensitively on the value inserted for $g_c^2/4$.

We propose below a method for the determination of the critical exponent β which does not require to know $g_c^2/4$ from the very beginning. No information about the critical exponent ν is also required. Following the Kouvel—Fisher idea which was used for the analysis of experimental data in solid-state physics [11], we consider the function

$$K(x) = \frac{Q_L}{\partial Q_L/\partial x}. \quad (13)$$

According to the results of [6] in the region $0.5 < x < 2$, relation (12) yields easily that

$$K(x) = \frac{x}{\beta}\left[1 + \frac{\text{const}}{N_\sigma^{y_1}} x^{y_1\nu}\right]. \quad (14)$$

Dropping the correction to the scaling term, we have

$$K(x) = \frac{x}{\beta} \quad (15)$$

in this region with the same accuracy as in (11). Taking into account the identity

$$\frac{\partial Q_L}{\partial x} = -\frac{g_c^2}{4} N_\sigma^{-\frac{(1-\beta)}{\nu}} \frac{\partial\langle|L|\rangle}{\partial(g^2/4)}, \quad (16)$$

we have immediately

$$K(x) = \frac{x}{\beta} = -\frac{N_\sigma^{\frac{1}{\nu}}}{g_c^2/4} \frac{\langle|L|\rangle}{\partial\langle|L|\rangle/\partial(g^2/4)}. \quad (17)$$

Finally, we obtain the simple but informative expression

$$\frac{\langle|L|\rangle}{\partial\langle|L|\rangle/\partial(g^2/4)} = \frac{1}{\beta}\left(\frac{g^2}{4} - \frac{g_c^2}{4}\right). \quad (18)$$

Eq. (18) means that, for the lattices with N_τ fixed and various N_σ, there are the intervals of $g^2/4$ where MC data for different N_σ must arrange around the same straight line. The inverse slop of this line represents the critical exponent β and the intersection with the $g^2/4$ axis yields the critical coupling.

In order to verify this approach we have used the high precision MC data produced by the Bielefeld group on the lattices $8^3 \times 4$, $12^3 \times 4$, and $18^3 \times 4$ [5, 10]. The calculation of derivatives with respect to $g^2/4$ has been made by using the parabolic fit for $\langle|L|\rangle$ (every parabola includes five MC points). This fit does not smooth the statistical fluctuations, but connects the data continuously and with the continuous first and second derivatives. The least minimal χ^2-fit of Eq. (18) ($\chi^2/N = 0.62$) includes 7 MC points in the region $0.396 < g^2/4 < 0.426$ ($2.345 < 4/g^2 < 2.527$) for $N_\sigma = 8$, 24 MC points in the region $0.414 < g^2/4 < 0.430$ ($2.325 < 4/g^2 < 2.415$) for $N_\sigma = 12$ and 16 MC points in the region $0.427 < g^2/4 < 0.433$ ($2.310 < 4/g^2 < 2.340$) for $N_\sigma = 18$. The result is

$$\beta = 0.326 \pm 0.003; \qquad 4/g_c^2 = 2.2988 \pm 0.0007. \quad (19)$$

We see that the critical exponent β is in excellent agreement with the 3D Ising model value and the inverse critical coupling $4/g_c^2$ coincides with that value which has been calculated in [7–10] by other method.

In conclusion, the following comments are in order. We have shown that the Kouvel—Fisher modification of FSS gives the extremely useful method for the calculation of the critical exponent β in SU(2) lattice gauge theory. This method allows one to investigate a finite lattice system at small, but finite $t = (g^2/4 - g_c^2/4)/(g_c^2/4)$ and large N_σ with the infinite-volume formulae (18). The critical coupling $g_c^2/4$ and critical exponent β are simultaneously determined as two parameters of the linear fit. It is clear that this approach may be applied to the calculation of all critical exponents in SU(2) gauge theory and to other models which undergo the second-order phase transition.

1. *Binder K.*//Z. Phys. **B43** (1981) 119.
2. *Barber M.N., Pearson R.B., Toussaint D., Richardson J.*//Phys. Rev. **B32** (1985) 1720.
3. *Ferrenberg A.M., Landau D.P.*//Phys. Rev. **B44** (1991) 5081.
4. *Caselle M., Hasenbusch M.*//J. Phys. A **30** (1997) 4963; Nucl. Phys. B (Proc. Suppl.) **63** (1998) 613.
5. *Engels J., Fingberg J., Weber M.*//Nucl. Phys. **B332** (1990) 737.
6. *Averchenkov V.A., Mogilevsky O.A.*//Acta phys. slov. **41** (1991) 102.
7. *Engels J., Fingberg J., Miller D.E.*//Nucl. Phys. **B387** (1992) 501.
8. *Engels J., Fingberg J., Mitrjushkin V.K.*//Phys. Lett. **B298** (1993) 154.
9. *Engels J., Mashkevich S., Scheideler T., Zinovjev G.*//Ibid. **B365** (1996) 219.
10. *Engels J., Scheideler T.*//Nucl. Phys. **B539** (1999) 557.
11. *Kouvel J.S., Fisher M.S.*//Phys. Rev. **A136** (1964) 1626; *Braun P., Fähnle M.*//J. Stat. Phys. **52** (1988) 775.

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ОБЧИСЛЕННЯ КРИТИЧНИХ ПАРАМЕТРІВ SU(2) КАЛІБРУВАЛЬНОЇ ТЕОРІЇ МЕТОДОМ КУВЕЛЯ—ФІШЕРА

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Р е з ю м е

Обчислено критичну константу зв'язку $4/g_c^2$ та критичну експоненту β для параметра порядку SU(2) калібрувальної теорії на гратці за допомогою методу скінченновимірного скейлінгу та методу, запропонованого Кувелем та Фішером для аналізу експериментальних даних. Використаний підхід, на відміну від стандартного, дає можливість обчислити $4/g_c^2$ та β одночасно як два параметри лінійного припасування монте-карлівських даних.