A DISCRETE NONLINEAR MODEL OF THREE COUPLED DYNAMICAL FIELDS INTEGRABLE BY THE CAUDREY METHOD

O. O. VAKHNENKO

Bogolyubov Institute for Theoretical Physics, Nat. Acad. Sci. of Ukraine
(14b, Metrolohichna Str., Kyïv 03143, Ukraine; e-mail: vakhnenko@bitp.kiev.ua),
Institute of Geophysics
(63b, Bohdan Khmel'nyts'kyy Str., Kyïv 01054, Ukraine)

UDC 530.182; 531.395

A nonlinear model of three coupled dynamical fields on an infinite regular chain is proposed. The system admits the zero curvature representation constituting the basis for its integration within the framework of inverse scattering transform. The associated auxiliary spectral problem is basically of the third order and gives rise to a fairly complicated subdivision into the regularity domains of Jost functions in the plane of a complex spectral parameter. As a result, both the direct and inverse scattering problems turn out to be substantially nontrivial. The Caudrey version of the direct and inverse scattering techniques for the needs of model integration is adapted. The simplest soliton solution is found.

Introduction

In a series of his articles [1—5], Caudrey developed a fairly constructive version of inverse scattering transform valid in principle for any one-dimensional scattering problem of an arbitrary order. In particular, the method enables one to integrate the nonlinear evolution equations associated with third-order differential (continuous) and finite difference (discrete) spectral problems in a manner substantially simpler [2, 4] as compared with other approaches [6] as well as to treat adequately a so-called loop-like soliton and multisoliton solutions [7, 8].

Bearing in mind the success of the Caudrey approach, we will introduce here a new third-order discrete spectral problem and will show how it generates an integrable dynamical system of three nonlinearly coupled fields on a regular infinite lattice. We will give a general sketch of the Caudrey method as applied to the model of interest and test it on a simplest solution.

The main features of the suggested model are as follows. The model describes the three-component nonlinear dynamical system on a regular one-dimensional lattice of the second order in time regarding to each field variable. Two of its components are mutually equivalent and, if required, may be combined approximately into a Toda-type field accompanied by some satellite field. The third component of the system is responsible for the angular valued field similar to that in the sine-Gordon model but with another type of nonlinearity and with the standard definitions of spatial and temporal coordinates. The couplings between the components are essentially nonlinear and displayed both in the kinetic and potential parts of a Lagrangian function.

1. Auxiliary Linear Problems

It is well known [9–11] that the compatibility condition

$$\frac{d}{d\tau}L(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z) \tag{1}$$

of two auxiliary linear problems

$$|u(n+1|z)\rangle \;=\; L(n|z)|u(n|z)\rangle, \tag{2}$$

$$\frac{d}{d\tau}|u(n|z)\rangle \;=\; A(n|z)|u(n|z)\rangle \tag{3}$$

permits both to restore the evolution operator $A(n|z)$ and to generate some nontrivial discrete nonlinear evolution system provided the spectral operator $L(n|z)$ is appropriately chosen. Here n stands for the discrete coordinate variable running through all integers from minus to plus infinity, τ denotes the time variable, while z marks the time independent spectral parameter.

In what follows, we adopt $L(n|z)$ and $A(n|z)$ to be 3×3 matrices, whereas $|u(n|z)\rangle$ is the three-component column vector

$$|u(n|z)\rangle \equiv \begin{pmatrix} \langle 1|u(n|z)\rangle \\ \langle 2|u(n|z)\rangle \\ \langle 3|u(n|z)\rangle \end{pmatrix}. \tag{4}$$

Taking $L(n|z)$ in the form

$L(n|z) =$

$$= \begin{pmatrix} p_{11}(n)+z+1/z & F_{12}(n) & p_{13}(n) \\ G_{21}(n) & 0 & G_{23}(n) \\ p_{31}(n) & F_{32}(n) & p_{33}(n)+z+1/z \end{pmatrix} \quad (5)$$

and looking for $A(n|z)$ in the form

$$A(n|z) = \begin{pmatrix} 0 & A_{12}(n) & 0 \\ A_{21}(n) & z+1/z & A_{23}(n) \\ 0 & A_{32}(n) & 0 \end{pmatrix}, \quad (6)$$

we observe that the positive result can be achieved under the reduction

$$F_{12}(n) = i\exp[+q_-(n)]\cos\alpha(n), \quad (7)$$

$$G_{21}(n) = i\exp[-q_-(n)]\cos\alpha(n), \quad (8)$$

$$G_{23}(n) = i\exp[-q_+(n)]\sin\alpha(n), \quad (9)$$

$$F_{32}(n) = i\exp[+q_+(n)]\sin\alpha(n); \quad (10)$$

$$p_{11}(n) = \dot q_-(n)\left[1-\sin^4\alpha(n)\right] -$$

$$-\dot q_+(n)\sin^2\alpha(n)\cos^2\alpha(n); \quad (11)$$

$$p_{13}(n) = \exp\left[+q_-(n)-q_+(n)\right]\times$$

$$\times\left[\dot q_-(n)\sin^3\alpha(n)\cos\alpha(n)+\right.$$

$$\left.+\,\dot q_+(n)\sin\alpha(n)\cos^3\alpha(n)-\dot\alpha(n)\right]; \quad (12)$$

$$p_{31}(n) = \exp\left[-q_-(n)+q_+(n)\right]\times$$

$$\times\left[\dot q_-(n)\sin^3\alpha(n)\cos\alpha(n)+\right.$$

$$\left.+\,\dot q_+(n)\sin\alpha(n)\cos^3\alpha(n)+\dot\alpha(n)\right], \quad (13)$$

$$p_{33}(n) = \dot q_+(n)\left[1-\cos^4\alpha(n)\right] -$$

$$-\dot q_-(n)\sin^2\alpha(n)\cos^2\alpha(n). \quad (14)$$

Here, $q_-(n)$, $q_+(n)$ and $\alpha(n)$ are nothing but the field variables of a desired nonlinear evolution model, while the overdot stands for the derivative with respect to time τ. In turn, the matrix elements $A_{jk}(n)$ of the evolution operator $A(n|z)$ are found to be

$$A_{12}(n) = -i\exp[+q_-(n)]\cos\alpha(n), \quad (15)$$
$$A_{21}(n) = -i\exp[-q_-(n-1)]\cos\alpha(n-1), \quad (16)$$
$$A_{23}(n) = -i\exp[-q_+(n-1)]\sin\alpha(n-1), \quad (17)$$
$$A_{32}(n) = -i\exp[+q_+(n)]\sin\alpha(n). \quad (18)$$

2. Nonlinear Three-component Dynamical Model

Inserting the just obtained expressions (7)–(14) and (15)–(18) into spectral (5) and evolution (6) matrices and manipulating with the compatibility (zero-curvature) relation (1), we come readily to the system of three dynamical equations for the field variables $q_+(n)$, $q_-(n)$, and $\alpha(n)$ on an infinite regular one-dimensional chain. For the sake of brevity, we prefer to write them in the standard Lagrangian form as

$$\frac{d}{d\tau}\left[\partial\mathscr{L}/\partial\dot q_+(n)\right] = \partial\mathscr{L}/\partial q_+(n), \quad (19)$$

$$\frac{d}{d\tau}\left[\partial\mathscr{L}/\partial\dot q_-(n)\right] = \partial\mathscr{L}/\partial q_-(n), \quad (20)$$

$$\frac{d}{d\tau}\left[\partial\mathscr{L}/\partial\dot\alpha(n)\right] = \partial\mathscr{L}/\partial\alpha(n) \quad (21)$$

with the Lagrangian function $\mathscr{L}$ given by

$$\mathscr{L} = \frac{1}{2}\sum_{m=-\infty}^{\infty}\dot q_+^2(m)\left[1+\cos^2\alpha(m)\right]\sin^2\alpha(m)+$$

$$+\frac{1}{2}\sum_{m=-\infty}^{\infty}\dot q_-^2(m)\left[1+\sin^2\alpha(m)\right]\cos^2\alpha(m)-$$

$$-\sum_{m=-\infty}^{\infty}\dot q_+(m)\dot q_-(m)\cos^2\alpha(m)\sin^2\alpha(m)-$$

$$-\sum_{m=-\infty}^{\infty}\dot\alpha^2(m)-\sum_{m=-\infty}^{\infty}\exp\left[+q_+(m+1)-q_+(m)\right]\times$$

$$\times\sin\alpha(m+1)\sin\alpha(m)-$$

$$-\sum_{m=-\infty}^{\infty}\exp\left[+q_-(m+1)-q_-(m)\right]\times$$

$$\times\cos\alpha(m+1)\cos\alpha(m). \quad (22)$$

According to the general rule, an equivalence between the zero curvature equation (1) and the nonlinear model of interest (19)–(22) following from the chosen specification (7)–(18) of spectral (5) and evolution (6) matrices opens the door for model (19)–(22) to be integrated by the method of inverse scattering transform. However, the particular realization of this scheme in our case does not look so simple inasmuch as three (rather than two) different eigenvalues constitute the spectrum of the limiting spectral matrix $L(-\infty|z)$ or adequately $L(+\infty|z)$. Therefore, we are bound to rely here upon the Caudrey approach [2, 4] although with some inevitable modifications.

ISSN 0503-1265. Ukr. J. Phys. 2003. V. 48, N 7

3. Gauge Transformed Auxiliary Linear Problems

Inspecting the limits of spectral matrix (5), (7)–(14) at both spatial infinities, we see that, in general, they do not coincide:

$$\lim_{n\to-\infty} L(n|z) \neq \lim_{n\to+\infty} L(n|z). \tag{23}$$

As a consequence, the direct application of the Caudrey theory to our problem is formally forbidden.

This type of inconvenience is known also for the Toda system [12] and can be rebuffed in principle by an appropriate gauge transformation:

$$|v(n|z)\rangle = S(n)|u(n|z)\rangle, \tag{24}$$
$$M(n|z) = S(n+1)L(n|z)S^{-1}(n). \tag{25}$$

The choice of the gauge matrix $S(n)$ is not unique and should meet the only basic condition

$$\lim_{n\to-\infty} M(n|z) = M(z) = \lim_{n\to+\infty} M(n|z), \tag{26}$$

where $M(z)$ can be taken as

$$M(z) = \begin{pmatrix} z+1/z & F_{12} & 0 \\ G_{21} & 0 & G_{23} \\ 0 & F_{32} & z+1/z \end{pmatrix}, \tag{27}$$

$$F_{12} = i\exp[+q_-]\cos\alpha, \tag{28}$$

$$G_{21} = i\exp[-q_-]\cos\alpha, \tag{29}$$

$$G_{23} = i\exp[-q_+]\sin\alpha, \tag{30}$$

$$F_{32} = i\exp[+q_+]\sin\alpha. \tag{31}$$

For the sake of definiteness, we adopt the gauge matrix to be

$$S(n) = \begin{pmatrix} S_{11}(n) & 0 & S_{13}(n) \\ 0 & 1 & 0 \\ S_{31}(n) & 0 & S_{33}(n) \end{pmatrix}, \tag{32}$$

where

$$S_{11}(n) = \exp\left[-q_-(n) + q_-\right]\cos[\alpha(n) - \alpha], \tag{33}$$

$$S_{13}(n) = +\exp\left[-q_+(n) + q_-\right]\sin[\alpha(n) - \alpha], \tag{34}$$

$$S_{31}(n) = -\exp\left[-q_-(n) + q_+\right]\sin[\alpha(n) - \alpha], \tag{35}$$

$$S_{33}(n) = \exp\left[-q_+(n) + q_+\right]\cos[\alpha(n) - \alpha]. \tag{36}$$

The use of the gauge transformed auxiliary linear problems

$$|v(n+1)|z)\rangle = M(n|z)|v(n|z)\rangle, \tag{37}$$

$$\frac{d}{d\tau}|v(n|z)\rangle = B(n|z)|v(n|z)\rangle, \tag{38}$$

and the gauge transformed zero-curvature relation

$$\dot{M}(n|z) = B(n+1|z)M(n|z) - M(n|z)B(n|z) \tag{39}$$

enables one to remove totally the principal theoretical obstacle described at the beginning of this section. Here, $M(n|z)$ and $B(n|z)$ denote the transformed spectral and evolution operators given by (25) and

$$B(n|z) = S(n)A(n|z)S^{-1}(n) + \dot{S}(n)S^{-1}(n), \tag{40}$$

respectively.

4. Eigenvalues of $M(z)$ and the Domains of their Subordination

According to Caudrey [2, 4], the main peculiarities of a particular inverse scattering problem are determined by the spectral properties of the limiting spectral matrix $M(z)$. In this context, the first step is to resolve the right

$$M(z)|v(z)\rangle = \zeta(z)|v(z)\rangle \tag{41}$$

and the left

$$\langle v^+(z)|M(z) = \langle v^+(z)|\zeta(z) \tag{42}$$

eigenvalue problems with $M(z)$ given by (27) — (31) and to perform the mutual comparison of all obtained eigenvalues

$$\zeta_1(z) = z, \tag{43}$$
$$\zeta_2(z) = z + 1/z, \tag{44}$$
$$\zeta_3(z) = 1/z \tag{45}$$

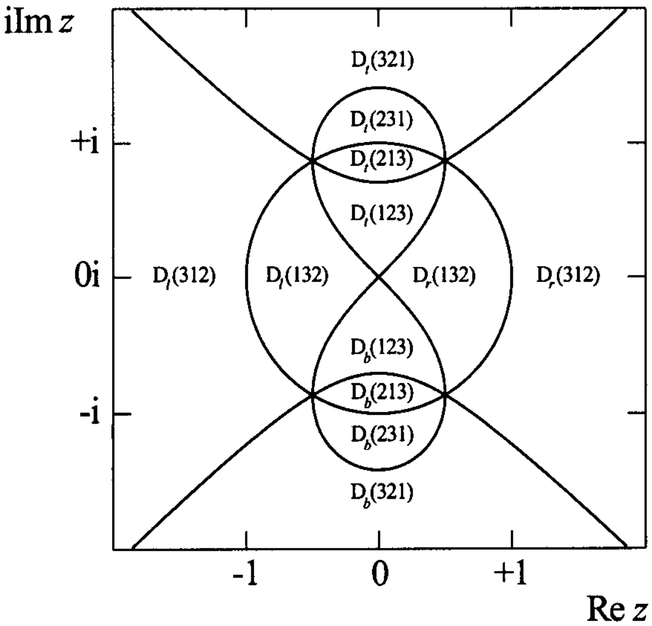
with respect of their moduli on the whole plane of complex spectral parameter z.

Thus, for the components $\langle k|v_j(z)\rangle$ of the right (column) eigenvectors $|v_1(z)\rangle$, $|v_2(z)\rangle$, and $|v_3(z)\rangle$, we find

$$\langle 1|v_1(z)\rangle = i\exp[+q_-]\cos\alpha, \tag{46}$$
$$\langle 2|v_1(z)\rangle = -1/z, \tag{47}$$
$$\langle 3|v_1(z)\rangle = i\exp[+q_+]\sin\alpha, \tag{48}$$

$$\langle 1|v_2(z)\rangle = +i\exp[+q_-]\sin\alpha, \tag{49}$$
$$\langle 2|v_2(z)\rangle = 0, \tag{50}$$
$$\langle 3|v_2(z)\rangle = -i\exp[+q_+]\cos\alpha, \tag{51}$$

$$\langle 1|v_3(z)\rangle = i\exp[+q_-]\cos\alpha, \tag{52}$$
$$\langle 2|v_3(z)\rangle = -z, \tag{53}$$
$$\langle 3|v_3(z)\rangle = i\exp[+q_+]\sin\alpha. \tag{54}$$



Subdivision into the regions $\{D(jkl)\}$ in the plane of complex spectral parameter z

For the components $\langle v_j^+(z)|k\rangle$ of the left (row) eigenvectors $\langle v_1^+(z)|$, $\langle v_2^+(z)|$, and $\langle v_3^+(z)|$, in turn, we have

$$\langle v_1^+(z)|1\rangle = i\exp[-q_-]\cos\alpha, \tag{55}$$
$$\langle v_1^+(z)|2\rangle = -1/z, \tag{56}$$
$$\langle v_1^+(z)|3\rangle = i\exp[-q_+]\sin\alpha, \tag{57}$$

$$\langle v_2^+(z)|1\rangle = -i\exp[-q_-]\sin\alpha, \tag{58}$$
$$\langle v_2^+(z)|2\rangle = 0, \tag{59}$$
$$\langle v_2^+(z)|3\rangle = +i\exp[-q_+]\cos\alpha, \tag{60}$$

$$\langle v_3^+(z)|1\rangle = i\exp[-q_-]\cos\alpha, \tag{61}$$
$$\langle v_3^+(z)|2\rangle = -z, \tag{62}$$
$$\langle v_3^+(z)|3\rangle = i\exp[-q_+]\sin\alpha. \tag{63}$$

The jth eigenvalue $\zeta_j(z)$ pertains equally both to the jth right $|v_j(z)\rangle$ and jth left $\langle v_j^+(z)|$ eigenvectors. As a result, the orthogonality relations

$$\frac{\langle v_j^+(z)|v_k(z)\rangle}{\langle v_j^+(z)|v_j(z)\rangle} = \delta_{jk} \tag{64}$$

are proved to be valid, where

$$\langle v_j^+(z)|v_k(z)\rangle \equiv \sum_{l=1}^{3}\langle v_j^+(z)|l\rangle\langle l|v_k(z)\rangle, \tag{65}$$

while j and k run from 1 to 3.

Insofar as all three eigenvalues $\zeta_1(z)$, $\zeta_2(z)$, and $\zeta_3(z)$ of $M(z)$ as functions of z are distinct (see (43)–(45)), the complex z plane can inevitably be divided into six different regions in accordance with six feasible chains of inequalities between their moduli $|\zeta_1(z)|$, $|\zeta_2(z)|$, and $|\zeta_3(z)|$. Speaking formally, the parameter z will be regarded as that belonging to the region $D(jkl)$ provided it satisfies the chains of inequalities

$$|\zeta_j(z)| < |\zeta_k(z)| < |\zeta_l(z)|, \tag{66}$$

where the sequence $\{jkl\}$ must be detectable among six possible permutations of $\{123\}$. Due to the evident equalities

$$|\zeta_j(z^*)| = |\zeta_j(z)| = |\zeta_j(-z)|, \tag{67}$$

each of the just defined regions is subdivided into two disconnected symmetric domains either in the top and bottom quadrants as

$$D(123) = D_t(123) + D_b(123), \tag{68}$$
$$D(213) = D_t(213) + D_b(213), \tag{69}$$
$$D(231) = D_t(231) + D_b(231), \tag{70}$$
$$D(321) = D_t(321) + D_b(321) \tag{71}$$

or predominantly in the left and right quadrants as

$$D(132) = D_l(132) + D_r(132), \tag{72}$$
$$D(312) = D_l(312) + D_r(312) \tag{73}$$

(see the Figure for clarity).

5. Direct Scattering Problem. The Advanced Caudrey Approach

The advanced version of the Caudrey approach to direct and inverse scattering problems is based upon his generalized definition of Jost functions [2, 4] which, in contrast to the standard ones [1, 3, 6], succeeds in avoiding any address to the conjugate spectral problem even in the highly complicated cases of spectral plane subdivision. As a result, both the direct and inverse scattering theories acquire the forms of essentially formalized procedures weakly sensitive to a particular choice of the spectral operator.

Following Caudrey [2, 4], we adopt the jth envelope Jost function

$$|\Phi_j(n|z)\rangle = [\zeta_j(z)]^{-n}\,|\varphi_j(n|z)\rangle \qquad (j=1,2,3) \tag{74}$$

associated with the gauge transformed auxiliary spectral problem (37) as a solution to the jth Fredholm summation equation

$$|\Phi_j(n|z)\rangle = |v_j(z)\rangle + \sum_{m=-\infty}^{\infty} K_j(n|z|m)||\Phi_j(m|z)\rangle \quad (75)$$

$(j = 1, 2, 3)$

with the jth kernel matrix $K_j(n|z|m)$ specified by the expression

$$K_j(n|z|m) = [\zeta_j(z)]^{m-n} [M(z)]^{n-m-1} \times$$
$$\times \left[\theta(n-m)I - \sum_{k=1}^{3} \theta\left(|\zeta_k(z)| - |\zeta_j(z)|\right) P_k(z)\right] \times$$
$$\times [M(m|z) - M(z)] \qquad (j = 1, 2, 3), \quad (76)$$

where $P_k(z)$ stands for the kth projection operator

$$P_k(z) \equiv \frac{|v_k(z)\rangle\langle v_k^+(z)|}{\langle v_k^+(z)|v_k(z)\rangle} \qquad (k = 1, 2, 3), \quad (77)$$

I denotes the unity 3×3 matrix, and

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \le 0 \end{cases}. \quad (78)$$

Then we call $|\varphi_j(n|z)\rangle$ to be the jth Jost function inasmuch as it meets all necessary demands of the usual definition, i.e. satisfies both the gauge transformed spectral equation (37) and the standard boundary condition

$$\lim_{n\to-\infty} [\zeta_j(z)]^{-n} |\varphi_j(n|z)\rangle = |v_j(z)\rangle \qquad (j = 1, 2, 3). \; (79)$$

An extra property affirming the boundedness of each envelope Jost function $|\Phi_j(n|z)\rangle$ $(j = 1, 2, 3)$ at $n \to +\infty$ follows from the respective Fredholm equation (75) after the use of the identity

$$\theta(n-m) \cdot I - \sum_{k=1}^{3} \theta\left(|\zeta_k(z)| - |\zeta_j(z)|\right) P_k(z) \equiv$$
$$\equiv \sum_{k=1}^{3} \theta\left(|\zeta_j(z)| - |\zeta_k(z)|\right) P_k(z) - \theta(m+1-n)I +$$
$$+ P_j(z) + \sum_{k=1}^{3} (1-\delta_{jk}) \left[1 - \theta(|\zeta_k(z)| - |\zeta_j(z)|) -\right.$$
$$\left. - \theta(|\zeta_j(z)| - |\zeta_k(z)|)\right] P_k(z) \quad (80)$$

in respective kernel matrix (76).

When proving the boundary properties of Jost functions at $n \to -\infty$ and $n \to +\infty$, an assumption about a sufficiently rapid decrease of $M(n|z) - M(z)$ to a zero 3×3 matrix at respective infinity of the coordinate n turns out to be rather convenient.

Actually the same assumption about the rapid vanishing of $M(n|z) - M(z)$ at $|n| \to \infty$ supports a sufficient condition for the Fredholm determinant $f_j(z)$ and a 3×3 matrix analog of the first Fredholm minor $F_j(n|z|m)$ $(j = 1, 2, 3)$ associated with the respective Fredholm equation (75) to exist in every spectral region defined in Section 4. Here, it should be particularly emphasized that our spectral parameter z has nothing to do with the auxiliary parameter of the standard Fredholm theory [13]. Moreover, each kernel matrix $K_j(n|z|m)$ $(j = 1, 2, 3)$ is seen (76) to be a piecewise function of the spectral parameter, i.e. to exhibit a jump discontinuity once the parameter z crosses the boundary between spectral domains. In principle, the similar discontinuities may be displayed also in $f_j(z)$ and $F_j(n|z|m)$ $(j = 1, 2, 3)$ inasmuch as these quantities are linked to the respective $K_j(n|z|m)$ through the basic Fredholm relations

$$F_j(n|z|m) - f_j(z)K_j(n|z|m) =$$
$$= \sum_{l=-\infty}^{\infty} F_j(n|z|l)K_j(l|z|m) \qquad (j = 1, 2, 3), \quad (81)$$

$$F_j(n|z|m) - f_j(z)K_j(n|z|m) =$$
$$= \sum_{l=-\infty}^{\infty} K_j(n|z|l)F_j(l|z|m) \qquad (j = 1, 2, 3). \quad (82)$$

However, within the interior of each individual spectral domain, the quantities $f_j(z)$ and $F_j(n|z|m)$ $(j = 1, 2, 3)$ have to be regular functions of the spectral parameter z [14] insofar as the same is certainly true for the respective kernel $K_j(n|z|m)$ (76). The limits of these functions as z approaches any interdomain boundary exist and are finite.

Thus, at all z providing $f_j(z) \neq 0$, the jth 3×3 matrix Fredholm resolvent

$$R_j(n|z|m) = \frac{F_j(n|z|m)}{f_j(z)} \qquad (j = 1, 2, 3) \quad (83)$$

is defined and the formal solution of the jth Fredholm equation (75) is given by

$$|\Phi_j(n|z)\rangle = |v_j(z)\rangle + \sum_{m=-\infty}^{\infty} R_j(n|z|m)|v_j(z)\rangle$$

$$(j = 1, 2, 3). \tag{84}$$

We can reckon this result (84) for the set of envelope Jost functions $\{|\Phi_j(n|z)\rangle\}$ as the main step in mapping the set of dynamical variables $\{q_+(n), q_-(n), \alpha(n)\}$ into the set of scattering data. The latter in the Caudrey terminology [1—4] is nothing but the generic information about the poles of Jost functions and their residues as well as about possible discontinuities of Jost functions in the complex z plane. Evidently the poles are determined by the zeros of Fredholm determinants while the jump singularities may happen only on the boundaries between the domains in the plane of complex spectral parameter (i.e. spectral domains).

6. Direct Scattering Transform. How to Achieve an Explicit Mapping

Strictly speaking, the mapping from the field amplitudes into the Jost functions given in its present form (84) suffers to be fairly implicit insofar as the field amplitudes $q_-(n), q_+(n), \alpha(n)$ remain to be explicitly traceable in all Jost functions under consideration. Unfortunately, this fact becomes evident only in the final stage of inversion from the scattering data to the field variables and can be plainly elucidated, e.g., in the extra property

$$\lim_{|\zeta_j(z)|\to\infty}\left[|\Phi_j(n|z)\rangle - S(n)S^{-1}(-\infty)|v_j(z)\rangle\right] = 0\cdot I$$

$$(j = 1, 2, 3) \tag{85}$$

necessary for the inversion to be fixed uniquely.

Looking at the fixing condition (85), we find that the situation with an implicit mapping can be readily handled via the simple back gauge transformation in all quantities of interest. Thus, instead of the jth Jost function $|\varphi_j(n|z)\rangle$ and jth envelope Jost function $|\Phi_j(n|z)\rangle$ $(j = 1, 2, 3)$, we have to use their back gauge transforms

$$|\xi_j(n|z)\rangle = S^{-1}(n)|\varphi_j(n|z)\rangle \qquad (j = 1, 2, 3) \tag{86}$$

and

$$|\Xi_j(n|z)\rangle = S^{-1}(n)|\Phi_j(n|z)\rangle \qquad (j = 1, 2, 3), \tag{87}$$

accordingly. The jth resolvent solution (84), in turn, is apparently converted to yield

$$|\Xi_j(n|z)\rangle = |u_j(z)\rangle + \sum_{m=-\infty}^{\infty} D_j(n|z|m)|u_j(z)\rangle$$

$$(j = 1, 2, 3), \tag{88}$$

where

$$|u_j(z)\rangle = S^{-1}(-\infty)|v_j(z)\rangle \qquad (j = 1, 2, 3) \tag{89}$$

stands for the jth right eigenvector of the operator

$$L(z) = S^{-1}(-\infty)M(z)S(-\infty) = L(-\infty|z) \tag{90}$$

and

$$D_j(n|z|m) = S^{-1}(n)R_j(n|z|m)S(-\infty)+$$

$$+\left[S^{-1}(n)S(-\infty) - I\right]\delta_{nm} \quad (j = 1, 2, 3) \tag{91}$$

denotes the jth back transformed resolvent. Finally, the jth fixing condition (85) is rewritten to be

$$\lim_{|\zeta_j(z)|\to\infty}[|\Xi_j(n|z)\rangle - |u_j(z)\rangle] = 0\cdot I \quad (j = 1, 2, 3). \tag{92}$$

It is worth noticing that, while making the mapping to be explicit, the back gauge transform saves all fundamental properties of Jost functions and envelope Jost functions to be carried over into their back gauge transformed counterparts. Therefore, it looks reasonable to treat $|\xi_j(n|z)\rangle$ and $|\Xi_j(n|z)\rangle$ as the jth Jost function and the jth envelope Jost function $(j = 1, 2, 3)$ associated directly with the original auxiliary spectral problem (2).

7. Scattering Data. Reflectionless Case

According to the general rule [2, 4], the scattering data can be identified with the generic information about the singularities of resolvent matrices. In this paper, we adopt such a definition and consider the so-called reflectionless case where the jump singularities of Jost functions on the boundaries between domains in the plane of complex spectral parameter are absent and the only informative singularities of Jost functions are the poles emanated from the zeros of respective Fredholm determinants.

In order to examine these poles, we have to know the Wronskian

$$\mathop{\mathrm{W}}_{k=1}^{3}\{|\xi_k(n|z)\rangle\} \equiv \det[\langle j|\xi_k(n|z)\rangle] \tag{93}$$

calculated on the Jost solutions of the original spectral problem (2). Applying the Wronskian operation (93) to the set of equalities

$$|\xi_k(n+1|z)\rangle = L(n|z)|\xi_k(n|z)\rangle \qquad (k = 1, 2, 3) \tag{94}$$

with the use of the asymptotic properties

$$\lim_{n\to-\infty}[\zeta_k(z)]^{-n}|\xi_k(n|z)\rangle = |u_k(z)\rangle \qquad (k=1,2,3) \quad (95)$$

and the equalities

$$L(z)|u_k(z)\rangle = \zeta_k(z)|u_k(z)\rangle \qquad (k=1,2,3), \quad (96)$$

we obtain

$$\mathop{\mathrm{W}}_{k=1}^{3}\{|\xi_k(n|z)\rangle\} =$$

$$= \mathop{\mathrm{W}}_{k=1}^{3}\{|u_k(z)\rangle\}[\det L(z)]^n \prod_{m=-\infty}^{n-1}\left\{\frac{\det L(m|z)}{\det L(z)}\right\}. \quad (97)$$

Now we can readily conclude that the right-hand side of (97) does not contain any specific information about the singularities of resolvent matrix (91) because the same is true separately for its cofactors,

$$\mathop{\mathrm{W}}_{k=1}^{3}\{|u_k(z)\rangle\} = (z-1/z)e^{+q_+(-\infty)+q_-(-\infty)}, \quad (98)$$

and

$$\det L(z) = \det L(m|z) = z + 1/z. \quad (99)$$

Thus, we have

$$\mathop{\mathrm{W}}_{k=1}^{3}\{\xi_k(n|z)\rangle\} = [z+1/z]^n[z-1/z]e^{+q_+(-\infty)+q_-(-\infty)}. (100)$$

To proceed further, we denote $z_j(r)$ to be the rth zero of the jth Fredholm determinant $f_j(z)$ and assume each zero to be simple, i.e.

$$f_j(z_j(r)) = 0, \qquad \lim_{z\to z_j(r)} df_j(z)/dz \neq 0$$

$$(r=1,2,3,\ldots,N_j;\ j=1,2,3), \quad (101)$$

and does not lie on any boundary between spectral domains. Then the rth residue of the jth Jost function $|\xi_j(n|z)\rangle$ can be defined as

$$|\mathrm{Res}\,[\xi_j(n|z_j(r))]\rangle \equiv \lim_{z\to z_j(r)}\{|\xi_j(n|z)\rangle[z-z_j(r)]\}$$

$$(r=1,2,3,\ldots,N_j; \qquad j=1,2,3), \quad (102)$$

which is usual for simple poles. To ensure the finiteness of residues, we have to adopt zeros belonging to different Fredholm determinants to be distinct.

This latter demand becomes natural when inspecting the set of relationships

$$|\mathrm{Res}\,[\xi_j(n|z_j(r))]\rangle = \lim_{z\to z_j(r)}\sum_{k=1}^{3}|\xi_k(n|z)\rangle\gamma_{kj}(r)[1-\delta_{jk}]$$

$$(r=1,2,3,\ldots,N_j; \qquad j=1,2,3), \quad (103)$$

where the coefficients $\gamma_{kj}(r)$ $(r=1,2,3,\ldots,N_j; j=1,2,3; k\neq j)$ and the locations of the poles $z_j(r)$ $(r=1,2,3,\ldots,N_j; j=1,2,3)$ are referred to as a discrete part of scattering data [2, 4]. Each of $N_1+N_2+N_3$ formulae (103) represents the superposing condition between the columns of a certain 3×3 matrix with zero-valued determinant

$$\mathop{\mathrm{W}}_{k=1}^{3}\Big\{|\mathrm{Res}\,[\xi_j(n|z_j(r))]\rangle\delta_{jk}+$$

$$+\lim_{z\to z_j(r)}|\xi_k(n|z)\rangle[1-\delta_{jk}]\Big\} = 0$$

$$(r=1,2,3,\ldots,N_j; \qquad j=1,2,3) \quad (104)$$

obtainable from the basic Wronskian (100) by the simple limiting operation $\lim_{z\to z_j(r)}\{\ldots[z-z_j(r)]\}$. The weak point in the suggested arguments lies in tacitly supposed coordinate independence of the superposing coefficients $\gamma_{kj}(r)$. However, it is precisely this hypothesis which proves to be the only plausible assumption giving rise to the self-consistent time evolution of scattering data and thus ensuring the noncontradictive character of the whole theory.

Another concretization of superposing coefficients

$$\gamma_{kj}(r) = \theta\left(|\zeta_k(z_j(r))| - |\zeta_j(z_j(r))|\right)\Gamma_{kj}(r)$$

$$(r1=,2,3,\ldots,N_j; \quad j=1,2,3; \quad k\neq j) \quad (105)$$

is a direct consequence of the asymptotic conditions

$$\lim_{n\to-\infty}|\Xi_j(n|z)\rangle = |u_j(z)\rangle \qquad (j=1,2,3), \quad (106)$$

which can be easily observed during the reconstruction of envelope Jost functions.

An additional information about the scattering data can be obtained by the symmetry analysis of envelope Jost functions and will be taken into account in the final formulae of their reconstruction. The symmetries of interest may be formally divided into two groups:

$$|\Xi_1(n|1/z)\rangle = |\Xi_3(n|z)\rangle, \quad (107)$$

$$|\Xi_2(n|1/z)\rangle = |\Xi_2(n|z)\rangle, \quad (108)$$

$$|\Xi_3(n|1/z)\rangle = |\Xi_1(n|z)\rangle \quad (109)$$

and

$$\langle k|\Xi_j(n|z^*)\rangle^* = (-1)^k\langle k|\Xi_j(n|z)\rangle$$

$$(j=1,2,3; \qquad k=1,2,3). \quad (110)$$

While the first group (107)—(109) becomes evident from the very forms of the original spectral operator (5) and the limiting eigenfunctions (89) written explicitly, the second one (110) should invoke an extra assumption about the reality of dynamical variables $q_-(n)$, $q_+(n)$, $\alpha(n)$ for its proper justification.

8. Reconstruction of Envelope Jost Functions in Terms of their Residues

Relying upon the evident correspondence

$$|\xi_j(n|z)\rangle = [\zeta_j(z)]^n\, |\Xi_j(n|z)\rangle \qquad (j = 1, 2, 3) \tag{111}$$

between the Jost functions $|\xi_j(n|z)\rangle$ and the envelope Jost functions $|\Xi_j(n|z)\rangle$ and using the fundamental formulae (103) for the residues $|\mathrm{Res}[\xi_j(n|z_j(r))]\rangle$ of Jost functions, we readily come to the similar formulae

$$|\mathrm{Res}\,[\Xi_j(n|z_j(r))]\rangle = $$
$$= \lim_{z\to z_j(r)} \sum_{k=1}^{3} |\Xi_k(n|z)\rangle \gamma_{kj}(r)\,[1-\delta_{jk}] \left[\frac{\zeta_k(z_j(r))}{\zeta_j(z_j(r))}\right]^n$$
$$(r = 1, 2, 3, \ldots, N_j; \qquad j = 1, 2, 3) \tag{112}$$

for the residues

$$|\mathrm{Res}\,[\Xi_j(n|z_j(r))]\rangle \equiv \lim_{z\to z_j(r)} \{|\Xi_j(n|z)\rangle[z - z_j(r)]\}$$
$$(r = 1, 2, 3, \ldots, N_j; \qquad j = 1, 2, 3) \tag{113}$$

of envelope Jost functions.

Now we have collected all necessary information (92), (106), (107)–(109), (110), (112) sufficient to reconstruct the general features of envelope Jost functions in reflectionless case. The methods of complex variable theory yield

$$|\Xi_1(n|z)\rangle = |u_1(z)\rangle + (1-\delta_{0N})\times$$
$$\times\left[\sum_{r=1}^{N} |\Xi_2(n|z_1(r))\rangle C_{21}(n|r)\frac{\zeta_1(z_1(r))}{\zeta_1(z)-\zeta_1(z_1(r))} + \right.$$
$$+\sum_{r=1}^{N} |\Xi_3(n|z_1(r))\rangle C_{31}(n|r)\frac{\zeta_1(z_1(r))}{\zeta_1(z)-\zeta_1(z_1(r))} +$$
$$+\sum_{r=1}^{N} |\Xi_2(n|z_1^*(r))\rangle C_{21}^*(n|r)\frac{\zeta_1(z_1^*(r))}{\zeta_1(z)-\zeta_1(z_1^*(r))} +$$
$$\left.+\sum_{r=1}^{N} |\Xi_3(n|z_1^*(r))\rangle C_{31}^*(n|r)\frac{\zeta_1(z_1^*(r))}{\zeta_1(z)-\zeta_1(z_1^*(r))}\right], \tag{114}$$

$$|\Xi_2(n|z)\rangle = |u_2(z)\rangle + (1-\delta_{0M})\times$$
$$\times\left[\sum_{r=1}^{M} |\Xi_1(n|\,\bar{z}_2(r))\rangle\, \bar{C}_{12}(n|r)\frac{\zeta_2(\bar{z}_2(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2(r))} + \right.$$
$$+\sum_{r=1}^{M} |\Xi_3(n|\,\bar{z}_2(r))\rangle\, \bar{C}_{32}(n|r)\frac{\zeta_2(\bar{z}_2(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2(r))} +$$
$$+\sum_{r=1}^{M} |\Xi_1(n|\,\bar{z}_2^*(r))\rangle\, \bar{C}_{12}^*(n|r)\frac{\zeta_2(\bar{z}_2^*(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2^*(r))} +$$
$$\left.+\sum_{r=1}^{M} |\Xi_3(n|\,\bar{z}_2^*(r))\rangle\, \bar{C}_{32}^*(n|r)\frac{\zeta_2(\bar{z}_2^*(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2^*(r))}\right], \tag{115}$$

$$|\Xi_2(n|z)\rangle = |u_2(z)\rangle + (1-\delta_{0M})\times$$
$$\times\left[\sum_{r=1}^{M} |\Xi_3(n|\,\overset{+}{z}_2(r))\rangle\, \overset{+}{C}_{32}(n|r)\frac{\zeta_2(\overset{+}{z}_2(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}_2(r))} + \right.$$
$$+\sum_{r=1}^{M} |\Xi_1(n|\,\overset{+}{z}_2(r))\rangle\, \overset{+}{C}_{12}(n|r)\frac{\zeta_2(\overset{+}{z}_2(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}_2(r))} +$$
$$+\sum_{r=1}^{M} |\Xi_3(n|\,\overset{+}{z}{}_2^*(r))\rangle\, \overset{+}{C}{}_{32}^*(n|r)\frac{\zeta_2(\overset{+}{z}{}_2^*(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}{}_2^*(r))} +$$
$$\left.+\sum_{r=1}^{M} |\Xi_1(n|\,\overset{+}{z}{}_2^*(r))\rangle\, \overset{+}{C}{}_{12}^*(n|r)\frac{\zeta_2(\overset{+}{z}{}_2^*(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}{}_2^*(r))}\right], \tag{116}$$

$$|\Xi_3(n|z)\rangle = |u_3(z)\rangle + (1-\delta_{0N})\times$$
$$\times\left[\sum_{r=1}^{N} |\Xi_1(n|z_3(r))\rangle C_{13}(n|r)\frac{\zeta_3(z_3(r))}{\zeta_3(z)-\zeta_3(z_3(r))} + \right.$$
$$+\sum_{r=1}^{N} |\Xi_2(n|z_3(r))\rangle C_{23}(n|r)\frac{\zeta_3(z_3(r))}{\zeta_3(z)-\zeta_3(z_3(r))} +$$
$$+\sum_{r=1}^{N} |\Xi_1(n|z_3^*(r))\rangle C_{13}^*(n|r)\frac{\zeta_3(z_3^*(r))}{\zeta_3(z)-\zeta_3(z_3^*(r))} +$$
$$\left.+\sum_{r=1}^{N} |\Xi_2(n|z_3^*(r))\rangle C_{23}^*(n|r)\frac{\zeta_3(z_3^*(r))}{\zeta_3(z)-\zeta_3(z_3^*(r))}\right], \tag{117}$$

where

$$C_{k1}(n|r) = \theta(|\zeta_k(z_1(r))| - |\zeta_1(z_1(r))|)\times$$

$$\times|\zeta_1(z_1(r))|)\left[\zeta_k(z_1(r))/\zeta_1(z_1(r))\right]^n C_{k1}(r)$$

$$(r=1,2,3,\ldots,N;\quad k=2,3;\quad z_1(r)z_3(r)=1), \quad (118)$$

$$\bar{C}_{k2}(n|r)=\theta(|\zeta_k(\bar{z}_2(r))|-|\zeta_2(\bar{z}_2(r))|)\times$$

$$\times\left[\zeta_k(\bar{z}_2(r))/\zeta_2(\bar{z}_2(r))\right]^n \bar{C}_{k2}(r)$$

$$(r=1,2,3,\ldots,M;\quad k=1,3;\quad \bar{z}_2(r)\stackrel{+}{z}_2(r)=1), \quad (119)$$

$$\stackrel{+}{C}_{k2}(n|r)=\theta(|\zeta_k(\stackrel{+}{z}_2(r))|-|\zeta_2(\stackrel{+}{z}_2(r))|)\times$$

$$\times\left[\zeta_k(\stackrel{+}{z}_2(r))/\zeta_2(\stackrel{+}{z}_2(r))\right]^n \stackrel{+}{C}_{k2}(r)$$

$$(r=1,2,3,\ldots,M;\quad k=3,1;\quad \stackrel{+}{z}_2(r)\bar{z}_2(r)=1), \quad (120)$$

$$C_{k3}(n|r)=\theta(|\zeta_k(z_3(r))|-|\zeta_3(z_3(r))|)\times$$

$$\times\left[\zeta_k(z_3(r))/\zeta_3(z_3(r))\right]^n C_{k3}(r)$$

$$(r=1,2,3,\ldots,N;\quad k=1,2;\quad z_3(r)z_1(r)=1). \quad (121)$$

Here, we have introduced a new numeration of poles originated from the manifested symmetries (107)–(110) of envelope Jost functions. In so doing, we had to assume that $N_1=2N=N_3$, $N_2=4M$ and

$$C_{21}(r) = C_{23}(r) \qquad (r=1,2,3,\ldots,N), \quad (122)$$
$$C_{31}(r) = C_{13}(r) \qquad (r=1,2,3,\ldots,N), \quad (123)$$
$$\bar{C}_{12}(r) = \stackrel{+}{C}_{32}(r) \qquad (r=1,2,3,\ldots,M), \quad (124)$$
$$\bar{C}_{32}(r) = \stackrel{+}{C}_{12}(r) \qquad (r=1,2,3,\ldots,M). \quad (125)$$

It is convenient to treat the quantities $C_{21}(r)$, $C_{31}(r)$, $z_1(r)$, $C_{23}(r)$, $C_{13}(r)$, $z_3(r)$ (where $r=1,2,3,\ldots,N$) and $\bar{C}_{12}(r)$, $\bar{C}_{32}(r)$,$\bar{z}_2(r)$, $\stackrel{+}{C}_{32}(r)$, $\stackrel{+}{C}_{12}(r)$, $\stackrel{+}{z}_2(r)$ (where $r=1,2,3,\ldots,M$) together with their complex conjugate counterparts as a new, more adequate, although overfilled set of scattering data rather than to decipher them in terms of the original set.

The general scheme of complete reconstruction of envelope Jost functions from the scattering data in reflectionless case becomes clear, since it is reduced actually to resolving the set of linear algebraic equations with respect to the quantities $|\Xi_j(n|z_1(r))\rangle$, $|\Xi_j(n|z_1^*(r))\rangle$ (where $j=2,3$; $r=1,2,3,\ldots,N$) and $|\Xi_j(n|\bar{z}_2(r))\rangle$, $|\Xi_j(n|\bar{z}_2^*(r))\rangle$ (where $j=1,3$; $r=1,2,3,\ldots,M$) or alternatively with respect to the quantities $|\Xi_j(n|\stackrel{+}{z}_2(r))\rangle$, $|\Xi_j(\stackrel{+}{z}{}_2^*(r))\rangle$ (where $j=3,1$; $r=1,2,3,\ldots,M$) and $|\Xi_j(n|z_3(r))\rangle$, $|\Xi_j(n|z_3^*(r))\rangle$ (where $j=1,2$; $r=1,2,3,\ldots,N$). However, its practical implementation is determined by the additional condition (173) supporting the self-consistency of field variables through the interdependence of scattering data (see Section 10 for more details).

9. Time Dependencies of Scattering Data

Below we derive the evolution equations for the scattering data as they were defined in the previous section.

As the first step, it is reasonable to suppose that $\dot{M}(z)=0\cdot I$. Then, at both spatial infinities $|n|\to\infty$, the gauge transformed zero-curvature relation (39) yields

$$B(z)M(z)=M(z)B(z), \quad (126)$$

where the limiting value

$$B(z)=\lim_{|n|\to\infty}B(n|z) \quad (127)$$

of the gauge transformed evolution operator (40) is given by

$$B(z)=(z+1/z)\cdot I-M(z). \quad (128)$$

Thus, the operators $B(z)$ and $M(z)$ are commutative and hence possess the same two sets of eigenvectors, namely the right (46)–(54) and left (55)–(63) ones. However, the eigenvalues

$$\eta_1(z) = 1/z, \quad (129)$$
$$\eta_2(z) = 0, \quad (130)$$
$$\eta_3(z) = z \quad (131)$$

of $B(z)$ and eigenvalues (43)–(45) of $M(z)$ are seen to be distinct. Of course, the jth eigenvalue $\eta_j(z)$ pertains equally well both to the jth right $|v_j(z)\rangle$ and jth left $\langle v_j^+(z)|$ eigenvectors, i.e.

$$B(z)|v_j(z)\rangle = \eta_j(z)|v_j(z)\rangle \qquad (j=1,2,3), \quad (132)$$
$$\langle v_j^+(z)|B(z) = \langle v_j^+(z)|\eta_j(z) \qquad (j=1,2,3). \quad (133)$$

Further, in parallel with the right Jost functions $|\varphi_j(n|z)\rangle$ satisfying to

$$M(n|z)|\varphi_j(n|z)\rangle=|\varphi_j(n+1|z)\rangle \qquad (j=1,2,3), \quad (134)$$

$$\lim_{n\to-\infty}[\zeta_j(z)]^{-n}|\varphi_j(n|z)\rangle=|v_j(z)\rangle \quad (j=1,2,3), \quad (135)$$

we will take advantage of the left Jost functions $\langle\varphi_j^+(n|z)|$ satisfying the relations

$$\langle\varphi_j^+(n+1|z)|M(n|z) = \langle\varphi_j^+(n|z)| \quad (j=1,2,3), \qquad (136)$$

$$\lim_{n\to-\infty}\langle\varphi_j^+(n|z)|[\zeta_j(z)]^n = \langle v_j^+(z)| \quad (j=1,2,3). \qquad (137)$$

The property of orthogonality

$$\langle\varphi_j^+(n|z)|\varphi_k(n|z)\rangle = \langle v_j^+(z)|v_j(z)\rangle\delta_{jk}$$

$$(j=1,2,3;\ k=1,2,3) \qquad (138)$$

will be also necessary.

The results of the previous two paragraphs enable us to prove that

$$\frac{d}{d\tau}|\varphi_j(n|z)\rangle = [B(n|z)-\eta_j(z)\cdot I]\,|\varphi_j(n|z)\rangle$$

$$(j=1,2,3). \qquad (139)$$

Indeed, differentiating (134) with respect to time τ with the subsequent substitute of $\dot{M}(n|z)$ from the zero-curvature relation (39), we conclude that

$$\frac{d}{d\tau}|\varphi_j(n|z)\rangle - B(n|z)|\varphi_j(n|z)\rangle =$$

$$=\sum_{k=1}^{3}|\varphi_k(n|z)\rangle d_{kj}(z) \quad (j=1,2,3) \qquad (140)$$

insofar as the left-hand side of (140) was revealed to satisfy the gauge transformed spectral equation (37). Then operating onto (140) with $\langle\varphi_i^+(n|z)|$ from left to right and taking the limit at $n\to-\infty$, we obtain

$$d_{ij}(z) = -\eta_i(z)\delta_{ij} \qquad (i=1,2,3;\ j=1,2,3), \qquad (141)$$

which in combination with (140) yields (139).

In terms of envelope Jost functions, we evidently have

$$\frac{d}{d\tau}|\Phi_j(n|z)\rangle = [B(n|z)-\eta_j(z)\cdot I]\,|\Phi_j(n|z)\rangle$$

$$(j=1,2,3). \qquad (142)$$

The simple manipulation with (127), (133) and (142) gives rise to

$$\lim_{n\to+\infty}\frac{d}{d\tau}\langle v_j^+(z)|\Phi_j(n|z)\rangle = 0 \qquad (j=1,2,3). \qquad (143)$$

This property (143) together with those early written down (127), (133), and (142) comprise the main tool in establishing the evolution equations for scattering data. In so doing, we have to prepare the reconstructions for $\langle v_j^+(z)|\Phi_j(n|z)\rangle$ $(j=1,2,3)$ from reconstructions (114)–(117) for $|\Xi_j(n|z)\rangle$ $(j=1,2,3)$ by means of the gauge transformation

$$|\Phi_j(n|z)\rangle = S(n)|\Xi_j(n|z)\rangle \qquad (j=1,2,3) \qquad (144)$$

and then to make two tricks assuming that $d[S(\infty)S^{-1}(-\infty)]/d\tau = 0$.

The first one consists in calculation of

$$\lim_{n\to+\infty}\ \lim_{\zeta_j(z)\to\zeta_j(z_j(s))}\left\{\Big[\zeta_j(z)-\zeta_j(z_j(s))\Big]^2\times\right.$$

$$\left.\times\frac{d}{d\tau}\langle v_j^+(z)|\Phi_j(n|z)\rangle\right\} \qquad (145)$$

$$(j=1,2,3;\quad s=1,2,3,\ldots,(\delta_{j1}+\delta_{j3})N+\delta_{j2}M),$$

where we will understand under $z_2(s)$ either $\bar{z}_2(s)$ or $\overset{+}{z}_2(s)$ depending on which of two equivalent expressions (115) or (116) for $|\Xi_2(n|z)\rangle$ is involved. As a result, we obtain

$$dz_1(s)/d\tau = 0 = dz_3(s)/d\tau \quad (s=1,2,3,\ldots,N), \quad (146)$$

$$d\,\bar{z}_2(s)/d\tau = 0 = d\,\overset{+}{z}_2(s)/d\tau \quad (s=1,2,3,\ldots,M). \ (147)$$

After the time independence of pole locations was established, the second trick consisting in the calculation of

$$\lim_{n\to+\infty}\ \lim_{\zeta_j(z)\to\zeta_j(z_j(s))}\left\{\Big[\zeta_j(z)-\zeta_j(z_j(s))\Big]\times\right.$$

$$\left.\times\frac{d}{d\tau}\langle v_j^+(z)|\Phi_j(n|z)\rangle\right\}$$

$$(j=1,2,3;\quad s=1,2,3,\ldots,(\delta_{j1}+\delta_{j3})N+\delta_{j2}M) \ (148)$$

might be done. As a result, we obtain

$$dC_{k1}(s)/d\tau = \Big[\eta_k(z_1(s))-\eta_1(z_1(s))\Big]C_{k1}(s)$$

$$(k=2,3;\ s=1,2,3,\ldots,N), \qquad (149)$$

$$d\,\bar{C}_{k2}(s)/d\tau = \Big[\eta_k(\bar{z}_2(s))-\eta_2(\bar{z}_2(s))\Big]\,\bar{C}_{k2}(s)$$

$$(k=1,3;\ s=1,2,3,\ldots,M), \qquad (150)$$

$$d\,\overset{+}{C}_{k2}(s)/d\tau = \Big[\eta_k(\overset{+}{z}_2(s))-\eta_2(\overset{+}{z}_2(s))\Big]\,\overset{+}{C}_{k2}(s)$$

$$(k = 3, 1;\ s = 1, 2, 3, \ldots, M), \tag{151}$$

$$dC_{k3}(s)/d\tau = \Big[\eta_k(z_3(s)) - \eta_3(z_3(s))\Big] C_{k3}(s)$$

$$(k = 1, 2, ;\ s = 1, 2, 3, \ldots, N). \tag{152}$$

The evolution equations for the complex conjugate part of scattering data can be obtained either by the same procedures with the replacement $z_j(s)$ into $z_j^*(s)$ or by the sheer complex conjugation of (146), (147), and (149)—(152).

10. Reconstruction of Field Amplitudes. General Scheme

Inspecting reconstructions (114)–(117) of envelope Jost functions $|\Xi_j(n|z)\rangle$ $(j = 1, 2, 3)$, we observe that they can be expanded in Laurent-type series in the regions of their regularity. In particular,

$$|\Xi_1(n|z)\rangle = |u_1(z)\rangle + \sum_{m=1}^{\infty} |U_1(n|m)\rangle \left[\zeta_1(z)\right]^{-m}$$

$$(z \in D(231) + D(321)), \tag{153}$$

$$|\Xi_2(n|z)\rangle = |u_2(z)\rangle + \sum_{m=1}^{\infty} |U_2(n|m)\rangle \left[\zeta_2(z)\right]^{-m}$$

$$(z \in D(132) + D(312)), \tag{154}$$

$$|\Xi_3(n|z)\rangle = |u_3(z)\rangle + \sum_{m=1}^{\infty} |U_3(n|m)\rangle \left[\zeta_3(z)\right]^{-m}$$

$$(z \in D(123) + D(213)). \tag{155}$$

These expansions (153)—(155) enable us to reconstruct the field amplitudes $q_-(n)$, $q_+(n)$, and $\alpha(n)$ in terms of expansion coefficients

$$\langle k|U_j(n|1)\rangle \equiv U_{kj}(n|1) \qquad (k = 1, 3;\ j = 1, 2, 3) \tag{156}$$

or, more precisely, in terms of their time derivatives.

For this purpose, we must invoke the equalities

$$|\Xi_j(n+1|z)\rangle\zeta_j(z) = L(n|z)|\Xi_j(n|z)\rangle \quad (j = 1, 2, 3) \tag{157}$$

for the back gauge transformed envelope Jost functions $\big|\Xi_j(n|z)\rangle$ on the one hand, and the nonlinear dynamical equations written in their authentic form

$$\dot{p}_{jk}(n) = X_{jk}(n+1) - X_{jk}(n) \quad (j = 1, 3;\ k = 1, 3), \tag{158}$$

on the other. Here, the notations

$$X_{11}(n) = e^{+q_-(n) - q_-(n-1)} \cos\alpha(n) \cos\alpha(n-1), \tag{159}$$

$$X_{13}(n) = e^{+q_-(n) - q_+(n-1)} \cos\alpha(n) \sin\alpha(n-1), \tag{160}$$

$$X_{31}(n) = e^{+q_+(n) - q_-(n-1)} \sin\alpha(n) \cos\alpha(n-1), \tag{161}$$

$$X_{33}(n) = e^{+q_+(n) - q_+(n-1)} \sin\alpha(n) \sin\alpha(n-1) \tag{162}$$

are implied.

Collecting the lowest terms in equalities (157) expanded in accordance with formulae (153)–(155), we might come to the interim result

$$p_{11}(n) =$$

$$= -i\left[U_{11}(n+1|1) - U_{11}(n|1)\right] e^{-q_-(-\infty)} \cos\alpha(-\infty) -$$

$$-i\left[U_{12}(n+1|1) - U_{12}(n|1)\right] e^{-q_-(-\infty)} \sin\alpha(-\infty), \tag{163}$$

$$p_{13}(n) =$$

$$= -i\left[U_{13}(n+1|1) - U_{13}(n|1)\right] e^{-q_+(-\infty)} \sin\alpha(-\infty) +$$

$$+i\left[U_{12}(n+1|1) - U_{12}(n|1)\right] e^{-q_+(-\infty)} \cos\alpha(-\infty), \tag{164}$$

$$p_{31}(n) =$$

$$= -i\left[U_{31}(n+1|1) - U_{31}(n|1)\right] e^{-q_-(-\infty)} \cos\alpha(-\infty) -$$

$$-i\left[U_{32}(n+1|1) - U_{32}(n|1)\right] e^{-q_-(-\infty)} \sin\alpha(-\infty), \tag{165}$$

$$p_{33}(n) =$$

$$= -i\left[U_{33}(n+1|1) - U_{33}(n|1)\right] e^{-q_+(-\infty)} \sin\alpha(-\infty) +$$

$$+i\left[U_{32}(n+1|1) - U_{32}(n|1)\right] e^{-q_+(-\infty)} \cos\alpha(-\infty), \tag{166}$$

which should be inserted into the dynamical equations (158). After an appropriate summation, the result for the quantities $X_{jk}(n)$ $(j = 1, 3; k = 1, 3)$ looks as follows:

$$X_{11}(n) = \cos^2\alpha(-\infty) - i\Big[\dot{U}_{11}(n|1)\cos\alpha(-\infty) +$$

$$+ \dot{U}_{12}(n|1)\sin\alpha(-\infty)\Big] e^{-q_-(-\infty)}, \tag{167}$$

$$X_{13}(n) =$$

$$= e^{+q_-(-\infty) - q_+(-\infty)} \cos\alpha(-\infty)\sin\alpha(-\infty) -$$

$$-i\Big[\dot{U}_{13}(n|1)\sin\alpha(-\infty) - \dot{U}_{12}(n|1)\cos\alpha(-\infty)\Big] \times$$

$$\times e^{-q_+(-\infty)}, \tag{168}$$

$$X_{31}(n) =$$

$$= e^{+q_+(-\infty) - q_-(-\infty)} \sin\alpha(-\infty)\cos\alpha(-\infty) -$$

$$-i\left[\dot{U}_{31}(n|1)\cos\alpha(-\infty)+\dot{U}_{32}(n|1)\sin\alpha(-\infty)\right]\times$$
$$\times e^{-q_-(-\infty)}, \qquad (169)$$

$$X_{33}(n)=\sin^2\alpha(-\infty)-i\left[\dot{U}_{33}(n|1)\sin\alpha(-\infty)-\right.$$
$$\left.-\dot{U}_{32}(n|1)\cos\alpha(-\infty)\right]e^{-q_+(-\infty)}. \qquad (170)$$

Here, we should remember that neither the authentic dynamical equations (158) nor the combinations of field variables (159)–(162) are independent. In order to overcome this obstacle when tackling the nonlinear dynamical system, we were forced to introduce the three-field parametrization (7)—(10). However, we do not know yet whether some universal parametrization for the expansion coefficients (156) or alternatively for the scattering data themselves is possible. For this reason, we restrict ourselves by the mere statement that the property

$$X_{11}(n)X_{33}(n)=X_{13}(n)X_{31}(n) \qquad (171)$$

(evident from definitions (159)–(162)) in combination with the properties

$$U_{k1}(n|m)=U_{k3}(n|m), \quad (k=1,3; \quad m=1,2,3,\ldots,\infty) \qquad (172)$$

(evident from the symmetry conditions (107), (109)) give rise to the following requirement:

$$i\dot{U}_{12}(n|1)e^{+q_+(-\infty)}\sin\alpha(-\infty)-\dot{U}_{13}(n|1)\dot{U}_{32}(n|1)=$$
$$=i\dot{U}_{32}(n|1)e^{+q_-(-\infty)}\cos\alpha(-\infty)-$$
$$-\dot{U}_{31}(n|1)\dot{U}_{12}(n|1). \qquad (173)$$

Inverting definitions (159)–(162) and supposing the quantities $X_{jk}(n)$ to be given by formulae (167)–(170) under all mentioned restrictions (172), (173), we come to the formal reconstruction of the field amplitudes

$$\exp\left[+2q_+(n)-2q_+(n-1)\right]=$$
$$=\frac{\left[X_{13}(n+1)X_{33}(n)+X_{11}(n+1)X_{13}(n)\right]\left[X_{33}(n)X_{31}(n-1)+X_{31}(n)X_{11}(n-1)\right]}{X_{13}(n+1)X_{31}(n-1)}, \qquad (174)$$

$$\exp\left[+2q_-(n)-2q_-(n-1)\right]=$$
$$=\frac{\left[X_{31}(n+1)X_{11}(n)+X_{33}(n+1)X_{31}(n)\right]\left[X_{11}(n)X_{13}(n-1)+X_{13}(n)X_{33}(n-1)\right]}{X_{31}(n+1)X_{13}(n-1)}, \qquad (175)$$

$$\cos 2\alpha(n)=\frac{X_{11}(n+1)X_{13}(n)-X_{13}(n+1)X_{33}(n)}{X_{11}(n+1)X_{13}(n)+X_{13}(n+1)X_{33}(n)}. \qquad (176)$$

11. Simplest Solution

Although the full reconstruction of envelope Jost functions could be done in principle according to the scheme described in Section 8, an extra interplay between their expansion coefficients (173) should always be taken into account or at least verified by the final results. We believe such an interplay may be linked with some intrinsic symmetries of envelope Jost functions. However, at present, any extra symmetries except for (107)–(110) have not yet been revealed, and the very question about their existence remains to be open.

Nevertheless, bearing in mind the formal criteria already adopted for the envelope Jost functions and requiring the solutions of the nonlinear model to be finite, the correct concretizations concerning the scattering data can be made sometimes.

As an example, let us analyze the simplest constructive case $N=1$, $M=0$, where the partial reconstructions (114)–(117) for the envelope Jost functions are taken to be

$$|\Xi_1(n|z)\rangle=|u_1(z)\rangle+$$
$$+|\Xi_3(n|z_1(1))\rangle C_{31}(n|1)\frac{\zeta_1(z_1(1))}{\zeta_1(z)-\zeta_1(z_1(1))}+$$
$$+|\Xi_3(n|z_1^*(1))\rangle C_{31}^*(n|1)\frac{\zeta_1(z_1^*(1))}{\zeta_1(z)-\zeta_1(z_1^*(1))}, \qquad (177)$$

$$|\Xi_2(n|z)\rangle| = u_2(z)\rangle. \quad (178)$$

The expression for $|\Xi_3(n|z)\rangle$ is dropped out since it carries the same information as (177).

The reconstruction of $|\Xi_1(n|z)\rangle$ will be complete provided the quantities $|\Xi_3(n|z_1(1))\rangle$ and $|\Xi_3(n|z_1^*(1))\rangle$ are found. The set of equations for their finding is evidently as follows:

$$\begin{aligned} &+ \ |\Xi_3(n|z_1(1))\rangle \left[1 - C_{31}(n|1)\frac{\zeta_1(z_1(1))}{\zeta_1(z_3(1)) - \zeta_1(z_1(1))}\right] - \\ &- \ |\Xi_3(n|z_1^*(1))\rangle C_{31}^*(n|1)\frac{\zeta_1(z_1^*(1))}{\zeta_1(z_3(1)) - \zeta_1(z_1^*(1))} = \\ &= \ |u_3(z_1(1))\rangle, \end{aligned} \quad (179)$$

$$\begin{aligned} &- \ |\Xi_3(n|z_1(1))\rangle C_{31}(n|1)\frac{\zeta_1(z_1(1))}{\zeta_1(z_3^*(1)) - \zeta_1(z_1(1))} + \\ &+ \ |\Xi_3(n|z_1^*(1))\rangle \left[1 - C_{31}^*(n|1)\frac{\zeta_1(z_1^*(1))}{\zeta_1(z_3^*(1)) - \zeta_1(z_1^*(1))}\right] = \\ &= \ |u_3(z_1^*(1))\rangle. \end{aligned} \quad (180)$$

To proceed further, the parametrization

$$z_1(1) \ = \ \exp[-\mu - ik], \quad (181)$$

$$z_3(1) \ = \ \exp[+\mu + ik] \quad (182)$$

by real parameters $\mu > 0$ and k seems to be reasonable. Then $C_{31}(n|1)$ can be written in the form

$$C_{31}(n|1) = \exp[+2\mu n + 2ikn]C_{31}(1) \quad (183)$$

with

$$C_{31}(1) = e^{-2\tau \mathrm{sh}(\mu+ik) - 2\mu x(0) + \ln(2\mathrm{sh}\mu) - 2i\beta(0)}, \quad (184)$$

where the parameters $x(0)$ and $\beta(0)$ are real.

In order the functions $q_-(n)$, $q_+(n)$, $\alpha(n)$ to be finite, the determinant of the linear set (179), (180) must be of constant sign. This demand is achievable under restrictions $k = \pi\nu$, $\cos 2\beta(0) < 0$, where $\nu = 1, 2$ and $\beta(0) = \pi/3$ without any loss of generality.

The rest of calculations is straightforward. The result is as follows:

$$q_-(n) \ = \ \ln\frac{\mathrm{ch}[\mu(n + 1/2 - x(\tau))]}{\mathrm{ch}[\mu(n - 1/2 - x(\tau)]} + \mu + q_-(-\infty), \quad (185)$$

$$q_+(n) \ = \ \ln\frac{\mathrm{ch}[\mu(n + 1/2 - x(\tau))]}{\mathrm{ch}[\mu(n - 1/2 - x(\tau))]} + \mu + q_+(-\infty), \quad (186)$$

$$\alpha(n) \ = \ \alpha(-\infty), \quad (187)$$

where

$$x(\tau) = \tau\frac{\mathrm{sh}\mu}{\mu}\cos(\pi\nu) + x(0) \qquad (\nu = 1, 2). \quad (188)$$

Conclusion

Summarizing, we have developed the nonlinear model describing three coupled dynamical subsystems on a regular one-dimensional lattice. Despite the highly nontrivial coupling in its kinetic and potential parts, the system as a whole admits the standard Lagrange formulation and can be readily rewritten in the Hamiltonian form as well. Moreover, it was obtained as a compatibility condition of two auxiliary linear matrix equations and hence can be integrated by the method of inverse scattering transform. This method turns out to be rather complicated even in the framework of the advanced Caudrey approach both due to the third order of the respective spectral problem and because of the hidden symmetries of Jost functions yet to be revealed explicitly. We have tried to give as broad information about the model as now possible in order to make it more reliable for the further investigation. It is interesting to note that even the simple result on a one-soliton solution has enabled us to test the whole inverse scattering machinery and to conclude that it should be built up on back gauge transformed envelope Jost functions $|\Xi_j(n|z)\rangle$ rather than on the originally introduced ones $|\Phi_j(n|z)\rangle$.

The work has been supported in parts by the Science and Technology Center in Ukraine, Project No 1747; and by the National Academy of Sciences of Ukraine, Grant 0102U002332.

1. *Caudrey P. J.*// Phys. Lett A. – 1980. – **79**. – P. 264.
2. *Caudrey P.J.*// Physica D. – 1982. – **6**. – P. 51.
3. *Caudrey P.J.*// Proc. Roy. Irish Acad. Sect. A. – 1983. – **83**. – P. 23.
4. *Caudrey P.J.*// Wave Phenomena: Modern Theory and Applications/ Ed. by C. Rogers and T.B. Moodie. – Amsterdam: Elsevier, 1984. – P. 221.
5. *Yan Y. and Caudrey P.J.*//Inverse Problems. – 1990. – **6**. – P. 153.
6. *Kaup D.J.*// Stud. Appl. Math. – 1980. – **62**. – P. 189.
7. *Vakhnenko V.O. and Parkes E.J.*// Dopovidi NAN Ukrainy. – 2001. – N7. – P. 81.
8. *Vakhnenko V.O. and Parkes E.J.*// Chaos, Solitons and Fractals. – 2002. – **13**. – P. 1819.
9. *Ablowitz M.J. and Ladik J.F.*// J. Math. Phys. – 1976. – **17**. – P. 1011.
10. *Vakhnenko O.O.*//J. Phys. A: Math. Gen. – 1999. – **32**. – P. 5735.
11. *Vakhnenko O.O.*// J. Math. Phys. – 2002. – **43**. – P. 2587.

12. *Takhtajan L.A. and Faddeev L.D.*// Hamil'tonov Podkhod v Teorii Solitonov. – Moscow: Nauka, 1986.
13. *Lovitt W.V.*// Linear Integral Equations. – New York: McGraw-Hill, 1924.
14. *Smyrnov V.I.*// Kurs Vysshey Matematiky, vol. 4, part I. – Moscow: Nauka, 1974.

ДИСКРЕТНА НЕЛІНІЙНА МОДЕЛЬ ТРЬОХ ЗВ'ЯЗАНИХ ДИНАМІЧНИХ ПОЛІВ, ІНТЕГРОВНА МЕТОДОМ КОДРІ

О.О. Вахненко

Р е з ю м е

Запропоновано нелінійну модель трьох зв'язаних динамічних полів на нескінченному регулярному дискретному ланцюжку. Система допускає представлення нульової кривизни, що є основою для її інтегрування методом оберненої задачі розсіювання. Допоміжна спектральна задача, асоційована з нелінійною системою, є задачею третього порядку, що приводить до досить складного поділу площини комплексного спектрального параметра на області регулярності функцій Йоста. Внаслідок цього і пряма, і обернена задачі розсіювання стають суттєво нетривіальними і тому розглядаються в рамках підходу, розробленого Кодрі. Загальну схему Кодрі адаптовано до потреб інтегрування запропонованої моделі та знайдено її найпростіший розв' язок.

ДИСКРЕТНАЯ НЕЛИНЕЙНАЯ МОДЕЛЬ ТРЕХ СВЯЗАННЫХ ДИНАМИЧЕСКИХ ПОЛЕЙ, ИНТЕГРИРУЕМАЯ МЕТОДОМ КОДРИ

А.А. Вахненко

Р е з ю м е

Предложена нелинейная модель трех связанных динамических полей на бесконечной регулярной дискретной цепочке. Система допускает представление нулевой кривизны, являющееся основанием для ее интегрирования методом обратной задачи рассеяния. Вспомогательная спектральная задача, ассоциированная с нелинейной системой, является задачей третьего порядка, что ведет к достаточно сложному делению плоскости комплексного спектрального параметра на области регулярности функций Йоста. Вследствие этого и прямая, и обратная задачи рассеяния оказываются существенно нетривиальными и поэтому рассматриваются в рамках подхода, разработанного Кодри. Проведена адаптация общей схемы Кодри для интегрирования предложенной модели и найдено ее простейшее решение.