SYMMETRIES OF UPPER HYBRID ELECTRON PLASMA WAVES

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A symmetry group of the integro-differential equations describing nonlinear upper hybrid waves in magnetized electron plasma is found. It is shown that the extension of the symmetry in the cold plasma limit allows us to build the general solution in this case. The results are compared to the symmetry properties of electron Langmuir waves.

Introduction

In recent decades, the Lie group method has been applied to explore many physically interesting nonlinear problems in gas dynamics, plasma physics, etc. [1 — 3]. Furthermore, extensions of the classical Lie algorithm to the integro-differential systems of equations of kinetic theory were proposed in [4, 5]. In this work, we generalize the results obtained previously [4] for electron plasma high frequency longitudinal waves in the absence of an external magnetic field to the case where the external magnetic field is present, i.e. for upper hybrid waves. In Section 1, the corresponding nonlinear integro-differential model equations are derived. In Section 2, symmetry transformations obtained for upper hybrid waves are presented together with their extension to the cold electron plasma case which made it possible to obtain a general solution in this case. For the sake of completeness, similar previous results for Langmuir waves are presented in Section 3. In Section 4, the obtained results are discussed and conclusions are made.

1. Model

Let us consider a high-frequency plasma motion with constant ion background density n_0. In this case, the Vlasov — Maxwell integro-differential system of equations holds:

$$\partial f/\partial t + \mathbf{v}\cdot\partial f/\partial \mathbf{r} -$$

$$-(e/m)[\mathbf{E} + (1/c)(\mathbf{v}\times\mathbf{B})]\partial f/\partial \mathbf{v} = 0,$$

$$\nabla\times\mathbf{E} + (1/c)\partial\mathbf{B}/\partial t = 0,$$

$$\nabla\cdot\mathbf{E} = 4\pi e(n_0 - \int f d\mathbf{v}),$$

$$\nabla\times\mathbf{B} = (1/c)\partial\mathbf{E}/\partial t - (4\pi e/c)\int \mathbf{v} f d\mathbf{v}, \nabla\cdot\mathbf{B} = 0, \quad (1)$$

where $f(t,\mathbf{r},\mathbf{v})$ is the electron distribution function. Let us assume that plasma is subjected to the action of a constant external magnetic field

$$\mathbf{B} = B_0\mathbf{e}_z,\ B_0 = \text{const},$$

so the electron cyclotron frequency is equal to $\omega_{ce} = eB_0/mc$. Let us restrict ourselves to transverse plasma movements (see, e.g., [6]):

$$\mathbf{E} = E(t,\,x)\,\mathbf{e}_x,\ \partial/\partial z = 0,\ \partial/\partial y = 0.$$

The electron distribution function f has been integrated over v_z and no longitudinal current is present, $\int v_z f dv_z = 0$. In this way, we obtain the simplified system of equations describing upper hybrid waves in the electron plasma:

$$\partial f/\partial t + v_x\partial f/\partial x - (e/m)E\partial f/\partial v_x +$$

$$+\omega_{ce}(v_x\partial f/\partial v_y - v_y\partial f/\partial v_x) = 0,$$

$$\partial E/\partial x = 4\pi e(n_0 - \int f d\mathbf{v}),$$

$$\partial E/\partial t = 4\pi e\int v_x f d\mathbf{v}, \quad (2)$$

where $d\mathbf{v} = dv_x dv_y$. This simplified system, however, still remains integro-differential. To aloid a possible nonlinear generation of the magnetic field, the external current density must be added:

$$\mathbf{j}_0 = j_0\mathbf{e}_y,\ j_0 = e\int v_y f d\mathbf{v}.$$

This statement completes the formulation of the model equations for upper hybrid waves.

2. Symmetries and Solutions for Upper Hybrid Waves

Continuous symmetry transformations for the integro-differential system (2) can be found by using the indirect method exploring the symmetry of an infinite set of equations for the moments of the distribution function f [4], as well as by the direct method developed recently in [5]. After some complicated but straightforward algebra, we obtain the following Lie symmetry group admitted by system (2):

$$X_1 = \partial/\partial t,\ X_2 = \partial/\partial x,$$

$$X_3 = \partial/\partial v_y - (m/e)\omega_{ce}\partial/\partial E,$$

$$X_4 = x\partial/\partial x + E\partial/\partial E + v_x\partial/\partial v_x +$$

$$+ v_y\partial/\partial v_y - 2f\partial/\partial f,$$

$$X_5 = \cos(\omega_{\rm uh}t)\partial/\partial x - \omega_{\rm uh}\sin(\omega_{\rm uh}t)\partial/\partial v_x +$$

$$+\omega_{ce}\cos(\omega_{\rm uh}t)\partial/\partial v_y + (m/e)\omega_{\rm uh}^2\cos(\omega_{\rm uh}t)\partial/\partial E,$$

$$X_6 = \sin(\omega_{\rm uh}t)\partial/\partial x + \omega_{\rm uh}\cos(\omega_{\rm uh}t)\partial/\partial v_x +$$

$$+\omega_{ce}\sin(\omega_{\rm uh}t)\partial/\partial v_y + (m/e)\omega_{\rm uh}^2\sin(\omega_{\rm uh}t)\partial/\partial E, \qquad (3)$$

where $\omega_{pe}^2 = 4\pi e^2 n_0/m$ is the electron Langmuir frequency and $\omega_{\rm uh}^2 = \omega_{pe}^2 + \omega_{ce}^2$ is the frequency of upper hybrid oscillations.

Considerable extension of the symmetry takes place in the cold electron plasma limit, i.e., for distribution functions of the form

$$f(t,\, x,\, v_x,\, v_y) \equiv n(t,\, x)\delta(v_x - u(t,\, x))\delta(v_y - v(t,\, x)).(4)$$

Equations (2) are reduced to the partial differential equations for the functions u , v, and E of the variables t and x:

$$\partial u/\partial\tau = -(e/m)E - \omega_{ce}v,$$

$$\partial v/\partial\tau = \omega_{ce}u,\ \partial E/\partial\tau = 4\pi e n_0 u, \qquad (5)$$

where $\partial/\partial\tau = \partial/\partial t + u\partial/\partial x$. Their solution by the use of Lagrangian variables is made possible by adding the equation for $x(\tau)$,

$$\partial x/\partial\tau = u, \qquad (6)$$

to (5) and treating x , $E,$ u, and v as the functions of τ and the initial value of $x(\tau)$, i.e. $x_0 = x(0)$. It is readily seen that

$$\partial^2 u/\partial\tau^2 + \omega_{\rm uh}^2 u = 0,$$

and the general solution is as follows:

$$x = I_1\cos(\omega_{\rm uh}\tau) + I_2\sin(\omega_{\rm uh}\tau) + I_3,$$

$$u = \omega_{\rm uh}[-I_1\sin(\omega_{\rm uh}\tau) + I_2\cos(\omega_{\rm uh}\tau)],$$

$$E = 4\pi e n_0[I_1\cos(\omega_{\rm uh}\tau) + I_2\sin(\omega_{\rm uh}\tau) + I_4],$$

$$v = \omega_{ce}[I_1\cos(\omega_{\rm uh}\tau) + I_2\sin(\omega_{\rm uh}\tau)] - (\omega_{pe}^2/\omega_{ce})I_4, \quad (7)$$

where $\mathbf{I} = \{I_1, ..., I_4\}$ are the functions of x_0 which are determined by the initial conditions, e.g. $I_2(x_0) = u(0, x_0)/\omega_{\rm uh}$. In this way, the complicated expression for $x,$ $E,$ u and v presented in [7] can be obtained. From the point of view of the symmetry approach, this possibility of finding the general solution in Lagrangian variables is related to the fact that system (5), (6) can be presented in the form

$$\partial\mathbf{I}/\partial\tau = 0 \qquad (8)$$

which is invariant, as is readily seen, under a wide class of transformations depending on arbitrary functions $\mathbf{F}$ and G:

$$\mathbf{I}' = \mathbf{F}(\mathbf{I}),\ \tau' = G(\tau,\, \mathbf{I}). \qquad (9)$$

Transformations (9) allow us to obtain the general solution of system (5), (6) starting from the trivial zero solution. In fact, the trivial solution

$$I_2 = 0,\ ...,\ I_4 = 0$$

is generalized by transformation (9) to

$$F_2(I_1,\, ...,\, I_4) = 0,\ ...,\ F_4(I_1 .\, ...,\, I_4) = 0,$$

with arbitrary functions F_2,...,F_4. Then we can express a solution in the form

$$I_2 = g_2(I_1),\ ...,\ I_4 = g_4(I_1).$$

Finally, we determine the functions g_2 to g_4 by the general initial conditions for $u,$ v and $E,$ and this will reproduce the solution obtained in Lagrangian variables. So the possibility of solving the cold electron plasma equations is due to the sufficiently large extension of the model symmetry.

3. Langmuir Electron Plasma Waves

Let us review shortly the previous results obtained in [4] for longitudinal electron plasma waves. In this case, i.e. for the system of equations (see, e.g. [7])

$$\partial f/\partial t + \upsilon_z \partial f/\partial z - (e/m)E_z\, \partial f/\partial \upsilon_z = 0,$$

$$\partial E_z/\partial z = 4\pi e(n_0 - \int f\, d\upsilon_z),$$

$$\partial E_z/\partial t = 4\pi e \int \upsilon_z\, f\, d\upsilon_z, \tag{10}$$

where the distribution function f has been integrated over υ_z and υ_y, the symmetry group is as follows [4]:

$$X_1 = \partial/\partial t,\ X_2 = \partial/\partial z,$$

$$X_3 = \cos(\omega_{pe}t)\partial/\partial z - \omega_{pe}\, \sin(\omega_{pe}t)\partial/\partial \upsilon_z +$$

$$+(m/e)\omega_{pe}^2\, \cos(\omega_{pe}t)\partial/\partial E_z,$$

$$X_4 = \sin(\omega_{pe}t)\partial/\partial z + \omega_{pe}\, \cos(\omega_{pe}t)\partial/\partial \upsilon_z +$$

$$+(m/e)\omega_{pe}^2\, \sin(\omega_{pe}t)\partial/\partial E_z,$$

$$X_5 = z\partial/\partial z + E_z \partial/\partial E_z + \upsilon_z \partial/\partial \upsilon_z - f\partial/\partial f. \tag{11}$$

In the cold electron plasma limit,

$$f(t,\, z,\, \upsilon_z) \equiv n(t,\, z)\delta(\upsilon_z - w(t,\, z)), \tag{12}$$

system (6) is reduced to the partial differential equations

$$\partial w/\partial \tau = -(e/m)E_z,\ \ \partial E_z/\partial \tau = 4\pi e n_0 w, \tag{13}$$

where $\partial/\partial \tau = \partial/\partial t + w\partial/\partial z$. In this case, invariants $\mathbf{I} = \{I_1,\, I_2,\, I_3\}$ have the form

$$I_1 = z - E_z/4\pi\, e\, n_0,$$

$$I_2 = (w/\omega_{pe})\, cos\, (\omega_{pe}\tau) + (E_z/4\pi\, e\, n_0)\, \sin(\omega_{pe}\tau),$$

$$I_3 = (w/\omega_{pe})\sin(\omega_{pe}\tau) - (E_z/4\pi e n_0)\, \cos(\omega_{pe}\tau). \tag{14}$$

The invariance under transformations (9) made it possible to obtain the general solution of (13) from the trivial one or to solve this system in Lagrangian variables as it was done in [7].

Conclusion

In this work, we have studied the invariance of the integro-differential equations for the nonlinear upper hybrid waves (2). The continuous symmetry group (3) has been obtained. Among the symmetries are present the time and space homogeneity (X_1 and X_2), nonrelativistic remnant of the Lorentz transform in the y direction ((X_3), similarity transform (X_4) related to the fact that no assumptions are made in Eqs. (2) *a priori* about the plasma temperature and thus no characteristic thermal velocity is contained in the system. Transformations X_5 and X_6 are the specific ones for the upper hybrid waves. They mean that spatially homogeneous upper hybrid oscillations can be included in an arbitrary solution of the model as, e.g., the nonlinear reaction of the system to a rapid homogeneous external current flash.

It is shown also that, in the cold electron plasma limit (4), the symmetry extension made allows one to obtain the general solution which is equivalent to the better known procedure of solving the equations in Lagrangian variables.

Comparison of the above-mentioned results with symmetries and solutions for the more simple theory of electron plasma high frequency waves in the absence of the external magnetic field shows a very close qualitative analogy.

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СИМЕТРІЇ ВЕРХНЬОГІБРИДНИХ ЕЛЕКТРОННИХ ХВИЛЬ У ПЛАЗМІ

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Одержано групу симетрії системи інтегро-диференціальних рівнянь нелінійних верхньогібридних хвиль у замагніченій електронній плазмі. Показано, що розширення симетрії у випадку холодної електронної плазми дозволяє знайти загальний розв'язок. Одержані результати порівняно з властивостями симетрії електронних ленгмюрівських хвиль.

СИММЕТРИИ ВЕРХНЕГИБРИДНЫХ ЭЛЕКТРОННЫХ ВОЛН В ПЛАЗМЕ

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Найдена группа симметрии системы интегро-дифференциальных уравнений нелинейных верхнегибридных волн в замагниченной электронной плазме. Показано, что расширение симметрии в случае холодной электронной плазмы дает возможность получить общее решение системы. Полученные результаты сравниваются со свойствами симметрии электронных ленгмюровских волн.