ON IRREDUCIBLE PARTIALS OF THE RICCI TENSOR TRACELESS PART IN A FINITE SPACE-TIME REGION IN GENERAL RELATIVITY [1]

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The Riemann tensor irreducible part $E_{iklm} = \frac{1}{2}(g_{il}S_{km} + g_{km}S_{il} - g_{im}S_{kl} - g_{kl}S_{im})$ constructed from the metric tensor g_{ik} and traceless part of the Ricci tensor $S_{ik} = R_{ik} - \frac{1}{4}g_{ik}R$ is expanded into bilinear combinations of bivectorial fields being eigenfunctions of E. Field equations for the bivectors induced by Bianchi identities are studied, and it is shown that, in general case, it will be the 3-parametric local symmetry group of a Yang—Mills field.

Introduction

It is well known that the Einstein equations in General Relativity join together pure geometric quantities on the left side with physical quantities (energy-momentum tensor of matter) on the right one.

But this fact means that geometry puts very rigid restrictions on the energy-momentum tensor and therefore on configurations of all physical fields. Any permitted mode of a physical field has the correspondent eigenmode of gravitational field, otherwise this mode should be prohibited.

We may study geometry types using curvature classifications. There are two types of curvature classifications: the classification of the Ricci tensor by J. Plebansky [2] and the Petrov classification of the Weyl tensor [3]. Both based on studying the eigenvectors of some tensors at a given point of space-time. But the eigenvectors of the Ricci tensor have no immediate physical sense, and Weyl tensor types say a little about sources of gravitational field because it is not affected the Einstein equations.

On the other hand, the Rainich—Misner—Wheeler unified theory of electromagnetic field [1, 4] is not a classification at all. However, it allow one to represent the curvature of a very restricted class of space-times as a construction of field quantities in a finite region of space-time. Meaning of Rainich conditions is discussed in the second section.

The next section is devoted to eigenbivectors of the irreducible part E_{iklm} of the Riemann tensor and its differential properties. Such an approach allows us to generalize the already unified theory for a sourceless SU(2) Yang—Mills field in the fourth section.

In the last section, the general case of gravitational field sources is discussed. It is shown that it should be the 3-parametric local symmetry group (maybe noncompact or degenerated) of a Yang—Mills field with or without sources.

There are five appendices: on bivectors, on curvature properties, on electromagnetic energy-momentum tensor structure, on existence of conformal transformation provided vanishing the scalar curvature, and details of awkward calculations.

In all tensor expressions, latin indices run over (0,1,2,3), greek indices — (1,2,3). Semicolon means covariant derivation.

1. Rainich Conditions

If curvature satisfies the conditions

$$R^i_m R^m_k = \frac{1}{4}\delta^i_k R_{mn}R^{mn}, \tag{1.1}$$

$$R = 0, \tag{1.2}$$

known as Rainich conditions, then it is possible to express irreducible part of the Riemann tensor E_{iklm} defined by Eq. (B.3) (see Appendix B) in the following form:

$$E_{iklm} = \frac{1}{2}(f_{ik}f_{lm} + \tilde{f}_{ik}\tilde{f}_{lm}), \tag{1.3}$$

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where f_{ik} is a bivector and $\tilde{f}_{ik}$ is its dual (see Appendix A) which satisfy the sourceless Maxwell equations $f^{ik}_{;k} = 0, \tilde{f}^{ik}_{;k} = 0$.

Contraction of (1.3) gives

$$S_{ik} = \frac{1}{2}(f_{in}f^n_k + \tilde{f}_{in}\tilde{f}^n_k) \tag{1.4}$$

which is identical with the Einstein equation. Really counting (1.2), there is the Einstein tensor on the left side and the energy-momentum tensor of an electromagnetic field on the right. So we have the self-consistent system of electromagnetic and gravitational fields.

It is easy to show (see Appendix C) that the Rainich conditions (1.1) and conditions for the rank of a matrix $\mathfrak{S}$ to be equal to 1 are the same.

In next section, the general case of a matrix $\mathfrak{S}$ will be studied.

2. Eigenbivectors of E_{iklm}

Matrices A and S from (B.1) are constructed from vierbein components of the Ricci tensor traceless part S_{ab}

$$S = \begin{pmatrix} S_{11}-S_{00} & S_{12} & S_{13} \\ S_{12} & S_{22}-S_{00} & S_{23} \\ S_{12} & S_{23} & S_{33}-S_{00} \end{pmatrix}, \tag{2.1}$$

$$A = \begin{pmatrix} 0 & S_{03} & -S_{02} \\ -S_{03} & 0 & S_{01} \\ S_{02} & -S_{01} & 0 \end{pmatrix}. \tag{2.2}$$

Let us define $\mathfrak{S} = S - iA$ – a Hermitian matrix.

Eigenvectors $\mathfrak{F}$ of the matrix $\mathfrak{S}$ satisfy the equations

$$\mathfrak{S}\mathfrak{F} = \lambda\mathfrak{F},$$

$$E_{iklm}f^{lm}_P = \lambda f_{ik}.$$

A Hermitian matrix always has real eigenvalues and it is possible to express the matrix $\mathfrak{S}$ through its eigenvectors

$$\mathfrak{S}_{\alpha\beta} = \sum_{\iota=1}^{3}\epsilon_\iota \underset{\iota}{\bar{\mathfrak{F}}}_{\alpha}\underset{\iota}{\bar{\mathfrak{F}}}_{\beta}, \tag{2.3}$$

$$E_{iklm} = \sum_{\iota=1}^{3}\frac{\epsilon_\iota}{2}(\underset{\iota}{f}_{ik}\underset{\iota}{f}_{lm} + \underset{\iota}{\tilde{f}}_{ik}\underset{\iota}{\tilde{f}}_{lm}), \tag{2.4}$$

$$S_{ik} = \sum_{\iota=1}^{3}\frac{\epsilon_\iota}{2}(\underset{\iota}{f}_{ia}\underset{\iota}{f}^a_k + \underset{\iota}{\tilde{f}}_{ia}\underset{\iota}{\tilde{f}}^a_k), \tag{2.5}$$

where $\epsilon_\iota = sign(\lambda_\iota) = \begin{cases} -1 & ,\lambda_\iota < 0 \\ 0 & ,\lambda_\iota = 0 \\ 1 & ,\lambda_\iota > 0 \end{cases}$.

S_{ik} looks like the energy-momentum tensor of a Yang—Mills field with a 3-parametric local symmetry group. If the group is compact and nondegenerated, then it is the SU(2) or O(3) group.

If scalar curvature R is zero or if R is nonzero but we applied a conformal transformation described in the Appendix D, then Bianchi identities (B.6, B.7) give

$$S^{ik}_{;k} = 0, \tag{2.6}$$

$$C^{;m}_{iklm} = E^{;m}_{iklm} = \frac{1}{2}(S_{kl;n} - S_{kn;l}). \tag{2.7}$$

The second equation is a consequence of the first one, so it is enough to use the first equation.

After the substitution S_{ik} from (2.5), we get

$$\sum_{\iota=1}^{3}\epsilon_\iota(\underset{\iota}{f}_{ia}\underset{\iota}{f}^{ak}_{;k} + \underset{\iota}{\tilde{f}}_{ia}\underset{\iota}{\tilde{f}}^{ak}) = 0. \tag{2.8}$$

A more general expression for the divergence $\underset{\iota}{f}^{ik}_{;k}$ satisfiying Eq. (2.8) is

$$\underset{1}{f}^{ik}_{;k} = -\epsilon_1\underset{1}{\tilde{f}}^{ik}\underset{1}{\xi}_k - \epsilon_2\underset{2}{\tilde{f}}^{ik}\underset{3}{B}_k - \epsilon_3\underset{3}{\tilde{f}}^{ik}\underset{2}{B}_k + \epsilon_2\underset{2}{f}^{ik}\underset{3}{A}_k -$$

$$-\epsilon_3\underset{3}{f}^{ik}\underset{2}{A}_k, \tag{2.9}$$

$$\underset{2}{f}^{ik}_{;k} = -\epsilon_2\underset{2}{\tilde{f}}^{ik}\underset{2}{\xi}_k - \epsilon_3\underset{3}{\tilde{f}}^{ik}\underset{1}{B}_k - \epsilon_1\underset{1}{\tilde{f}}^{ik}\underset{3}{B}_k + \epsilon_3\underset{3}{f}^{ik}\underset{1}{A}_k -$$

$$\epsilon_1\underset{1}{f}^{ik}\underset{3}{A}_k, \tag{2.10}$$

$$\underset{3}{f}^{ik}_{;k} = -\epsilon_3\underset{3}{\tilde{f}}^{ik}\underset{3}{\xi}_k - \epsilon_1\underset{1}{\tilde{f}}^{ik}\underset{2}{B}_k - \epsilon_2\underset{2}{\tilde{f}}^{ik}\underset{1}{B}_k + \epsilon_1\underset{1}{f}^{ik}\underset{2}{A}_k -$$

$$-\epsilon_2\underset{2}{f}^{ik}\underset{1}{A}_k, \tag{2.11}$$

$$\underset{1}{\tilde{f}}{}^{ik}_{;k} = +\epsilon_1 \underset{1}{f}{}^{ik} \underset{1}{\xi}{}_k + \epsilon_2 \underset{2}{f}{}^{ik} \underset{3}{B}{}_k + \epsilon_3 \underset{3}{f}{}^{ik} \underset{2}{B}{}_k + \epsilon_2 \underset{2}{\tilde{f}}{}^{ik} \underset{3}{A}{}_k -$$

$$-\epsilon_3 \underset{3}{\tilde{f}}{}^{ik} \underset{2}{A}{}_k, \tag{2.12}$$

$$\underset{2}{\tilde{f}}{}^{ik}_{;k} = +\epsilon_2 \underset{2}{f}{}^{ik} \underset{2}{\xi}{}_k + \epsilon_3 \underset{3}{f}{}^{ik} \underset{1}{B}{}_k + \epsilon_1 \underset{1}{f}{}^{ik} \underset{3}{B}{}_k + \epsilon_3 \underset{3}{\tilde{f}}{}^{ik} \underset{1}{A}{}_k -$$

$$-\epsilon_1 \underset{1}{\tilde{f}}{}^{ik} \underset{3}{A}{}_k, \tag{2.13}$$

$$\underset{3}{\tilde{f}}{}^{ik}_{;k} = +\epsilon_3 \underset{3}{f}{}^{ik} \underset{3}{\xi}{}_k + \epsilon_1 \underset{1}{f}{}^{ik} \underset{2}{B}{}_k + \epsilon_2 \underset{2}{f}{}^{ik} \underset{1}{B}{}_k + \epsilon_1 \underset{1}{\tilde{f}}{}^{ik} \underset{2}{A}{}_k -$$

$$-\epsilon_2 \underset{2}{\tilde{f}}{}^{ik} \underset{1}{A}{}_k. \tag{2.14}$$

Quantities A_k looks like Yang—Mills potentials, but the dependence of f_{ik} upon A_k is unknown, so they are simply vectorial coefficients.

3. Already Unified Theory of a SU(2) Yang—Mills Field

Let $\epsilon_\iota = 1$, $\xi_k = 0$, $B_k = 0$, then the second divergence of bivectors f^{ik} gives

$$\underset{2}{f}{}^{ik}(\underset{3}{A}{}_{k;i} + \underset{1}{A}{}_i \underset{2}{A}{}_k) = \underset{3}{f}{}^{ik}(\underset{2}{A}{}_{k;i} - \underset{1}{A}{}_i \underset{3}{A}{}_k),$$
$$\underset{3}{f}{}^{ik}(\underset{1}{A}{}_{k;i} + \underset{2}{A}{}_i \underset{3}{A}{}_k) = \underset{3}{f}{}^{ik}(\underset{2}{A}{}_{k;i} - \underset{2}{A}{}_i \underset{1}{A}{}_k),$$
$$\underset{1}{f}{}^{ik}(\underset{2}{A}{}_{k;i} + \underset{3}{A}{}_i \underset{1}{A}{}_k) = \underset{2}{f}{}^{ik}(\underset{1}{A}{}_{k;i} - \underset{3}{A}{}_i \underset{2}{A}{}_k).$$

Interpreting these expressions as identities and using antisymmetry of f_{ik}, we obtain the usual definitions of SU(2) Yang—Mills field tensors:

$$f_{ik} = A_{k;i} - A_{i;k} + [A_i, A_k].$$

Then the system of equations (2.9) becomes

$$f^{ik}_{;k} + [A_k, f^{ik}] = 0$$

which are sourceless SU(2) Yang—Mills field equations [5].

The Einstein equations are already satisfied.

4. Field Equations in the General Case

Now we returning to the general case of eigenbivectors. All expansive calculations are moved into Appendix E.

The second divergence of (2.9)—(2.14) gives (E.16)—(E.18). It is not so easy to express eigenbivectors f_{ik} through their potentials like those in the previous section.

Expressions (E.16)—(E.18) as well as bivectors Ξ (E.13)—(E.15) are invariants of the gauge group of dual rotation (E.19)—(E.21).

It is possible to fix a gauge requiring (E.26). Such a way of gauge fixing defines 3 new scalar fields ϕ_ι with

$$\phi_1 + \phi_2 + \phi_3 = 0.$$

In this gauge, (E.16)—(E.18) take the form (E.27)—(E.29). Now interpreting these equations as identities, we obtain expressions for eigenbivectors. They are consistent only when (E.33)—(E.38) are true.

Let us define

$$F_{ik} = A_{k;i} - A_{i;k} + [A_i, A_k].$$

Then the first 3 equations of system (2.9)—(2.14) take the form

$$F^{ik}_{;k} + [A_k, F^{ik}] = J^i \tag{4.1}$$

which are 3-parametric group Yang—Mills field equations.

The last three equations of system (2.9)—(2.14) take the form

$$\tilde{F}^{ik}_{;k} + [A_k, \tilde{F}^{ik}] = K^i = 0. \tag{4.2}$$

These equations with consistency conditions (E.33)—(E.38) are interpreted as field equations for the sources of a Yang—Mills field.

Here, the vectors J^k and K^k are the sums of all terms (2.9)—(2.14) not included into (4.1), (4.2) with opposite sign.

Conclusions

It is shown that the GR Einstein equations allow, as a source of the gravitational field, nothing but a Yang—Mills field with 3-parametric symmetry group with or without sources. This means that any other sets of fields must mimic to demonstrate the same behaviour and the energy-momentum tensor as eigenmodes of the gravitational field, otherwise they will be prohibited.

Taking into account that the gauge Yang—Mills fields are key-stone ingredients in all unified gauge theories of elementary particles, we consider them above-revealed structural connection with the gauge gravity field as the very important relation. It opens a new possibility to include the gauge gravity field into inified gauge theories as well.

The nature and properties of sources of a Yang—Mills field require more detailed and careful researches.

APPENDIX A.
Bivectors and Its Vierbein Components

Orthogonal vierbein h_i^a is defined by the following expressions:

$$h_{ia}h_k^a = g_{ik};$$
$$h_a^i h_{ib} = \eta_{ab} = diag\,(\ 1, \quad -1, \quad -1, \quad -1,\ ), \qquad \text{(A.1)}$$

where g_{ik} is the metric tensor.

Bivector is an antisymmetric tensor $f_{ik} = -f_{ki}$. Vierbien components of the bivector $f_{ab} = h_a^i h_b^k f_{ik}$,

$$f_{ab} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & -h_3 & h_2 \\ -e_2 & h_3 & 0 & -h_1 \\ -e_3 & -h_2 & h_1 & 0 \end{pmatrix}.$$

Using the usual remapping of bivector indices

A	1	2	3	4	5	6
ik	01	02	03	32	13	21

,

it is possible to write the same bivecor as a real 6-vector or as a complex 3-vector $F = (e_1, e_2, e_3, h_1, h_2, h_3)$, $\mathfrak{F} = (e_1 + ih_1, e_2 + ih_2, e_3 + ih_3)$.

A dual bivector defined as

$$\tilde{f}_{ik} \equiv \frac{\sqrt{-g}}{2}\epsilon_{iklm}f^{lm},$$

where g is the determinant of the metric tensor g_{ik} and ϵ_{iklm} is the absolutely antisymmetric Levi—Civita pseudotensor, has components

$$\tilde{f}_{ab} = \begin{pmatrix} 0 & -h_1 & -h_2 & -h_3 \\ h_1 & 0 & -e_3 & e_2 \\ h_2 & e_3 & 0 & -e_1 \\ h_3 & -e_2 & e_1 & 0 \end{pmatrix},$$

$$\tilde{F} = (-h_1, -h_2, -h_3, e_1, e_2, e_3),$$

$$\tilde{\mathfrak{F}} = (-h_1 + ie_1, -h_2 + ie_2, -h_3 + ie_3).$$

The useful identity for the bivectors X_{ik} and Y_{lm} reads

$$X_{ia}Y_k{}^a - \tilde{X}_{ka}\tilde{Y}_i{}^a = \frac{1}{2}g_{ik}X_{ab}Y^{ab}.$$

It is possible to define so-called dual rotations with parameter φ as

$$f_{ik} \to f_{ik}\cos\varphi - \tilde{f}_{ik}\sin\varphi,$$

$$\tilde{f}_{ik} \to f_{ik}\sin\varphi + \tilde{f}_{ik}\cos\varphi.$$

Vierbein components of a parity conjugated contravariant bivector are the same as covariant vierbein components of the original one:

$$Pf^{ab} = f_P^{ab} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & -h_3 & h_2 \\ -e_2 & h_3 & 0 & -h_1 \\ -e_3 & -h_2 & h_1 & 0 \end{pmatrix}.$$

Contraction of any self-dual bivector $f_{ik}^{(+)} \equiv f_{ik} - i\tilde{f}_{ik}$ with any antiself-dual bivector $g_{ik}^{(-)} \equiv g_{ik} + i\tilde{g}_{ik}$ is zero, $f_{ik}^{(+)}g^{(-)ik} = 0$.

APPENDIX B.
Curvature Tensor and Its Properties

The Riemann tensor is defined as

$$R^i_{klm} = \frac{\partial\Gamma^i_{km}}{\partial x^l} - \frac{\partial\Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl},$$

where $\Gamma^i_{nl} = \frac{1}{2}g^{ij}(\frac{\partial g_{kj}}{x^l} + \frac{\partial g_{jl}}{x^k} - \frac{\partial g_{kl}}{x^j})$ is a Christoffel symbol of the second kind.

B1. Algebraic Properties

The Riemann tensor has the following symmetries:

$$R_{iklm} = -R_{kilm} = -R_{ikml},$$
$$R_{iklm} = R_{lmik},$$
$$R_{iklm} + R_{imkl} + R_{ilmk} = 0,$$

so it has 20 independent components.

Contractions of the Riemann tensor are known as the Ricci tensor and scalar curvature:

$$R_{ik} = R^l_{ilk}, \quad R_{ik} = R_{ki}, \quad R = R^i_i.$$

Using a bivectorial remapping of the first and second pairs of indices of the Riemann tensor, it is possible to rewrite it as a symmetric 6x6 matrix:

$$R_{iklm} \to R_{AB} = R_{BA} = \begin{pmatrix} M & N \\ N & -M \end{pmatrix} + \begin{pmatrix} S & A \\ -A & S \end{pmatrix}, \qquad \text{(B.1)}$$

where M, N, S, A — 3x3 matrices and

$$M_{\alpha\beta} = M_{\beta\alpha},\ N_{\alpha\beta} = N_{\beta\alpha},$$
$$S_{\alpha\beta} = S_{\beta\alpha},\ A_{\alpha\beta} = -A_{\beta\alpha},$$
$$A, B = 1\ldots 6, \qquad \alpha, \beta = 1\ldots 3;$$
$$M_{11} + M_{22} + M_{33} = \frac{R}{2},$$
$$N_{11} + N_{22} + N_{33} = 0.$$

The Riemann tensor is reducible into the following irreducible parts:

$$R_{iklm} = C_{iklm} + E_{iklm} + G_{iklm}, \quad (B.2)$$

where C_{iklm} is the so-called conformaly invariant Weyl tensor and

$$E_{iklm} = \frac{1}{2}\left(g_{il}S_{km} + g_{km}S_{il} - g_{im}S_{kl} - g_{kl}S_{im}\right); \quad (B.3)$$

$$G_{iklm} = \frac{R}{12}\left(g_{il}g_{km} - g_{im}g_{kl}\right); \quad (B.4)$$

$S_{ik} \equiv R_{ik} - \frac{R}{4}g_{ik}$ — Ricci tensor traceless part.

Matrices M and N of B.1 are constructed from components of the Weyl tensor C_{iklm} and scalar curvature R and matrices A and S — from components of E_{iklm} (or S_{ik}).

B2. Differential Properties

The Riemann tensor satisfies the Bianchi identities

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0 \quad (B.5)$$

and contracted Bianchi identities

$$R^m_{ikl;m} + R_{ik;l} - R_{il;k} = 0, \quad (B.6)$$

$$\left(R^i_k - \frac{1}{2}R\delta^i_k\right)_{;i} = 0. \quad (B.7)$$

APPENDIX C.
Structure of the Energy-momentum Tensor of an Electromagnetic Field

The energy-momentum tensor of an electromagnetic field is defined by the following expression:

$$T_{ik} = -f_{ia}f^a_k + \frac{1}{4}g_{ik}f_{ab}f^{ab} = -\frac{1}{2}(f_{ia}f^a_k + \widetilde{f_{ia}}\widetilde{f^a_k}).$$

It is possible to express its vierbein components through electromagnetic field components either in a real bivector form or in a complex 3-dimensional vector $\mathfrak{F} = (e_1+ih_1, e_2+ih_2, e_3+ih_3)$ and a complex conjugated vector $\bar{\mathfrak{F}} = (e_1-ih_1, e_2-ih_2, e_3-ih_3)$ in the following way:

$$T_{00} = \frac{1}{2}(e_1^2+e_2^2+e_3^2+h_1^2+h_2^2+h_3^2) =$$

$$= \frac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_1 + \bar{\mathfrak{F}}_2\mathfrak{F}_2 + \bar{\mathfrak{F}}_3\mathfrak{F}_3),$$

$$T_{11} = \frac{1}{2}(-e_1^2+e_2^2+e_3^2-h_1^2+h_2^2+h_3^2) =$$

$$= \frac{1}{2}(-\bar{\mathfrak{F}}_1\mathfrak{F}_1 + \bar{\mathfrak{F}}_2\mathfrak{F}_2 + \bar{\mathfrak{F}}_3\mathfrak{F}_3),$$

$$T_{22} = \frac{1}{2}(e_1^2-e_2^2+e_3^2+h_1^2-h_2^2+h_3^2) =$$

$$= \frac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_1 - \bar{\mathfrak{F}}_2\mathfrak{F}_2 + \bar{\mathfrak{F}}_3\mathfrak{F}_3),$$

$$T_{33} = \frac{1}{2}(e_1^2+e_2^2-e_3^2+h_1^2+h_2^2-h_3^2) =$$

$$= \frac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_1 + \bar{\mathfrak{F}}_2\mathfrak{F}_2 - \bar{\mathfrak{F}}_3\mathfrak{F}_3),$$

$$T_{01} = -e_2h_3 + h_2e_3 = \frac{i}{2}(\bar{\mathfrak{F}}_2\mathfrak{F}_3 - \bar{\mathfrak{F}}_3\mathfrak{F}_2),$$

$$T_{02} = e_1h_3 - h_1e_3 = \frac{i}{2}(-\bar{\mathfrak{F}}_1\mathfrak{F}_3 + \bar{\mathfrak{F}}_3\mathfrak{F}_1),$$

$$T_{03} = -e_1h_2 + h_1e_2 = \frac{i}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_2 - \bar{\mathfrak{F}}_2\mathfrak{F}_1),$$

$$T_{12} = -e_1e_2 - h_1h_2 = -\frac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_2 + \bar{\mathfrak{F}}_2\mathfrak{F}_1),$$

$$T_{13} = -e_1e_3 - h_1h_3 = -\frac{1}{2}(\bar{\mathfrak{F}}_1\mathfrak{F}_3 + \bar{\mathfrak{F}}_3\mathfrak{F}_1),$$

$$T_{23} = -e_2e_3 - h_2h_3 = -\frac{1}{2}(\bar{\mathfrak{F}}_2\mathfrak{F}_3 + \bar{\mathfrak{F}}_3\mathfrak{F}_2).$$

It is evident that the previous formulae are expressible in the 3x3 Hermitian matrix form:

$$\mathfrak{S} = \begin{pmatrix} T_{11}-T_{00} & T_{12}+iT_{03} & T_{13}-iT_{02} \\ T_{12}-iT_{03} & T_{22}-T_{00} & T_{23}+iT_{01} \\ T_{12}+iT_{02} & T_{23}-iT_{01} & T_{33}-T_{00} \end{pmatrix} =$$

$$= -\begin{pmatrix} \bar{\mathfrak{F}}_1\mathfrak{F}_1 & \bar{\mathfrak{F}}_1\mathfrak{F}_2 & \bar{\mathfrak{F}}_1\mathfrak{F}_3 \\ \bar{\mathfrak{F}}_2\mathfrak{F}_1 & \bar{\mathfrak{F}}_2\mathfrak{F}_2 & \bar{\mathfrak{F}}_2\mathfrak{F}_3 \\ \bar{\mathfrak{F}}_3\mathfrak{F}_1 & \bar{\mathfrak{F}}_3\mathfrak{F}_2 & \bar{\mathfrak{F}}_3\mathfrak{F}_3 \end{pmatrix}.$$

The matrix $\mathfrak{S}$ has rank 1, i.e. all its subdeterminants are zero. It is easy to prove that the former statement is equivalent to the so-called Rainich conditions [1, 4]:

$$T_{ia}T^a_k = \frac{1}{4}g_{ik}T_{ab}T^{ab}.$$

APPENDIX D.
On the Existence of a Conformal Transformation Provided Vanishing the Scalar Curvature

Consider the Riemannian space V_4 with the metric g_{ik}, Riemann tensor R_{iklm}, Ricci tensor $R_{ik} = R^a_{iak}$, and scalar curvature $R = R^a_a \not\equiv 0$. We shall find a conformal transformation

$$g_{ik} \to \bar{g}_{ik} = \varphi g_{ik},$$

$$R_{iklm} \to \bar{R}_{iklm},$$

$$R_{ik} \to \bar{R}_{ik},$$

$$R \to \bar{R} = 0,$$

which provides the vanishing of $\bar{R}$. The Riemann tensor of the conformal metric is

$$\bar{R}_{iklm} = \varphi R_{iklm} + \frac{1}{2}\left(g_{im}\varphi_{kl} + g_{kl}\varphi_{im} - g_{il}\varphi_{km} - g_{km}\varphi_{il}\right) -$$

$$-\frac{3}{4\varphi}\left(g_{im}\varphi_k\varphi_l + g_{kl}\varphi_i\varphi_m - g_{il}\varphi_k\varphi_m - g_{km}\varphi_i\varphi_l\right) +$$

$$+\frac{1}{4\varphi}\left(g_{im}g_{kl} - g_{km}g_{il}\right)\varphi_n\varphi^n,$$

where $\varphi_i \equiv \nabla_i\varphi$ $\varphi_{ik} \equiv \nabla_i\nabla_k\varphi$. Then

$$\bar{R}_{ik}R =_{ik} -\frac{\varphi_{ik}}{\varphi} - \frac{1}{2\varphi}g_{ik}\nabla_n\nabla^n\varphi + \frac{3}{2\varphi^2}\varphi_i\varphi_k,$$

$$\bar{R} = R - \frac{3}{\varphi}\nabla_n\nabla^n\varphi + \frac{3}{2\varphi^2}\varphi_n\varphi^n.$$

Equating $\bar{R}$ to zero and making the substitution $\varphi = \psi^2$, we obtain the so-called conformal scalar field equation [6]:

$$\nabla_i\nabla^i\psi - \frac{1}{6}R\psi = 0.$$

APPENDIX E.
Detailed Calculations

To reduce some expressions, let us introduce complex field variables

$$\underset{\iota}{\mathfrak{A}}_{i} = \underset{\iota}{A}_{i} + i\underset{\iota}{B}_{i}, \tag{E.1}$$

$$\underset{\iota}{\mathfrak{F}}_{ik} = \underset{\iota}{f}_{ik} + i\underset{\iota}{\tilde{f}}_{ik}, \tag{E.2}$$

$$\underset{\iota}{\mathfrak{H}}_{ik} = \underset{\iota}{\Phi}_{ik} + i\underset{\iota}{\Theta}_{ik}, \tag{E.3}$$

so $\tilde{\mathfrak{F}} = -i\mathfrak{F}$.

Then (2.9)—(2.14) becomes

$$\underset{1}{\mathfrak{F}}^{ik}_{;k} = i\epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{1}{\xi}_{k} + \epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{3}{\mathfrak{A}}_{k} - \epsilon_3\underset{3}{\mathfrak{F}}^{ik}\underset{2}{\mathfrak{A}}^*_{k}, \tag{E.4}$$

$$\underset{2}{\mathfrak{F}}^{ik}_{;k} = i\epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{2}{\xi}_{k} + \epsilon_3\underset{3}{\mathfrak{F}}^{ik}\underset{1}{\mathfrak{A}}_{k} - \epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{3}{\mathfrak{A}}^*_{k}, \tag{E.5}$$

$$\underset{3}{\mathfrak{F}}^{ik}_{;k} = i\epsilon_3\underset{1}{\mathfrak{F}}^{ik}\underset{3}{\xi}_{k} + \epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{2}{\mathfrak{A}}_{k} - \epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{1}{\mathfrak{A}}^*_{k}, \tag{E.6}$$

where * means complex conjugation.

Let us introduce a complex bivectorial field $\mathfrak{H}$

$$\underset{1}{\mathfrak{H}}^{ik} = \underset{1}{\mathfrak{A}}_{[k;i]} + \epsilon_1(\underset{2}{\mathfrak{A}}^*_{[i}\underset{3}{\mathfrak{A}}^*_{k]} - i\underset{1}{\xi}'_{[i}\underset{1}{\mathfrak{A}}_{k]}), \tag{E.7}$$

$$\underset{2}{\mathfrak{H}}^{ik} = \underset{2}{\mathfrak{A}}_{[k;i]} + \epsilon_2(\underset{3}{\mathfrak{A}}^*_{[i}\underset{1}{\mathfrak{A}}^*_{k]} - i\underset{2}{\xi}'_{[i}\underset{2}{\mathfrak{A}}_{k]}), \tag{E.8}$$

$$\underset{3}{\mathfrak{H}}^{ik} = \underset{3}{\mathfrak{A}}_{[k;i]} + \epsilon_3(\underset{1}{\mathfrak{A}}^*_{[i}\underset{2}{\mathfrak{A}}^*_{k]} - i\underset{3}{\xi}'_{[i}\underset{3}{\mathfrak{A}}_{k]}), \tag{E.9}$$

where [ ] means alternation,

$$\epsilon_1\underset{1}{\xi}' = \epsilon_2\underset{2}{\xi} - \epsilon_3\underset{3}{\xi}, \tag{E.10}$$

$$\epsilon_2\underset{2}{\xi}' = \epsilon_3\underset{3}{\xi} - \epsilon_1\underset{1}{\xi}, \tag{E.11}$$

$$\epsilon_3\underset{3}{\xi}' = \epsilon_1\underset{1}{\xi} - \epsilon_2\underset{2}{\xi}, \tag{E.12}$$

and a real field Ξ:

$$\underset{1}{\Xi}_{ik} = \underset{1}{\xi}_{[k;i]} - 2\epsilon_2\underset{3}{A}_{[i}\underset{3}{B}_{k]} + 2\epsilon_3\underset{2}{A}_{[i}\underset{2}{B}_{k]}, \tag{E.13}$$

$$\underset{2}{\Xi}_{ik} = \underset{2}{\xi}_{[k;i]} - 2\epsilon_3\underset{1}{A}_{[i}\underset{1}{B}_{k]} + 2\epsilon_1\underset{3}{A}_{[i}\underset{3}{B}_{k]}, \tag{E.14}$$

$$\underset{3}{\Xi}_{ik} = \underset{3}{\xi}_{[k;i]} - 2\epsilon_1\underset{2}{A}_{[i}\underset{2}{B}_{k]} + 2\epsilon_2\underset{1}{A}_{[i}\underset{1}{B}_{k]}. \tag{E.15}$$

Vanishing the second divergence of any bivector gives

$$\epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{3}{\mathfrak{H}}_{ik} - \epsilon_3\underset{3}{\mathfrak{F}}^{ik}\underset{2}{\mathfrak{H}}^*_{ik} + i\epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{1}{\Xi}_{ik} = 0, \tag{E.16}$$

$$\epsilon_3\underset{3}{\mathfrak{F}}^{ik}\underset{1}{\mathfrak{H}}_{ik} - \epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{3}{\mathfrak{H}}^*_{ik} + i\epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{2}{\Xi}_{ik} = 0, \tag{E.17}$$

$$\epsilon_1\underset{1}{\mathfrak{F}}^{ik}\underset{2}{\mathfrak{H}}_{ik} - \epsilon_2\underset{2}{\mathfrak{F}}^{ik}\underset{1}{\mathfrak{H}}^*_{ik} + i\epsilon_3\underset{3}{\mathfrak{F}}^{ik}\underset{3}{\Xi}_{ik} = 0. \tag{E.18}$$

Transformations of the fields under dual rotations read

$$\underset{\iota}{\mathfrak{F}} \to e^{-i\epsilon_\iota\alpha_\iota}\underset{\iota}{\mathfrak{F}}, \tag{E.19}$$

$$\underset{\iota}{\mathfrak{A}} \to e^{-i\epsilon_\iota\alpha'_\iota}\underset{\iota}{\mathfrak{A}}, \tag{E.20}$$

$$\underset{\iota}{\mathfrak{H}} \to e^{-i\epsilon_\iota\alpha'_\iota}\underset{\iota}{\mathfrak{H}}, \tag{E.21}$$

where

$$\epsilon_1\underset{1}{\alpha}' = \epsilon_2\underset{2}{\alpha} - \epsilon_3\underset{3}{\alpha}, \tag{E.22}$$

$$\epsilon_2\underset{2}{\alpha}' = \epsilon_3\underset{3}{\alpha} - \epsilon_1\underset{1}{\alpha}, \tag{E.23}$$

$$\epsilon_3\underset{3}{\alpha}' = \epsilon_1\underset{1}{\alpha} - \epsilon_2\underset{2}{\alpha}. \tag{E.24}$$

Ξ is invariant under dual rotations. It is evident that Eqs. (E.16)—(E.18) are also invariant.

Let

$$\frac{\varphi_2}{\varphi_3} = e^{\phi_1}, \quad \frac{\varphi_3}{\varphi_1} = e^{\phi_2}, \quad \frac{\varphi_1}{\varphi_2} = e^{\phi_3}, \tag{E.25}$$

$$\phi_1 + \phi_2 + \phi_3 = 0,$$

where φ_ι are arbitrary positive real scalar functions. To solve (E.16)—(E.18), it is enough to fix a gauge requiring

$$\epsilon_1\varphi_1\underset{1}{\mathfrak{F}}^{ik}\underset{1}{\Xi}_{ik} + \epsilon_2\varphi_2\underset{2}{\mathfrak{F}}^{ik}\underset{2}{\Xi}_{ik} + \epsilon_3\varphi_3\underset{3}{\mathfrak{F}}^{ik}\underset{3}{\Xi}_{ik} = 0. \tag{E.26}$$

Then

$$\epsilon_2\underset{2}{\mathfrak{F}}^{ik}(\underset{3}{\mathfrak{H}}_{ik} - ie^{-\phi_3}\underset{2}{\Xi}_{ik}) = \epsilon_3\underset{3}{\mathfrak{F}}^{ik}(\underset{2}{\mathfrak{H}}^*_{ik} + ie^{\phi_2}\underset{3}{\Xi}_{ik}), \tag{E.27}$$

$$\epsilon_3\underset{3}{\mathfrak{F}}^{ik}(\underset{1}{\mathfrak{H}}_{ik} - ie^{-\phi_1}\underset{3}{\Xi}_{ik}) = \epsilon_1\underset{1}{\mathfrak{F}}^{ik}(\underset{3}{\mathfrak{H}}^*_{ik} + ie^{\phi_3}\underset{1}{\Xi}_{ik}), \tag{E.28}$$

$$\epsilon_1\underset{1}{\mathfrak{F}}^{ik}(\underset{2}{\mathfrak{H}}_{ik} - ie^{-\phi_2}\underset{1}{\Xi}_{ik})] = \epsilon_2\underset{2}{\mathfrak{F}}^{ik}(\underset{1}{\mathfrak{H}}^*_{ik} + ie^{\phi_1}\underset{2}{\Xi}_{ik}). \tag{E.29}$$

So,

$$\epsilon_1\underset{1}{\mathfrak{F}}_{ik} = \underset{1}{\mathfrak{H}}_{ik} - ie^{-\phi_1}\underset{3}{\Xi}_{ik} = \underset{1}{\mathfrak{H}}^*_{ik} + ie^{\phi_1}\underset{2}{\Xi}_{ik}, \tag{E.30}$$

$$\epsilon_2\underset{2}{\mathfrak{F}}_{ik} = \underset{2}{\mathfrak{H}}_{ik} - ie^{-\phi_2}\underset{1}{\Xi}_{ik} = \underset{2}{\mathfrak{H}}^*_{ik} + ie^{\phi_2}\underset{3}{\Xi}_{ik}, \tag{E.31}$$

$$\epsilon_3\underset{3}{\mathfrak{F}}_{ik} = \underset{3}{\mathfrak{H}}_{ik} - ie^{-\phi_3}\underset{2}{\Xi}_{ik} = \underset{3}{\mathfrak{H}}^*_{ik} + ie^{\phi_3}\underset{1}{\Xi}_{ik}. \tag{E.32}$$

The set of consistency conditions of system (E.30)—(E.32) is

$$\underset{1}{\Theta}_{ik} = \frac{e^{\phi_1}}{2}\underset{2}{\Xi}_{ik} + \frac{e^{-\phi_1}}{2}\underset{3}{\Xi}_{ik}, \tag{E.33}$$

$$\underset{2}{\Theta}_{ik} = \frac{e^{\phi_2}}{2}\underset{3}{\Xi}_{ik} + \frac{e^{-\phi_2}}{2}\underset{1}{\Xi}_{ik}, \tag{E.34}$$

$$\underset{3}{\Theta}_{ik} = \frac{e^{\phi_3}}{2}\underset{1}{\Xi}_{ik} + \frac{e^{-\phi_3}}{2}\underset{2}{\Xi}_{ik}, \tag{E.35}$$

$$\underset{1}{\tilde{\Phi}}_{ik} = \frac{e^{\phi_1}}{2}\underset{2}{\Xi}_{ik} - \frac{e^{-\phi_1}}{2}\underset{3}{\Xi}_{ik}, \qquad (E.36)$$

$$\underset{2}{\tilde{\Phi}}_{ik} = \frac{e^{\phi_2}}{2}\underset{3}{\Xi}_{ik} - \frac{e^{-\phi_2}}{2}\underset{1}{\Xi}_{ik}, \qquad (E.37)$$

$$\underset{3}{\tilde{\Phi}}_{ik} = \frac{e^{\phi_3}}{2}\underset{1}{\Xi}_{ik} - \frac{e^{-\phi_3}}{2}\underset{2}{\Xi}_{ik}. \qquad (E.38)$$

Now

$$\epsilon_1 \underset{1}{f}_{ik} = \underset{1}{\Phi}_{ik} = \underset{1}{A}_{[k;i]} + \epsilon_1(\underset{2}{A}_{[i}\underset{3}{A}_{k]} - \underset{2}{B}_{[i}\underset{3}{B}_{k]} + \underset{1}{\xi'}_{[i}\underset{1}{B}_{k]}), \qquad (E.39)$$

$$\epsilon_2 \underset{2}{f}_{ik} = \underset{2}{\Phi}_{ik} = \underset{2}{A}_{[k;i]} + \epsilon_2(\underset{3}{A}_{[i}\underset{1}{A}_{k]} - \underset{3}{B}_{[i}\underset{1}{B}_{k]} + \underset{2}{\xi'}_{[i}\underset{2}{B}_{k]}), \qquad (E.40)$$

$$\epsilon_3 \underset{3}{f}_{ik} = \underset{3}{\Phi}_{ik} = \underset{3}{A}_{[k;i]} + \epsilon_3(\underset{1}{A}_{[i}\underset{2}{A}_{k]} - \underset{1}{B}_{[i}\underset{2}{B}_{k]} + \underset{3}{\xi'}_{[i}\underset{3}{B}_{k]}). \qquad (E.41)$$

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ПРО РОЛЬ БЕЗСЛІДОВОЇ ЧАСТИНИ ТЕНЗОРА РІЧЧІ В НЕЗВІДНИХ СТРУКТУРАХ ЗАГАЛЬНОЇ ТЕОРІЇ ВІДНОСНОСТІ В СКІНЧЕННІЙ ПРОСТОРОВО-ЧАСОВІЙ ОБЛАСТІ

Ю. Семенов

Р е з ю м е

Незвідну частину $E_{iklm} = \frac{1}{2}(g_{il}S_{km} + g_{km}S_{kl} - g_{im}S_{kl} - g_{kl}S_{im})$ тензора Рімана R_{iklm}, яка побудована за допомогою метричного тензора g_{ik} та безслідової частини $S_{ik} = R_{ik} - \frac{1}{4}g_{ik}R$ тензора Річчі R_{ik}, розкладено на білінійні комбінації бівекторних полів, що є власними функціями тензора E_{iklm}. Вивчено польові рівняння для цих бівекторів, що індуковані тотожностями Біанки. Показано, що в загальному випадку існує трипараметрична локальна симетрія янг-мілсівського поля.

О РОЛИ БЕССЛЕДОВОЙ ЧАСТИ ТЕНЗОРА РИЧЧИ В НЕСВОДИМЫХ СТРУКТУРАХ ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ В КОНЕЧНОЙ ПРОСТРАНСТВЕННО-ВРЕМЕННОЙ ОБЛАСТИ

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Несводимая часть $E_{iklm} = \frac{1}{2}(g_{il}S_{km} + g_{km}S_{kl} - g_{im}S_{kl} - g_{kl}S_{im})$ тензора Римана R_{iklm}, построенная с помощью метрического тензора g_{ik} и нулевой части $S_{ik} = R_{ik} - \frac{1}{4}g_{ik}R$ тензора Риччи R_{ik}, разложена на билинейные комбинации бивекторных полей, являющихся собственными функциями тензора E_{iklm}. Изучены полевые уравнения для бивекторов, индуцированных тождествами Бианки. Показано, что в общем случае существует трехпараметрическая локальная симметрия янг-миллсовского поля.