FRACTIONAL KINETICS FOR ANOMALOUS DIFFUSION AND RELAXATION

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Recently, kinetic equations with partial fractional derivatives have attracted attention as a tool for the description of anomalous relaxation and diffusion phenomena. We present a short review on the modern status of fractional kinetic equations. The topics considered are as follows:
— derivation of fractional kinetic equations with space fractional derivative;
— anomalous diffusion and relaxation;
— non-Boltzmann stationary states.
Applications of the general theory to plasma physics problems are proposed.

Introduction

Recently, kinetic equations with fractional space and time derivatives have attracted attention as a tool for the description of anomalous diffusion and relaxation phenomena, see, e.g., [1, 2 and references on earlier studies therein]. In these phenomena, the laws of normal diffusion (ordinary Brownian motion) are altered, e.g., the mean squared displacement no longer increases linearly with time but instead grows slower (subdiffusion) or faster (superdiffusion) than the linear function. Furthermore, in contrast to exponential relaxation, which is the distinctive feature of ordinary Brownian motion, fractional kinetics may exhibit non-exponential slow relaxation. Another distinctive feature of fractional kinetics is non-Boltzmann stationary states which occur in the open systems far from equilibrium. Thus, fractional kinetics covers a large area of anomalous dynamics called also "strange kinetics", the term which was introduced in [3] and originally referred to the dynamics of Hamiltonian systems displaying superdiffusion in the limit of weak chaos [4]. Now, the term "strange kinetics" is employed in a general sense implying a variety of topics which are connected with deviations from exponential or Gaussian laws, and deviations from fast decaying correlations [5]. Here, one is faced with the existence of long-range time and space correlations, disorder and cooperativity, shared by systems studied by physicists, chemists, engineers, and many more. We refer to the Special Issue of Chemical Physics, Vol.284, Nos.1–2 (November 1, 2002) which is exclusively devoted to the topic "Strange Kinetics".

Theoretically, strange kinetics and anomalous dynamics are intimately connected to the description based on random walks in continuous time, generalized master, Langevin and Fokker—Planck equations. Recently, it has become clear that many of these theoretical tools are mathematically related to the expanding area of fractional differential equations. It was found that fractional kinetic equations may be viewed as the long-time and long-space limit of continuous-time random walks, a model that was successfully applied to describe anomalous diffusion phenomena in many areas, e.g., turbulence [6], disoreder media [7], intermittent chaotic systems [8], underground water pollution [9], etc. However, the kinetic equations have two advantages over the random walk approach: first, they allow one to explore various boundary conditions (e.g., reflecting and/or absorbing) and, second, to study diffusion and/or relaxation phenomena in external fields. Both possibilities are difficult to realize in the random walk schemes.

The different way to obtain fractional kinetic equations is the use of the Langevin equations. In present review, just this approach is used.

There are three types of fractional kinetic equations: the first one, describing Markovian processes, contains equations with fractional space or velocity derivative; the second one, describing non-Markovian processes, contains equations with fractional time derivative; and the third class, naturally, contains both fractional space and time derivatives, as well. The recent review [1] was devoted mainly to time fractional kinetic equations which are used for studying relaxation and diffusion phenomena close to equilibrium. In the present paper, we deal mainly with space and velocity fractional kinetic equations which are believed to describe a certain class of diffusion and relaxation phenomena in open systems far from equilibrium.

1. Derivation of Fractional Kinetic Equation

1.1. Lévy noises

In what follows, we will start from the Langevin equation with a white Lévy noise. Various models of dynamical systems driven by non-Gaussian Lévy noises obeying the Lévy statistics are used for the description of anomalous random processes and related anomalous diffusion phenomena [10—14, 35]. These models are based on Generalized Central Limit Theorem, according to which Lèvy stable probability distributions are the limit ones for properly normalized sums of random variables with diverging variance [15, 16]. It implies that just Lévy distributions, similarly to the Gaussian one, naturally occur when the evolution of a system or the result of an experiment are determined by the sum of a large number of random factors.

Lévy stable probability density functions (PDFs) are classified by their Lévy index α which lies between 0 and 2. The case $\alpha = 2$ corresponds to the Gaussian PDF. For Lèvy indices ranging in the interval $0 < \alpha < 2$, the Lévy stable PDFs possess power-law tails of the form $\propto |x|^{-\alpha-1}$ This means that moments of order $q \geq \alpha$ diverge. Therefore, there exist large "outliers" or peaks in the Lévy noises which appear due to "fat" tails of the PDFs, see Fig.1. On the contrary, these peaks do not exist in the Gaussian noise, since they are prohibited by rapidly decaying tails of the Gaussian PDF, see the bottom sample. Because of such a drastic difference between two types of the noises, the statistical behaviours of the systems driven by them differ greatly one from the other.

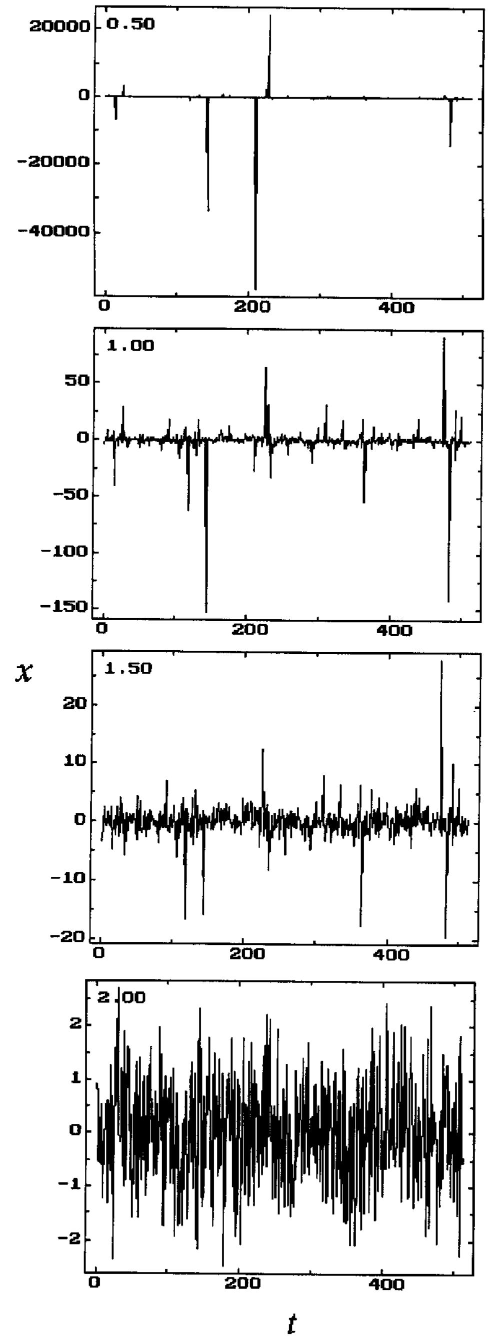

Fig. 1. Lévy noises with the Lévy indices (from top to bottom) α = 0.50, 1.00, 1.50, and 2.00

1.2. Kinetic equation for ordinary Lévy motion

Before getting a space fractional kinetic equation, it is expedient, from a methodical viewpoint [17], to get an equation for the probability density function (PDF) $f(x,t)$ of the random process called α-stable process or ordinary Lévy motion. This is a non-stationary process, whose increments are self-affine, stationary, independent and distributed by the Lévy stable law [18]. A characteristic function has the form

$$\widehat{f}(k,t) = \int_{\infty}^{-\infty} dx f(x,t)\exp(ikx) = \exp(-D\,|k|^{\alpha}\,t), \qquad (1)$$

where α is the Lévy index, $0 < \alpha \leq 2$, $D > 0$, and $D^{1/\alpha}$ is called a scale parameter. Here we restrict ourselves with one-dimensional stable processes with symmetric PDFs. The generalizations see in [19—21]. At $\alpha = 2$, we arrive at the particular case of the Wiener process, or ordinary Brownian motion. It follows from Eq.(1) that $\widehat{f}(k,t)$ obeys the equation

$$\frac{\partial \widehat{f}}{\partial t} = -D\,|k|^{\alpha}\,\widehat{f}, \quad \widehat{f}(k,0) = 1. \qquad (2)$$

We use the Riesz fractional derivative which may be defined, for a "sufficiently well-behaved" function $\phi(x), x \in \mathbf{R}$, as the (pseudo-differential) operator characterized in its Fourier representation by

$$\int_{-\infty}^{\infty} dx \exp(ikx)\frac{d^{\alpha}}{d\,|x|^{\alpha}}\phi(x) = -\,|k|^{\alpha}\,\widehat{\phi}(k)\,,$$

$$k \in R,\ \alpha > 0 \qquad (3)$$

(for the rigorous definition of the Riesz fractional derivative in terms of Riemann—Liouville derivatives, see, e.g., [22]. Note that $d^2/d\,|x|^2 = d^2/dx^2$, but $d/d\,|x| = d/dx$). Now, with the use of Eqs.(1)—(3), the evolution equation for the PDF of the α-stable process can be written as

$$\frac{\partial f}{\partial t} = D\frac{\partial^{\alpha} f}{\partial |x|^{\alpha}}. \qquad (4)$$

For $\alpha = 2$, Eq.(4) is the usual diffusion equation. For $0 < \alpha < 2$, it describes anomalous superdiffusion, see Sect.3 below.

1.3. Fractional Einstein—Smoluchowski Equation

We start from the integral equation for the PDF of a Markovian stochastic process,

$$f(x,t+\Delta t) = \int d(\Delta x) f(x-\Delta x,t)\psi(x-\Delta x;\Delta x,\Delta t), \qquad (5)$$

where $\psi(x;\Delta x,\Delta t)$ is the transition probability, that is, the probability for $x(t)$ to get an increment Δx during an interval Δt . The starting Langevin equation is

$$\frac{dx}{dt} = -\frac{1}{m\gamma}\frac{dU}{dx} + Y_{\alpha}(t), \qquad (6)$$

where U is the potential energy, m is the particle mass, γ is the friction coefficient, and $Y_{\alpha}(t)$ is a stationary white Lévy noise. Similarly to the definition of a white Gaussian noise, white Lévy noise can be defined such that an integral of $Y_{\alpha}(t)$ over some time lag is an $\alpha-$stable process with the characteristic function given by Eq.(1). Then, we get from Eq.(6), by integrating during a time interval Δt, which is shorter than time intervals during which physical parameters change appreciably:

$$\Delta x = -\frac{\Delta t}{m\gamma}\frac{dU}{dx} + L(\Delta t), \quad L(\Delta t) = \int_{t}^{t+\Delta t} dt^{'}\,Y_{\alpha}(t^{'}). \qquad (7)$$

The transition probability follows from Eq.(7),

$$\psi(x;\Delta x,\Delta t) =$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\exp\left[-ik\left(\Delta x + \frac{\Delta t}{m\gamma}\frac{dU}{dx}\right) - D|k|^{\alpha}\Delta t\right]. \qquad (8)$$

We insert Eq.(8) into Eq.(5), expand the left- and right-hand sides of the equation in Taylor series in Δt and then take the limit $\Delta t \to 0$. As the result, we get FESE,

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\frac{dU/dx}{m\gamma}f\right) + D\frac{\partial^{\alpha} f}{\partial |x|^{\alpha}}, \qquad (9)$$

which is reduced to the known Einstein-Smoluchowski equation at $\alpha = 2$. We also note that, in [14], by using the analogous procedure, the fractional Fokker—Planck equation (FFPE) was obtained, which contains the term with velocity fractional derivative, thus describing anomalous diffusion in the phase space.

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2. Anomalous Superdiffusion

Let us turn to Eq.(4). The characteristic function of the solution is given by Eq.(1). In the real space, the Lévy stable PDFs are expressed in terms of Fox' H functions [23]. Such a representation of all stable PDFs was achieved in [24]. Mathematical details on H functions are presented in [25, 26]. However, in the present paper, we do not touch a real space representation for an arbitrary α.

Since the variance and higher moments of integer order diverge for the stable PDFs, as statistical means characterizing the properties of these processes, the moments of fractional orders can be used [10, 27]. In order to guarantee the reality, they must be defined for the modulus of a stochastic variable. Therefore, in case of force-free relaxation, the moments of fractional orders are

$$M_x\left(t;q,\alpha\right) \equiv \int\limits_{-\infty}^{\infty} dx\,|x|^q\, f\left(x,t\right) =$$

$$= \left\{\ ((Dt)^{1/\alpha}/\nu)^q C(q;\alpha), \quad 0 < q < \alpha\infty, \quad q \geq \alpha\ \right\} \tag{10}$$

for $0 < \alpha < 2$, whereas

$$M_x\left(t;q,2\right) = \frac{(Dt)^{q/2}}{\nu^2} C(q;2) \tag{11}$$

for $\alpha = 2$ and an arbitrary q, where

$$C(q;\alpha) = \int\limits_{-\infty}^{\infty} dx_2\,|x_2|^q \int \frac{dx_1}{2\pi} \exp(-ix_1x_2 - |x_1|^\alpha).$$

The coefficient $C(q;\alpha)$ can be evaluated with the use of generalized function theory [10]:

$$C(q;\alpha) = \frac{2}{\pi q} \sin\left(\frac{\pi q}{2}\right) \Gamma(1+q) \Gamma\left(1 - \frac{q}{\alpha}\right),$$

$$0 < q < \alpha. \tag{12}$$

Equations (10)—(12) have a direct physical consequence for the description of anomalous diffusion. Indeed, for ordinary Brownian motion, the typical displacement $\delta x(t)$ of a particle may be written through the second moment as

$$\delta x(t) = M_x^{1/2}(t;2,2) \propto t^{1/2}.$$

One may note from Eq.(11) that, for normal diffusion, $M_x^{1/q}(t;q,2) \propto t^{1/2}$at any q and, thus, any order of the moment can serve as the measure of a normal diffusion rate:

$$\delta x(t) \approx M_x^{1/q}(t;q,2) \propto t^{1/2},$$

if one is interested in the time-dependence of the characteristic displacement, but not in the value of the prefactor. We recall that usually just the time-dependence, but not the prefactor, serves as an indicator of normal or anomalous diffusion [7]. In analogy with the Brownian case, it follows from Eq.(10) that the quantity $M_x^{1/q}(t;q,\alpha)$ at $0 < \alpha < 2$ and any $q < \alpha$ can serve as the measure of an anomalous superdiffusion rate:

$$\delta x(t) \approx M_x^{1/q}(t;q,\alpha) \propto t^{1/\alpha}, \quad 0 < q < \alpha < 2. \tag{13}$$

We give a remark on the above, which is concerned with alternative ways for characterizing the rate of superdiffusion. Indeed, instead of introducing displacement Δx, there exist two similar ways.

In [28], as a measure of the width of a diffusion packet, the length $R_p(t)$ of a segment containing fixed probability p is used,

$$\int\limits_{|x|<R_p(t)} f(x,t)dx = p.$$

Using the characteristic function of a 1-dimensional Lévy stable process with symmetric PDF and changing variables, we get

$$\int\limits_{|x|<R_p(t)} dx \int\limits_{\infty}^{-\infty} \frac{dk}{2\pi} \exp(-ikx - |k|^\alpha) = p.$$

Since the right-hand side of this equation does not depend on t,

$$R_p(t) \propto t^{1/\alpha}, \tag{14}$$

which is in accordance with Eq. (13).

Another way to extract the scaling operationally is enclosing the "walker" in an "imaginary growing box" [13]:

$$\langle x^2(t)\rangle_L \approx \int\limits_{L_1 t^{1/\alpha}}^{L_2 t^{1/\alpha}} dx x^2 f(x,t) \propto t^{2/\alpha}. \tag{15}$$

This procedure has been also implemented numerically [13]. Of course, the scaling result (15) should not be confused with the mean square displacement, which is infinite. However, for $\alpha > 1$, the squared absolute mean

$$\langle|x|^2\rangle = \left(\int\limits_{\infty}^{-\infty} dx x f(x,t)\right)^2$$

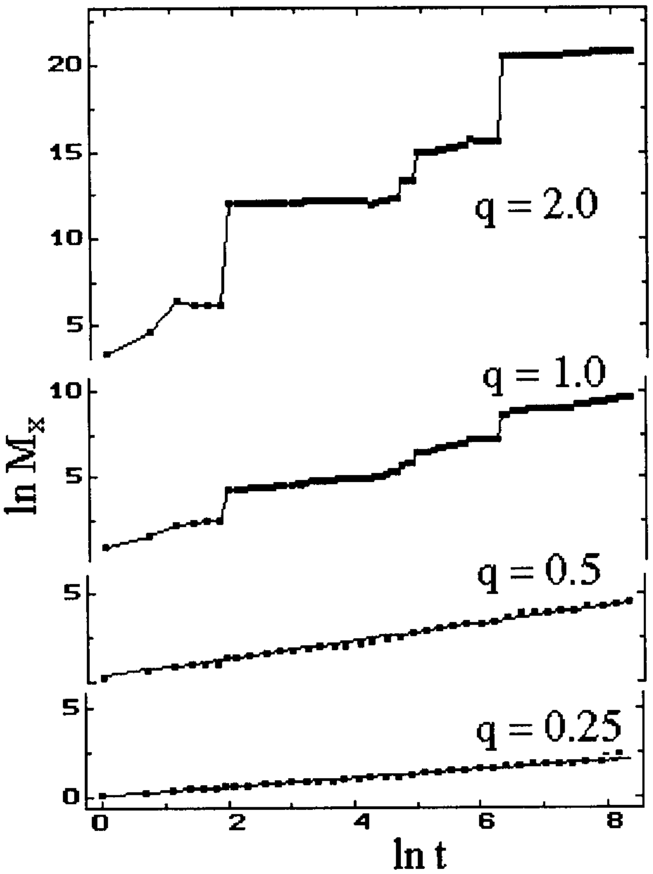

Fig.2. Force-free relaxation in the framework of FESE. The moment M_x versus t at different values of moment exponent q. The Lévy index α is 1

converges and is proportional to $t^{2/\alpha}$, which is also in accordance with (14), (15).

Thus, all three ways of extracting anomalous scaling give the same result.

The numerical simulation is based on the solution of the Langevin equation (6). Here and below, the stochastic source $Y_\alpha(t)$ is represented in numerical simulation as a discrete approximation of a "white Lévy noise", that is, as a stationary consequence of independent identically distributed variables having symmetric stable PDF with the Lévy index α and the scale parameter equal to 1. The model of white Lévy noise is described in [29] in more details. In a force-free problem, we estimate numerically the moments $M_x(t; q, \alpha)$ by averaging over realizations of $x(t)$. The details of simulations are described in [14].

In Fig. 2, we show $M_x(t; q, 1)$ versus t at different q on a log—log scale. At $q < \alpha = 1$, the dependence is well fitted by a straight line whose slope allows one to define the diffusion exponent. At $q \geq \alpha$, the theoretical value of the moment is infinite, and the moment strongly fluctuates in numerical simulation, thus it is unable to get the diffusion exponent.We note that the increase of the number of trajectories does not lead to the damping of fluctuations seen in the top figure.

3. Harmonic Lévy Oscillator

Let $U = ax^2/2$, $a = m\omega^2$ in Eq.(9) which is solved with the initial condition $f(x, 0) = \delta(x)$. The corresponding equation for the characteristic function has the solution [13]

$$\widehat{f}(k, t) = \exp(-D_{\rm osc}(t)\,|k|^\alpha), \tag{16}$$

where

$$D_{\rm osc}(t) = \frac{D\gamma}{\alpha\omega^2}\left(1 - \exp\left(-\frac{\alpha\omega^2}{\gamma}t\right)\right). \tag{17}$$

It follows from Eqs.(16) and (17) that $x(t)$ is asymptotically stable at small times, $t << \tau_x = \gamma/\alpha\omega^2$. At large times, $t >> \tau_x$, the process $x(t)$ becomes stationary with the stable PDF possessing the following properties: PDF is unimodal with a maximum being at the origin, and (ii) it has a slowly decaying tail such that the variance diverges.

The fractional moments are used to characterize relaxation and diffusion phenomena:

$$\langle |x|^q \rangle = D_{\rm osc}^{q/\alpha}(t) C(q; \alpha), \tag{18}$$

where $C(q; \alpha)$ is determined by Eq.(12). Numerical simulation of the relaxation of a harmonic Lévy oscillator is based on the numerical solution of the Langevin equation(6) with a subsequent estimation of the q-th moment [14]. The results are shown in Fig.3 in dimensional variables for different values of ω, $q = 0.25, \alpha = 1, \gamma = 1$. The values obtained in the numerical simulation are shown by black points whereas the solid line demonstrates the result obtained with Eqs.(17), (18). Vertical arrows show τ_x, after which the process $x(t)$ becomes stationary. The results of the numerical simulation based on the Langevin approach agree with the theory based on the kinetic equation on both the non-stationary and stationary stages of evolution. Another linear stochastic system, namely, a plane rotator driven by the Lévy noise was considered in [36].

4. Quartic Lévy Oscillator

Let us consider the simple example of a non-linear quartic Cauchy oscillator with the potential $U = bx^4/4$ and the Lévy index $\alpha = 1$. In the dimensionless variables $x^{'} = x/x_0, t^{'} = t/t_0$ such that $x_0 = (m\gamma D/b)^{1/3}, t_0 = x_0/D$, the stationary PDF can be easily obtained from Eq.(9):

$$f(x) = \pi^{-1}(1 - x^2 + x^4)^{-1}. \tag{19}$$

Note that $f(x)$ has two important properties: (i) bimodal structure, that is, the PDF has a local minimum at $x_{\min} = 0$ and two maxima at $x_{\max} = \pm 1/\sqrt{2}$, and (ii) the PDF has steep power-law asymptotics at $x \to \pm\infty$, $f(x) \propto x^{-4}$, hence, the variance is finite. These properties are drastically different from the properties of stationary solutions for both a Brownian quartic oscillator and harmonic Lévy oscillator. It was shown in [33] that the steep power-law asymptotics and the bimodality are inherent in quartic Lévy oscillators with all α's such that $1 \leq \alpha < 2$. Moreover, this result was generalized in [32], namely, it was shown that, for symmetric potentials of the general form $U(x) \propto x^{2m+2}/(2m+2)$, $m = 0, 1, 2, ...$, the PDFs display a distinct bimodal character and have power-law tails which decay as

$$f(x) \approx \frac{C_\alpha}{|x|^{\alpha+2m+1}}, \tag{20}$$

where $C_\alpha = \pi^{-1}\Gamma(\alpha)\sin(\pi\alpha/2)$ is a "universal" constant in the sense that it does not depend on m. This property of the PDFs is illustrated by Fig.4, which represents the results of numerical modelling based on the solutions of the Langevin equations. In the left column, the potential energy functions with different m indices are shown by solid lines. The dotted lines indicate their curvatures. In the middle and in the right columns, the typical sample paths are shown for oscillators driven by the Gaussian noise (Brownian oscillator) and the Lévy noise with $\alpha = 1$ (Cauchy oscillator), respectively. Each row corresponds to the oscillator with the index m indicated in the left column. It is seen that the typical sample paths for all Brownian oscillators are nearly the same, consisting of small increments of the coordinate during each time step. This is the consequence of the exponential shape of stationary Boltzmann PDFs, which prohibits large increments. In contrast, Lévy flights with large increments are clearly distinct in the figures of the right column. These flights appear due to the power-law asymptotics of the stationary PDFs, which permit large values of increments to occur. The longest flights are

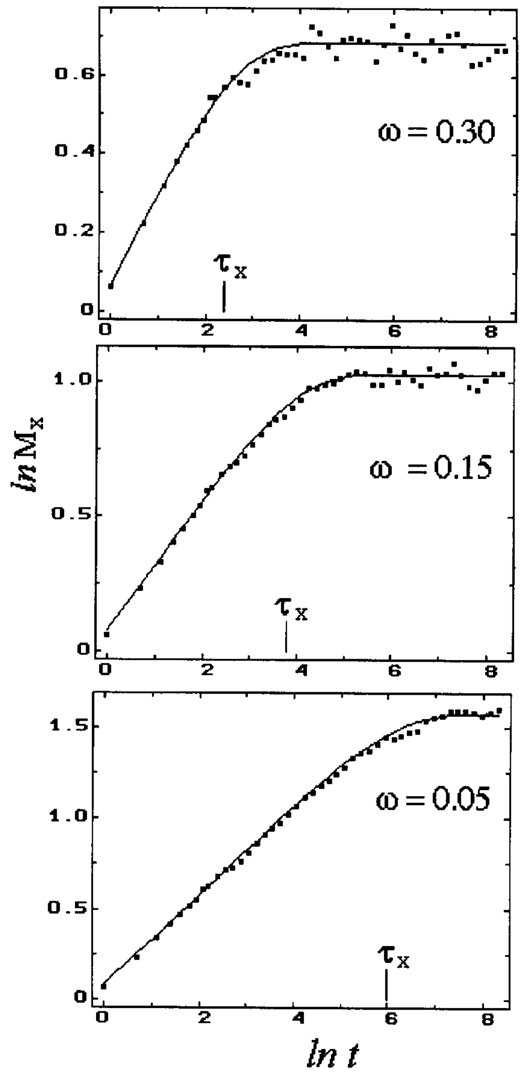

Fig.3. Relaxation of a harmonic Lévy oscillator: q-th moment versus t on a double logarithmic scale

realized in case of a harmonic Lévy oscillator, $m = 0$, because the PDF of a harmonic oscillator has the fattest tail. As follows from Eq.(20), with m increasing, the power-law asymptotics become steeper, therefore, the flights become shorter, that is, long flights occur more rarely. This effect is clearly seen in the right column, when comparing, for example, sample paths for a linear oscillator (at the top) with a strongly non-linear oscillator in the bottom. The quantitative relations for the family of non-linear Lévy oscillators were studied in [32] in detail. In [34], the peculiarities of the unimodal — bimodal transition during time evolution were studied for a quartic Lévy oscillator.

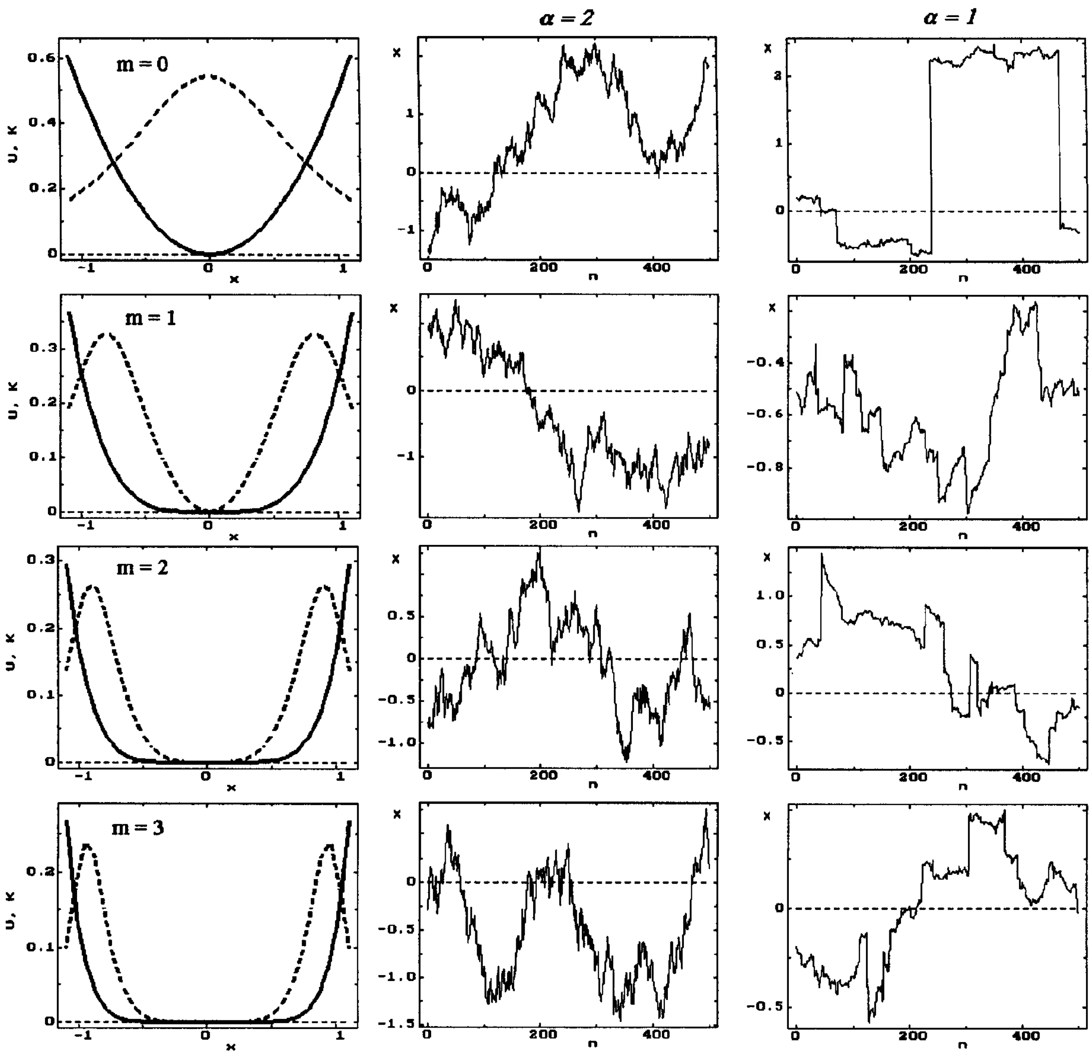

Fig.4. Left column: the potential energy functions $U = x^{2m+2}/(2m+2)$ (solid lines) and their curvatures (dotted lines) for different values of m: $m = 0$ (linear oscillator), and $m = 1, 2, 3$ (strongly non-linear oscillators). Middle column: typical sample paths of Brownian oscillators, $\alpha = 2$, with the potential energy functions shown on the left. Right column: typical sample paths of Lévy oscillators, $\alpha = 1$

5. Anharmonic Lévy Oscillator

Since a harmonic Lévy oscillator has unimodal stationary PDF, and a quartic Lévy oscillator has bimodal stationary PDF, one might expect that, for an anharmonic Lévy oscillator with the potential energy function $U = ax^2/2 + bx^4/4$, $a > 0$, $b > 0$, the bimodal-unimodal transition exists in a stationary state if the parameters a and b vary. We show it for a Cauchy anharmonic oscillator. Introducing the same dimensionless variables as in the previous Sections, setting $a' = at_0/m\gamma$ and omitting primes, we arrive at the equation for the characteristic function of the stationary PDF on the right semi-axis:

$$\frac{d^3 \hat{f}(k)}{dk^3} - a\frac{d\hat{f}(k)}{dk} = \hat{f}(k)\,, \quad \hat{f}(0) = 1,$$

$$\frac{d\hat{f}(0)}{dk} = 0, \ \ \hat{f}(k=\infty) = 0. \tag{21}$$

(Note that, in dimensionless variables, we deal with the potential energy function $U = ax^2/2 + x^4/4$; thus, only one parameter, a, remains.) The solution is $\hat{f}(k) = \left(-z^* e^{zk} + z e^{z^* k}\right)(z - z^*)^{-1}$, where z is the complex root of the characteristic equation $z^3 - az - 1 = 0$, that is, $z = -(u+w)/2 + i\sqrt{3}(u-w)/2$, where $u^3 = (1+\sqrt{1-4a^3/27})/2, w^3 = \left(1-\sqrt{1-4a^3/27}\right)/2$. We are interested in the unimodal-bimodal transition when the parameter a varies. Let a_c be the critical value, which we determine by using Eqs.(4.6)—(4.9). The condition for the transition is $d^2 f(0)/dx^2 = 0$, or equivalently, defining $J(a) = \int_0^\infty dk\, k^2 \hat{f}(k)$, $J(a_c) = 0$. If $J > 0$, the stationary PDF is unimodal; if $J < 0$, it is bimodal. Then, $sgn\, J = sgn\left(z^2 + z^{*2}\right)$ and, defininig $\zeta \equiv 4^{1/3} a_c/3$, we get

$$4\zeta = \left(1+\sqrt{1-\zeta^3}\right)^{2/3} + \left(1-\sqrt{1-\zeta^3}\right)^{2/3}. \tag{22}$$

The solution of Eq.(22) is $\zeta = 0.420$ and therefore $a_c = 0.794$. For $a > a_c$, the quadratic term in the potential energy function prevails, and the stationary PDF has one maximum at the origin. In contrast, for $a < a_c$, the quartic term dominates and dictates the shape of the PDF. As a result, the bimodal stationary PDF appears with a local minimum at the origin. Returning to the dimensional variables, we can rewrite the condition of transition in terms of a critical value b_c of the quartic term amplitude:

$$b_c = a^3/0.794^3(m\gamma D)^2. \tag{23}$$

This relation implies that increasing noise requires smaller anharmonicity to cause the bimodal stationary PDF. Thus, the bimodality results indeed from the combination of the Lévy character of the noise and the anharmonicity of the potential well. In Fig.5, the profiles of stationary PDFs are shown for an anharmonic Cauchy oscillator for different values of the coefficient a, from top to bottom: $a = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,$ and 1.0. The PDFs are obtained by the inverse Fourier transformation of characteristic functions. It is clear that the bimodality is most pronounced for $a = 0$, that is, for the quartic Cauchy oscillator. As the parameter a increases, the bimodal profile smoothes out, and, finally, it turns to a unimodal one. The generalization for arbitrary α was considered in [32].

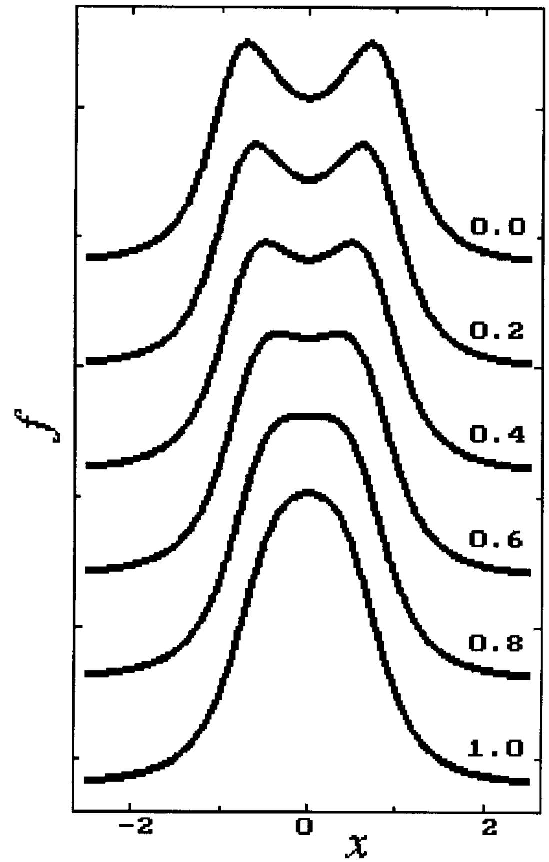

Fig. 5. Profiles of the stationary PDF's (obtained by the inverse Fourier transformation) are shown for an anharmonic Cauchy oscillator

6. Fractional Kinetics for Relaxation and Superdiffusion in a Magnetic Field

The various processes in space and thermonuclear plasmas could serve as important applications of fractional kinetics. Indeed, many of the current challenges in solar system plasmas as well as in plasmas of thermonuclear devices arise from fundamentally multiscale and nonlinear nature of plasma fluctuation

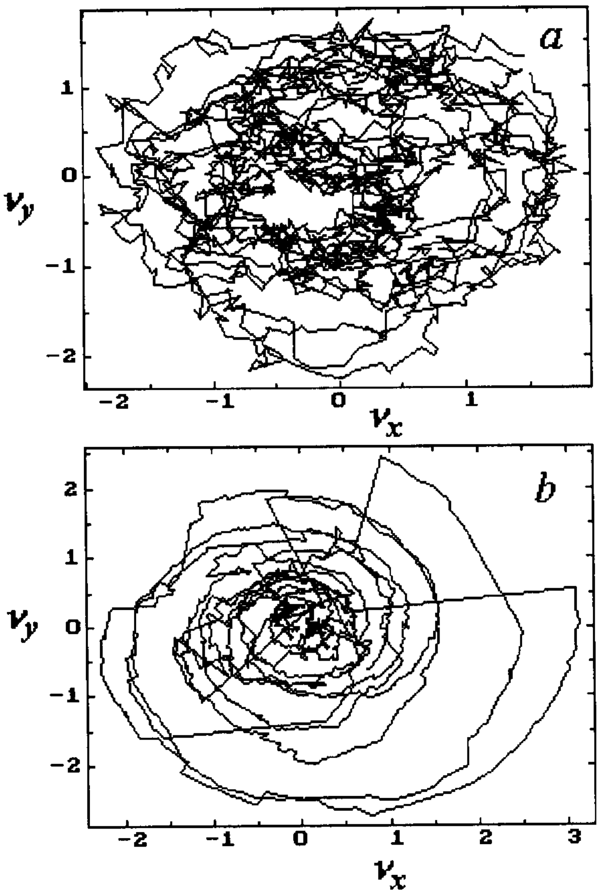

Fig. 6. Numerical solution to the Langevin equations (24). Trajectories on (v_x, v_y) plane: a) for $\alpha = 2.0$, b) for $\alpha = 1.2$

and wave processes, see, e.g., recent references on self-organized criticality, originally applied to plasma physics problems in [30], measurements of space plasma wave properties [31], measurements of chaotic transport in analogous geophysical and plasma systems showing anomalous diffusion and Levy statistics [37], data analysis of plasma edge fluctuations in different thermonuclear devices [38—40], etc. Very recently, it was shown that the fluctuations of the ion saturation current and floating potential measured in the boundary plasma of a Torsatron "Uragan 3M" can be described within the framework of non-Gaussian Lévy statistics [41]. Anomalous diffusion and plasma heating, particle acceleration and macroscopic transfer processes require to go beyond the "traditional" plasma kinetic theory. Fractional kinetics can be useful for describing such processes, just as it occurs in other fields of applications. Recently, a phenomenological fractional kinetic equation was proposed, which governs the distribution of density fluctuations in a tokamak [42]. In this Section, we consider a test charged particle with mass m and the charge e, embedded in a constant external magnetic field $\mathbf{B}$ and subjected to a stochastic electric field $\varepsilon(t)$. We also assume, as in the classical problem for a charged Brownian particle [43], that the particle is influenced by the linear friction force $-\nu m\mathbf{v}$, ν is the friction coefficient. For this particle, the Langevin equations of motion are

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}\,,$$

$$\frac{d\mathbf{v}}{dt} = \frac{e}{mc}[\mathbf{v}\times\mathbf{B}] - \nu\mathbf{v} + \frac{e}{m}\varepsilon. \tag{24}$$

The statistical properties of the field $\varepsilon(t)$ are assumed to be as follows.

1. $\varepsilon(t)$ is homogeneous and isotropic.
2. $\varepsilon(t)$ is a stationary white Lévy noise.

The first assumption is the usual one when dealing with the motion of a charged particle in a random electric field. The second assumption provides us with a simple and straightforward possibility, at least, from the methodical viewpoint, to consider abnormal diffusion and non-Maxwell stationary states, both properties are inherent in strongly non-equilibrium plasmas of the solar system and thermonuclear devices. Thus, if $\alpha < 2$, then, by applying the procedure described in detail for the one-dimensional case in [14], we arrive at the fractional Fokker—Planck equation for a charged particle in the constant magnetic field and random electric field:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \Omega\left[\mathbf{v}\times\mathbf{e}_z\right]\frac{\partial f}{\partial \mathbf{v}} =$$

$$= \nu\frac{\partial}{\partial \mathbf{v}}(\mathbf{v}f) - D\left(-\Delta_{\mathbf{v}}\right)^{\alpha/2} f, \tag{25}$$

where $\Omega = eB/mc$, $D = e^{\alpha} D_{\varepsilon}/m^{\alpha}$ and $\left(-\Delta_{\mathbf{v}}\right)^{\alpha/2}$ is the fractional Riesz derivative with respect to the velocity. This operator is defined through its Fourier transform as

$$\left(-\Delta_{\mathbf{v}}\right)^{\alpha/2} f(\mathbf{r},\mathbf{v},t) \div |\mathbf{k}|^{\alpha}\,\hat{f}(\kappa,\mathbf{k},t)\,, \tag{26}$$

where $\hat{f}$ is the characteristic function,

$$\hat{f}(\kappa,\mathbf{k},t) = \int\!\!\int d\mathbf{r}d\mathbf{v}\exp\left(i\kappa\mathbf{r} + i\mathbf{k}\mathbf{v}\right) f(\mathbf{r},\mathbf{v},t). \tag{27}$$

An explicit representation of the Riesz derivative is realized through a hypersingular integral, see [22] containing the detailed presentation of the Riesz differentiation. We also note that, at $\alpha = 2$, Eqs. (25), (26) are reduced to the Fokker—Planck equation for the Brownian motion.

FFPE (25) is studied in detail in [35]. It is found, in particular, that the stationary states are essentially non-Maxwellian ones and, at the diffusion stage of relaxation, the characteristic displacement of a particle grows superdiffusively with time and is inversely proportional to the magnetic field:

$$\Delta r \propto < r^q >^{1/q} \propto t^{1/\alpha}/B. \quad (28)$$

In [35], the analytical results are also compared with those of the numerical simulation based on the solution of the Langevin equations (24). As an example of numerical simulation, in Fig.6, we present the trajectories of a particle on the (v_x, v_y) plane for a) charged Brownian particle, $\alpha = 2$, and b) charged Lévy particle, $\alpha = 1.2$. Large "Lévy flights" are clearly seen in Fig.6,*b*. For quantitative results, see [35]

7. Summary

Let us summarize briefly the results reviewed above:

(i) fractional kinetic equations appear as a natural generalization of the basic kinetic equations of the theory of Brownian motion;

(ii) they have a solid probabilistic mathematical justification based on Generalized Central Limit Theorem;

(iii) space and/or velocity fractional kinetic equations can be a convenient tool for studying systems

— far from equilibrium,

— in external fields,

— with boundary-value problems;

(iv) the systems governed by fractional kinetic equations demonstrate "unusual" relaxation properties and non-Boltzmann stationary states,

thus giving

(v) interesting predictions for experiments far from equilibrium.

The problem is how to determine the value of the Lévy index and/or the order of fractional derivative. In some cases, it may be determined either from experimental data or from semiphenomenological supporting models. At the same time, the development of a consistent theory of essentially non-Gaussian fluctuations, leading to a "microscopic" justification of fractional kinetics, is of considerable interest. Finally, we mention that more involved fractional kinetic equations were recently proposed, which employ so-called distributed-order fractional derivatives and describe, in particular, the diffusion phenomena characterized by diffusion exponent varying with time [44].

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ДРОБОВА КІНЕТИКА АНОМАЛЬНОЇ ДИФУЗІЇ ТА РЕЛАКСАЦІЇ

А.В. Чечкін, В.Ю. Гончар

Р е з ю м е

Кінетичні рівняння в частинних дробових похідних нещодавно привернули до себе увагу як засіб описання аномальних релаксаційних та дифузійних явищ. Наведено короткий огляд сучасного стану кінетичних дробових рівнянь. Розглянуто такі питання:
—одержання дробових кінетичних рівнянь з просторовою дробовою похідною;
— аномальні дифузія і релаксація,
— небольцманівські стаціонарні стани.
Запропоновано кілька застосувань загальної теорії до проблем фізики плазми.

ДРОБНАЯ КИНЕТИКА АНОМАЛЬНОЙ ДИФФУЗИИ И РЕЛАКСАЦИИ

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Р е з ю м е

Кинетические уравнения с частными дробными производными привлекли недавно внимание как инструмент описания аномальных релаксационных и диффузионных явлений. Дан краткий обзор современного статуса дробных кинетических уравнений. Рассмотрены следующие вопросы:
— получение дробных кинетических уравнений с пространственной дробной производной;
— аномальные диффузия и релаксация;
— небольцмановские стационарные состояния.
Предложен ряд приложений общей теории к проблемам физики плазмы.