SPATIAL SOLITONS IN ANOMALOUS DISPERSIVE MEDIA WITH NONLOCAL NONLINEARITY

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Two-dimensional localized envelope soliton-like structures are investigated in anomalous dispersive media in the framework of the nonlinear Schrödinger equation including cubic and nonlocal nonlinearities. It is shown that a crucial role in the formation of stable localized structures is played by the fourth order dispersive effect.

Introduction

Most studies of solitons in dispersive nonlinear media are devoted to one-dimensional ($1d$) systems. Solitons are rather suitable objects for energy and information transport in optical [1, 2] and biological systems [3, 4]. At the same time, different two- ($2d$) or three-dimensional ($3d$) structures appears under natural conditions. For example, stable $2d$ nonlinear structures are typically observed when an electromagnetic wave propagates through some nonlinear medium (solid, liquid, gas, or plasma). It happens if wave diffraction (or dispersion) is balanced by nonlinear self-focusing leading to the formation of stationary nonlinear waveguides [5]. In the framework of the Nonlinear Schrödinger Equation (NSE), this balance is unstable and a $2d$ soliton is dispersed or collapsed. To eliminate the possibility of wave collapse, it is sufficient to take into account the saturation of nonlinearity when a basic dimensionless NSE equation takes the form

$$\mathrm{i}\frac{\partial\Psi}{\partial t} + D\Delta_\perp\Psi + \widehat{A}\left(|\Psi|^2\right)\Psi = 0, \qquad (1)$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and, in the case of not very high wave intensity $|\Psi|^2$,

$$\widehat{A}\left(|\Psi|^2\right) = B|\Psi|^2 + K|\Psi|^4 . \qquad (2)$$

Equation (1) describes the self-focusing and formation of stable waveguides if $BK < 0$ and $DB > 0$. This stabilizing effect is often explored in nonlinear optics [5]. One needs to generalize Eq. (1) including the fourth order dispersion effects, due to rather complex polarization properties of electromagnetic waves (for example, whistler waves) in magnetized plasmas [6]:

$$\mathrm{i}\frac{\partial\Psi}{\partial t} + D\Delta_\perp\Psi + P\Delta_\perp^2\Psi + \widehat{A}\left(|\Psi|^2\right)\Psi = 0. \qquad (3)$$

Stable $2d$ localized structures, described by Eq. (3) with a nonlinear term of the form (2), have been analyzed in [6] for two different cases, corresponding to the normal and anomalous dispersion regimes of stationary whistler waveguides ($DB > 0$ and $DB < 0$). These structures have been observed in laboratory experiments [7]. Two-dimensional localized structures have also been observed for intensive Langmuir waves in quiet plasma [8] and for Upper Hybrid (UH) waves in beam-plasma experiments in the anomalous dispersion region ($\omega < 2\Omega_e$) of UH waves [9, 10]. Existence of the stable $2d$ structures of Langmuir waves may be explained taking into account a saturable [11] or nonlocal [12] nonlinearity of the form

$$\widehat{A}\left(|\Psi|^2\right) = (B + C\Delta_\perp)|\Psi|^2 \qquad (4)$$

with $BC < 0$.

Importance of nonlocal interaction effects to describe short-scaled (compared to the skin electron depth c/ω_{pe}) UH structures (observed experimentally near the UH plasma resonance [13]) was first pointed out in [14]. However, in the normal dispersion region, Eq. (1), with (4), $BC < 0$, $BD > 0$ has soliton solution which collapses even in $1d$ case. With regard for the fourth order dispersive term ($\sim k_\perp^4 v_{Te}^4$) in UH wave dispersion:

$$\omega^2 = \omega_{\mathrm{UH}}^2 + \frac{3\omega_{pe}^2 k_\perp^2 v_{te}^2}{\omega^2 - 4\Omega_e^2} + \frac{15\omega_{pe}^2 k_\perp^4 v_{te}^4}{(\omega^2 - 4\Omega_e^2)(\omega^2 - 9\Omega_e^2)}, \qquad (5)$$

where $v_{te}^2 = T_e/m_e$, $k^2\rho_e^2 \ll 1$, $\omega_{\mathrm{UH}}^2 = \omega_{pe}^2 + \Omega_e^2$, $\rho_e = v_{te}/\Omega_e$, one can explain [15, 16] the experimental observation of UH solitons [13] in the region of normal dispersion ($2\Omega_e < \omega < 4\Omega_e$).

Nonlocal interaction plays an essential role also in the theory of matter waves or Bose-Einstein condensate [17, 18]. In particular, it was shown that a coherent state of atoms (localized both in real and momentum $2d$ spaces

[18]) can be described by Eq. (1) in the case of attractive interaction between atoms of the condensate with $BD > 0$ and with a rather general nonlocal nonlinearity of the integral form:

$$\widehat{A}\left(|\Psi|^2\right) = \int R(\vec{r}-\vec{\xi}) \left|\Psi(\vec{\xi})\right|^2 d\vec{\xi}. \qquad (6)$$

In isotropic and weakly nonlocal limits, expression (6) is reduced to (4) with $C = \frac{1}{2}\int \xi^2 R(\vec{\xi}) d\vec{\xi}$.

Other important application of GNSE (1) with $\widehat{A}\left(|\Psi|^2\right)$ of the forms (2) and (4) is a description of localized (self-trapped) quasi-particles in $2d$ discrete lattices due to exciton-phonon or electron-phonon interactions [2, 3, 4]. Numerical simulations predict the existence of stable moving $2d$ coherent structures (a quasi-particle wave function accompanied by lattice deformation) [19, 20]. Their stability against collapse (or pinning) has been explained using the continuum approach accounting for a saturable nonlinearity [19] or nonlocality ($BC > 0$) [20]. The case $BD > 0$, when $2d$ NSE requires additional terms to prevent the collapse of localized structures, has been investigated in [12, 15, 16, 18—20].

Localized structures in the case $BD < 0$ were not expected at all. As known, the coefficient D is proportional to the group velocity dispersion $D \sim \frac{\partial^2\omega}{\partial k_\perp^2}$, and the coefficient $B \sim \frac{\partial\delta\omega}{\partial|\Psi|^2}$, i.e., is determined by the dependence of the nonlinear frequency shift (or the refractive index for nonlinear optical media) on intensity. It is well known that, in accordance with the Lighthill criterion there is no modulational instability, at $BD < 0$ for ordinary NSE, and therefore solitons are not able to appear. However, as was mentioned above, $2d$ localized UH structures have been observed in some beam-plasma experiments in the region of anomalous dispersion with $BD < 0$. In such a case, stable whistler waveguides have also been observed. We show below that higher order linear dispersion effects [such that $DP > 0$ in Eq. (3)] and nonlinear dispersion effects with $\widehat{A}\left(|\Psi|^2\right)$ given by (4) with $BC > 0$ may lead to a formation of stable solitons. As was shown in [21], this is also true for $1d$ solitons. In the case where $\widehat{A}\left(|\Psi|^2\right)$ is given by (2), soliton solutions have been examined for $1d$ space in [22] and for $2d$ space in [6].

1. Basic Equations

We study the possibility for formation of stable solitons in the framework of the Generalized Nonlinear Schrödinger Equation (GNSE) including second-order and fourth-order dispersion effects and cubic and nonlocal nonlinearityies:

$$\mathrm{i}\frac{\partial\Psi}{\partial t} + D\Delta_\perp\Psi + P\Delta_\perp^2\Psi + B|\Psi|^2\Psi + C\Psi\Delta_\perp|\Psi|^2 = 0. \qquad (7)$$

In this paper, we concentrate on the so-called anomalous dispersive regime $\partial^2\omega/\partial k^2 < 0$ which corresponds to $DB < 0$, the signs of other coefficients being chosen in such a way:

$$PC < 0, PB < 0. \qquad (8)$$

In particular, such signs correspond to UH waves interacting with magnetosonic waves in the adiabatic approximation [14] for $\omega_{pe}^2 < 3\Omega_e^2$: $D < 0$ and $P < 0$, $B > 0, C > 0$. Any localized wavepacket envelope described by GNSE (7) conserves the following integrals of motion:

(i) number of quanta ("energy" or "beam power"):

$$N = \int |\Psi|^2 d^2\mathbf{r}, \qquad (9)$$

(ii) x and y components of momentum:

$$\vec{I} = -\frac{\mathrm{i}}{2}\int \left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) d^2\mathbf{r}, \qquad (10)$$

(iii) z-component of angular momentum:

$$\vec{M} = -\frac{\mathrm{i}}{2}\int \left(\Psi^*\left[\mathbf{r}\times\nabla\Psi\right] - \Psi\left[\mathbf{r}\times\nabla\Psi^*\right]\right) d^2\mathbf{r}, \qquad (11)$$

(iv) Hamiltonian:

$$H = D\int |\nabla\Psi|^2 d^2\mathbf{r} - P\int |\Delta_\perp\Psi|^2 d^2\mathbf{r} -$$
$$-\frac{1}{2}B\int |\Psi|^4 d^2\mathbf{r} + \frac{1}{2}C\int (\nabla|\Psi|^2)^2 d^2\mathbf{r} \equiv$$
$$\equiv DI_D - PI_P - \frac{1}{2}BI_B + \frac{1}{2}CI_C. \qquad (12)$$

Some insight to properties of the wave packet dynamics and the possibility of soliton solutions of Eq. (7) can be obtained using the virial integral identity

$$\frac{N}{8}\frac{d^2 r_{\mathrm{eff}}^2}{dt^2} = \int \Big\{ D^2|\nabla\Psi|^2 - 4DP|\Delta_\perp\Psi|^2 +$$
$$+4P^2|\nabla\Delta_\perp\Psi|^2 - \frac{1}{2}DB|\Psi|^4 - PB\left(|\nabla\Psi|^2 + 2\left|\frac{\partial\Psi}{\partial r}\right|^2\right)\times$$
$$\times r\frac{\partial|\Psi|^2}{\partial r} + CD\left(\nabla|\Psi|^2\right)^2 - PC\Delta_\perp|\Psi|^2\left[\Delta_\perp|\Psi|^2 -\right.$$

$$-6\left|\frac{\partial\Psi}{\partial r}\right|^2 - -3r\frac{\partial}{\partial r}\left|\frac{\partial\Psi}{\partial r}\right|^2 - \frac{3}{r}\frac{\partial}{\partial r}\left|\frac{\partial\Psi}{\partial\varphi}\right|^2 -$$

$$-\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial\Psi}{\partial r}\frac{\partial\Psi^*}{\partial\varphi}+c.c\right)\Bigg]\Bigg\}\,d^2\mathbf{r}, \qquad (13)$$

where the effective wave packet width is defined as follows:

$$r_{\text{eff}}^2 = \frac{1}{N}\int r^2\,|\Psi|^2\,d^2\mathbf{r}, \qquad (14)$$

If $P = 0$, this relation is reduced to [12]

$$\frac{N}{8}\frac{d^2 r_{\text{eff}}^2}{dt^2} = DH + \frac{CD}{2}\int\left(\nabla|\Psi|^2\right)^2 d^2\vec{r}. \qquad (15)$$

For the signs of coefficients (8), $CD < 0$ and so that any wave packet should collapse in the case $H < 0$. As shown in [12], a stable solution in the case $P = 0$ may exist only if $CD > 0$. Thus, for the problem under consideration with focusing nonlocal nonlinearity ($CD < 0$), it is crucial to take into account the fourth order dispersion effect [the term proportional to P in (3)] for the description of possible soliton solutions.

It is important to note that Hamiltonian (12) in the anomalous dispersive regime (8) is bounded from below at a fixed number of quanta even if $C = 0$, but $P \neq 0$. Really, taking into account the Hölder inequality

$$I_D \le (N I_P)^{1/2} \qquad (16)$$

and the well-known [23] integral inequality

$$I_B \le 2 N I_D, \qquad (17)$$

one obtains the following estimate for the Hamiltonian (remember that $CD \le 0$):

$$H \ge -|D|I_D + |P|I_P - \frac{1}{2}|B|I_B \ge$$

$$\ge -\left(|D| + |B|N\right)N^{1/2}I_P^{1/2} + |P|I_P. \qquad (18)$$

Thus, the following inequality can be found straightforwardly from (18):

$$H \ge -\frac{N}{4|P|}\left(|D| + |B|N\right)^2. \qquad (19)$$

As we will see in the next section, the Hamiltonian is negative for any localized steady-state solution of GNSE (21). Therefore, in accordance with the Lyapunov's theorem, there exists a stationary localized solution which corresponds to the Hamiltonian's global extremum.

2. Soliton Features in Anomalous Dispersive Regime

Let us consider localized stationary solutions of GNSE (7) of the form:

$$\Psi(\mathbf{r},t) = \widehat{\Psi}(\mathbf{r})e^{\mathrm{i}\Lambda t}, \qquad (20)$$

where Λ is a nonlinear frequency shift. Substituting (20) in GNSE (7), we obtain the partial differential equation for the function $\widehat{\Psi}(\mathbf{r})$ in $2d$ space:

$$-\Lambda\widehat{\Psi} + (D + P\Delta_\perp)\,\Delta_\perp\widehat{\Psi} +$$

$$+\widehat{\Psi}\left(B + C\widehat{\Psi}\Delta_\perp\right)|\widehat{\Psi}|^2 = 0. \qquad (21)$$

Multiplying Eq. (21) by $\widehat{\Psi}^*$ and integrating over space coordinates, the following relation for steady-state solutions may be obtained:

$$\Lambda N = P I_P - D I_D + B I_B - C I_C. \qquad (22)$$

The other integral relation is found multiplying Eq. (21) by $r^2\partial\widehat{\Psi}^*/\partial r$, integrating, and adding the complex conjugate:

$$\Lambda N = -P I_P + B I_B/2. \qquad (23)$$

Excluding Λ from (22), (23), one can write Hamiltonian (12) valid for stationary solutions as

$$H = -|P|I_P - |C|I_C/2. \qquad (24)$$

Further, we are looking for background radially-symmetric soliton-like solutions of Eq. (7) with zero momentum (10) and angular momentum (11) of the following form:

$$\Psi(\mathbf{r},t) = \psi(r)e^{\mathrm{i}\Lambda t}. \qquad (25)$$

Here, we are interested mostly in ordinary solitons (the phase of the wave function (25) does not depend on r). Therefore, the radial function $\psi(r)$ may be assumed to be real. Introducing the rescaling transformation: $r = \rho\sqrt{\frac{P}{D}}$, $\Lambda = \lambda\frac{D^2}{|P|}$, $\psi(r) = U(\rho)\sqrt{\frac{D^2}{B|P|}}$, one obtains the ordinary differential equation for a radial soliton profile:

$$-\lambda U - \Delta_\rho U - \Delta_\rho^2 U + U^3 + \sigma U\Delta_\rho U^2 = 0, \qquad (26)$$

where $\Delta_\rho = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}$, $\sigma = \frac{C/P}{B/D}$. Localized soliton solutions satisfy the boundary conditions

$$\lim_{\rho\to\infty} U(\rho) = 0,\ \lim_{\rho\to\infty}\left[\Delta_\rho U(\rho)\right] = 0, \qquad (27)$$

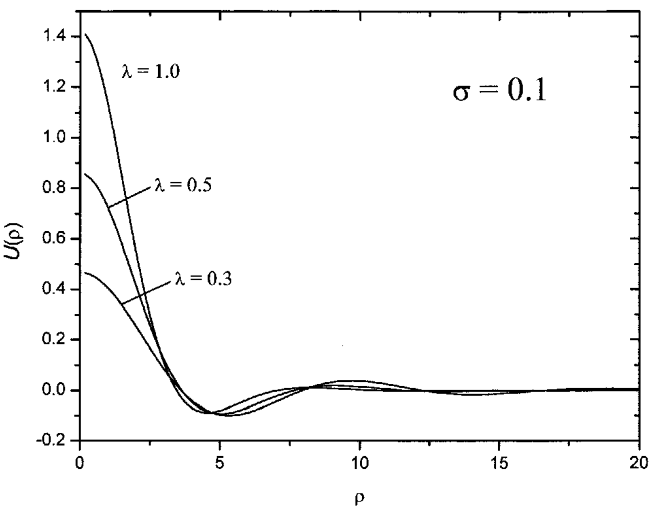

Fig. 1. Stationary numerical solutions of GNSE with different nonlinear frequency shifts in the anomalous dispersive regime ($\sigma = 0.1$)

$$\left.\frac{dU(\rho)}{d\rho}\right|_{\rho=0} = 0,\ \left.\frac{d}{d\rho}\Delta_\rho U(\rho)\right|_{\rho=0} = 0. \tag{28}$$

We have solved the boundary-value problem (26)-(28) numerically by the relaxation-like method in a spectral Hankel space with additional stabilizing factor (see [6] for details). Fig. 1 shows the typical radial profiles of a two-parameter (with parameters λ and σ) soliton family.

Some general soliton properties and conditions for stable soliton formation may be investigated analytically. First of all, we note that the nonlinear frequency shift λ is positive, as follows from Eq. (23). Moreover, λ for a localized solution exceeds $1/4$. It follows from the asymptotic behaviour of the soliton profile $U(\rho)$ at large ρ. The linearized Eq. (26) has solutions with asymptotes (at $r \to \infty$) of the form $h_i r^{-1/2}\exp(\mathrm{i}k_i r)$, where the wave numbers k_i obey the dispersion equation

$$-\Lambda + k_i^2|D| - k_i^4|P| = 0, \tag{29}$$

so that

$$k_i^2 = \frac{|D|}{2|P|}\left\{1 \pm \sqrt{1 - 4\Lambda|P|/D^2}\right\}, \tag{30}$$

Localized structures correspond to solutions (30) with $\mathrm{Im}k_i < 0$. Hence, one obtains the restriction $\Lambda > D^2/(4|P|)$ (or $\lambda > 1/4$ after a rescaling transformation). It is interesting to note that $\mathrm{Re}k_i \neq 0$, therefore a soliton profile always has oscillating tail (see Fig. 1). These tails are especially noticeable if $\lambda \approx 0.25$, when the decay rate ($\mathrm{Im}k_i$) tends to zero, but the frequency of oscillations remains finite: $\mathrm{Re}k_i \geq \sqrt{|D|/(2|P|)}$.

It is clear from Eq. (29) that, in the limiting case of the absence of the fourth-order dispersion effect ($P = 0$), there is no stationary localized solutions of GNSE in the anomalous dispersive regime (8) at all. Actually, Eq. (29) yields $k_i^2 = \Lambda/|D| > 0$ so that the decay rate is equal to zero ($\mathrm{Im}k_i$=0) if $P = 0$.

As well known, the stationary Eq. (21) can be obtained from the constrained variational problem for the Hamiltonian: $\delta(H + \Lambda N) = 0$, provided that N =const. So that the steady-state solution is a stationary point of the Hamiltonian at a fixed number of quanta.

In the framework of variational analysis, one approximates the exact solution by some ansatz which should takes into account the main soliton features. Obviously, the accuracy of variational analysis crucially depends on the choice of a trial function. First of all, one should choose a function localized at $r \to \infty$. Therefore, one parameter determines the decay rate and so characterizes the soliton width. The other important parameter accounts for a possible variation of the soliton phase. In the general case, the phase may be a function of r. Also, as we will see, it is necessary to take into account the oscillating nature of a soliton in media with fourth-order dispersion in the anomalous dispersive regime, especially for $\lambda \approx 1/4$. Due to these arguments, we have used the following trial function:

$$\psi_0(r) = hJ_0(\gamma r)\, e^{-\frac{1}{2}\mu_1^2 r^2(1+\mathrm{i}\mu_2)}, \tag{31}$$

where $J_0(x)$ is the Bessel function of zero order. The first variational parameter μ_1 characterizes the localization scale, the second parameter μ_2 is a phase curvature parameter (or a chirp), and the third one, $\mu_3 = \gamma/\mu_1$, characterizes the number of soliton oscillations on its characteristic width (μ^{-1}).

Therefore, the solution m_i, $i = 1, 2, 3$, of the set of equations

$$\frac{\partial H}{\partial \mu_i} = 0 \tag{32}$$

determines the soliton parameters corresponding to a stationary point of Hamiltonian (12).

For the soliton stability, one should demand that a stationary solution $\psi_0(r; m_1, m_2, m_3)$ realize Hamiltonian's extremum (maximum or minimum). In this case, a deviation from the point (m_1, m_2, m_3) in the space of parameters (μ_1, μ_2, μ_3) would lead to a change of the Hamiltonian, but such a change is forbidden by the conservation law (12). Hence, the soliton stability criterion in the framework of variational analysis coincides with the condition that the second-order differential of the Hamiltonian,

$$d^2H = \sum_{i,j=1}^{3} \frac{\partial^2 H}{\partial \mu_i \partial \mu_j}\bigg|_{\vec{\mu}=\vec{m}} d\mu_i d\mu_j, \quad (33)$$

is a quadratic form of fixed sign at the stationary point $\mu_i = m_i$.

Generally speaking, set (32) has two different types of solutions. The first of them with $m_2 = 0$ corresponds to ordinary solitons. This solution has a phase which does not depend on r. The second one with $m_2 \neq 0$ corresponds to a soliton with curved front (or a chirped soliton). Chirped solitons become stable only if $\sigma > \sigma_{\rm cr} \approx 1.45$. Thorough numerical and analytic study of solitons with curved front is the topic of our future investigation. In the following, we restrict our investigation by ordinary solitons.

Fig. 3 shows the number of quanta vs nonlinear frequency shift for stable ordinary solitons. It is seen that results of variational analysis are in a very good agreement with numerical modeling. The soliton width $r_{\rm eff}$ is a decreasing function of N. Note that, because of a nonlocal nonlinearity, it saturates at a finite value r_∞. The limiting value r_∞ increases with parameter σ. Note also that $r_{\rm eff}$ tends to infinity at a finite value of the number of quanta N_0. This threshold value N_0 can be found in the limit $m_3 \to \infty$. Really, as seen from Fig. 3, the parameter m_3 tends to infinity when $N \to N_0$ (see also [6]). Similar soliton behaviour has been found in [6], where the threshold value has been estimated to be $N_0 \approx 2.2$. As should be, this value does not depend on the type of a small-scale supplementary nonlinearity, because the threshold corresponds to the limit of very broad soliton solutions.

Note, that the trial function (31) does not depend on azimuthal angle φ. Therefore, our variational analysis predicts the existence of localized soliton-like structures which are stable with respect to a radially symmetric perturbation. The general stability analysis must be based on the direct simulation of the $2d$ nonstationary GNSE with perturbed steady-state solution as the initial condition.

We have solved GNSE (7) by means of the Split-Step Fourier Transform method. Let us consider the numerical scheme used for this purpose in more details. The main idea (see, e.g., [1, 24]) of the composition (or split-step) method is to split the composite nonlinear-dispersive operator into two pieces and to integrate two obtained equations separately. First of them has the exact solution, and the second can be solved much more easily than the original equation. The final result of the wave packet envelope evolution on one time step is given by a definite combination of these partial solutions.

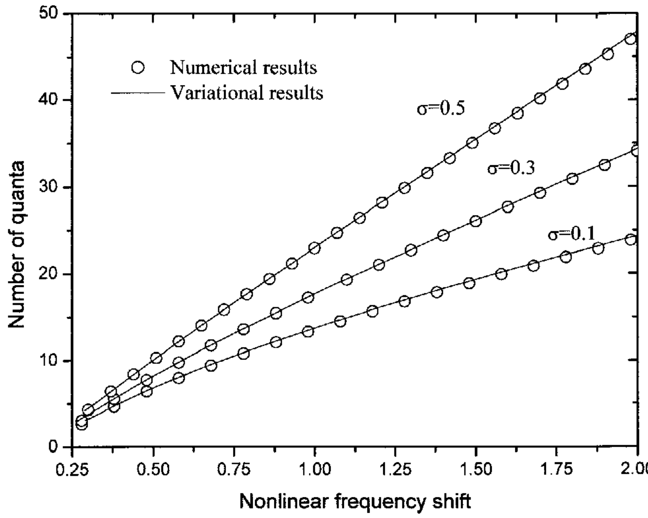

Fig. 2. Number of quanta vs nonlinear frequency shift for ordinary solitons in the anomalous dispersive regime. Solid curves for variational results, circles for numerical results

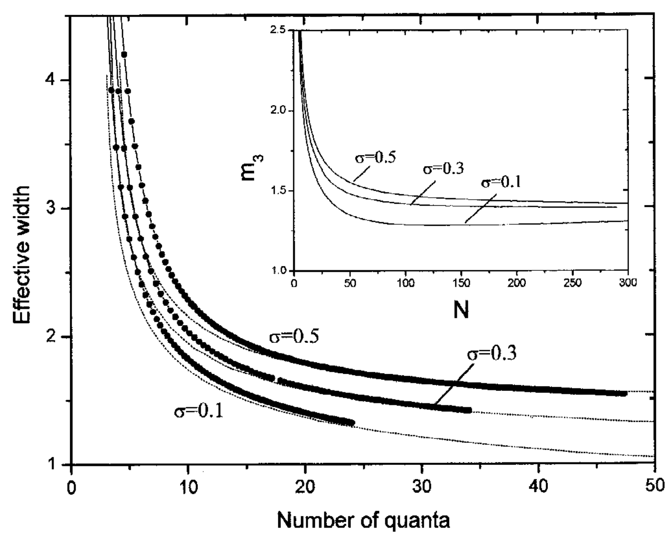

Fig. 3. Effective soliton width vs number of quanta for different values of the parameter σ. Dotted curves for variational results, curves with points for numerical results. The inserted figure: variational parameter which characterizes the number of oscillations on the soliton width vs number of quanta

Eq. (7) can be rewritten as follows:

$$\frac{\partial \Psi}{\partial t} = (\widehat{L} + \widehat{A})\Psi, \quad (34)$$

where the linear operator $\widehat{L} = \mathrm{i}(D + P\Delta_\perp)\Delta_\perp$ and the nonlinear operator $\widehat{A} = \mathrm{i}(B + C\Delta_\perp)|\Psi|^2$. The approximate solution, found by the split-step method accurately to the second order in the time step Δt, may be written formally as

$$\Psi(t + \Delta t) = \exp(\Delta t\widehat{L})\exp(\Delta t\widehat{A})\Psi(t). \tag{35}$$

Expression (35) means that the evolution of a wave packet from the moment t to $t + \Delta t$ is splitted into two steps.

It is only the nonlinearity that acts in the first step ($\widehat{L} = 0$). It is easy to show that the equation $\partial_t\Psi_A = \widehat{A}\Psi_A$ has exact solution of the form $\Psi_A(t + \Delta t) = \exp(\Delta t\widehat{A})\Psi_A(t)$.

In the second step, the dispersion acts alone ($\widehat{A} = 0$). We have integrated the corresponding linear evolution equation by means of the implicit Crank-Nicholson's scheme. It is very convenient to solve the equation with a linear operator in the Fourier spectral space:

$$\exp(\Delta t\widehat{L})\Psi(t) = \left\{F^{-1}\exp[\Delta t\widehat{L}(-\mathrm{i}\vec{k})]F\right\}\Psi(t), \tag{36}$$

where F denotes the $2d$ Fourier transform operator, which is carried out by the algorithm of Fast Fourier transformation. It makes the numerical evaluation of Eq. (36) very effective.

We have used the so-called symmetrized Split-Step Fourier method, when the temporal evolution from the moment t to $t + \Delta t$ is evaluated as follows:

$$\Psi(t + \Delta t) = \exp\left[\frac{\Delta t}{2}\widehat{L}\right]\exp\left[\Delta t\widehat{A}\right]\exp\left[\frac{\Delta t}{2}\widehat{L}\right]\Psi(t), \tag{37}$$

where $\Delta t\widehat{A}$ is taken at the midplane $t + \Delta t/2$. Scheme (37) gives the estimate of $\Psi(t + \Delta t)$ which is valid up to order Δt^3. Even higher accuracy of the split-step procedure may be obtained using higher-order splitting methods [24].

The results of the extensive series of numerical simulations may be summarized as follows. We have found that the evolution of perturbed solitons is quasiperiodic. The effective width and amplitude oscillate and remain finite, so that a soliton neither collapses, nor spreads out. Solitons are stable even with respect to rather large asymmetric perturbations.

3. Discussion

We have demonstrated that, in nonlinear media with the fourth-order dispersive effect, there exist stable bright envelope solitons. In the presence of an additional nonlocal nonlinearity, the soliton width is bounded from below: it saturates even for high-power wave beams. We have examined soliton features analytically by means of variational analysis which takes into account a possible variation of the soliton phase and the oscillatory nature of a soliton radial profile in the anomalous dispersive regime. We have compared the results obtained analytically with the results of numerical modeling. The soliton stability has been investigated analytically and was verified by direct numerical simulations.

Our model is relevant first of all for the theoretical explanation of the existence of stable localized UH structures in magnetized plasmas (see Introduction). However, the obtained results can be applied to the investigation of solitons in many other physical systems, where the fourth order dispersion effects and nonlocal nonlinearity are of great importance. In particular, the basic equation similar to Eq. (7) has been obtained in [25] in continuum limit of discrete NSE with competing short-range and long-range interaction (see Eq. (11) of [25]). In the so-called supercritical limit, this equation can be reduced to an equation similar to Eq. (7) with $DB < 0$, $DP > 0$, $DC < 0$ (see Eq. (25) of [25]). In [25], it was shown numerically that, in the supercritical case, there exists a stable soliton of staggered or oscillating form which is in qualitative agreement with our analysis in the previous section.

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ПРОСТОРОВІ СОЛІТОНИ В АНОМАЛЬНОМУ ДИСПЕРСІЙНОМУ СЕРЕДОВИЩІ З НЕЛОКАЛЬНОЮ НЕЛІНІЙНІСТЮ

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Р е з ю м е

Досліджено двовимірні солітони огинаючої в аномальному дисперсійному середовищі на основі нелінійного равняння Шрьодінгера із врахуванням нелокальної і кубічної нелінійностей. Показано, що визначальну роль у формуванні стійких локалізованих структур відіграють дисперсійні ефекти четвертого порядку.

ПРОСТРАНСТВЕННЫЕ СОЛИТОНЫ В АНОМАЛЬНОЙ ДИСПЕРСИОННОЙ СРЕДЕ С НЕЛОКАЛЬНОЙ НЕЛИНЕЙНОСТЬЮ

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Исследованы двумерные солитоны огибающей в аномальной диспергирующей среде на основе нелинейного уравнения Шрёдингера с учетом нелокальной и кубической нелинейностей. Показано, что определяющую роль в формировании устойчивых локализованных структур играют дисперсионные эффекты четвертого порядка.