WEIL REPRESENTATIONS OF THE CAYLEY—KLEIN HERMITIAN SYMPLECTIC CATEGORY[1]

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A method of the categorical extensions of Cayley—Klein groups is applied. The method uses the Cayley—Klein spaces as objects of the Cayley—Klein category endowed with all possible linear relations or bilinear forms as morphisms. The constructed induced representations of Cayley—Klein groups are transported to the constructions of Weil representations of the Cayley—Klein classical Hermitian categories by categorification. The explicit form of the Weil representations of the Cayley—Klein Hermitian symplectic category is given.

Introduction

In accordance to the unprecedented amount of information in the Cosmic Microwave Background data in the form of temperature and polarization power spectra [1], we suppose that the true cosmological model belongs to the class of models based on Cayley—Klein geometries. In the first part of the work, we follow the program of categorification and categorical extension developed in [2, 3] and apply it to the problem of categorification of the theory of a Cayley—Klein group $G(\mathbf{j})$ [4–6] as follows:

Let $G(\mathbf{j})$ be a Cayley—Klein group [4–6]. Then $G(\mathbf{j})$ is merely the visible part of a certain category **CK** which is invisible to the naked eye. More precisely, there exists a certain category **CK** (the *train* of the group $G(\mathbf{j})$) such that the group itself is the automorphism group of a certain object $V(\mathbf{j})$, while the Cayley—Klein semigroup $\Gamma(\mathbf{j})$ is the semigroup of endomorphisms of the same object. Furthermore, each representation ρ of $G'(\mathbf{j})$ on a space $H(\mathbf{j})$ can be extended to a representation of the category **CK**. In other words, for each object $W(\mathbf{j})$ of the category **CK**, we can construct a linear space $T(W(\mathbf{j}))$ and, for each morphism $P : W(\mathbf{j}) \to W'(\mathbf{j})$, we can construct a linear operator $\tau(P) : T(W(\mathbf{j})) \to T\ (W'(\mathbf{j}))$ such that, for any morphisms $P : W(\mathbf{j}) \to W'(\mathbf{j})$ and $Q : W'(\mathbf{j}) \to W''(\mathbf{j})$, we have

$\tau(QP) = \tau(Q)\tau(P)$

with $T(V(\mathbf{j})) = H(\mathbf{j})$, and the operators $\tau(g)$ and $\rho(g)$ are the same for all $g \in G(\mathbf{j})$.

We note that all the spaces $T(W(\mathbf{j}))$ and all the operators $\tau(g)$ "grow out of" the one and only representation ρ of $G(\mathbf{j})$ and the one and only space $H(\mathbf{j})$.

To explain this idea of an analog of a structure on a category, let us consider first the category of linear relations. The objects of this category are linear spaces over a field $\mathbb{F}$ (and we will suppose them to be finite-dimensional). The morphisms $P : V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ are the linear relations, that is, the subspaces P of $V(\mathbf{j}) \oplus W(\mathbf{j})$.

Sometimes such subspaces are the graphs of linear operators from $V(\mathbf{j})$ into $W(\mathbf{j})$, but in general this is not the case.

If $P : V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ and $Q : W(\mathbf{j}) \rightrightarrows Y(\mathbf{j})$ are linear relations, then their product $QP : V(\mathbf{j}) \rightrightarrows Y(\mathbf{j})$ is defined as follows: $(v, y) \in V(\mathbf{j}) \oplus Y(\mathbf{j})$ is contained in the subspace QP if there exists $\omega \in W(\mathbf{j})$ such that $(v, \omega) \in P$ and $(\omega, y) \in Q$ (and this is how one would want to define the product of "multivalued maps").

The following are defined for a linear relation P: $V(\mathbf{j}) \rightrightarrows W(\mathbf{j})$ in the same way as for an operator:

(a) the kernel $\ker P$ — the set of all $v \in V(\mathbf{j})$ such that $(v, 0) \in P$;

(b) the image $\mathrm{im}P$ — the projection of P onto $W(\mathbf{j})$;

(c) the domain of definition $D(P)$ — the projection of P onto $V(\mathbf{j})$.
In addition, we define

(d) the indefiniteness $\mathrm{Indef}(P)$; this is the set of $\omega \in W$ such that $(0, \omega) \in P$; if P is the graph of an operator, then $\mathrm{Indef}(P) = 0$;

(e) the rank $\mathrm{rk}(P)$: $\mathrm{rk}(P) = \dim D(P) - \dim\ker P = \dim \mathrm{im}P - \dim \mathrm{Indef}P = \dim P - \dim\ker P - \dim \mathrm{Indef}P$.

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This paper is organized as follows. In the first section, we define the dual numbers, real Cayley—Klein spaces, and real Cayley—Klein classical groups. In the second section, we construct the induced representations of Cayley—Klein orthogonal groups. In the third section, we consider categorification as a "bridge" allowing to transport constructions from real Cayley—Klein classical groups to Cayley—Klein classical Hermitian categories. In the fourth and fifth sections, Weil representations of the Cayley—Klein symplectic category are obtained.

Provided that the true cosmological model belongs to the class of models studied here as Cayley—Klein geometries with constant curvatures, the Cosmic Microwave Background data will enable us to constrain several combinations of cosmological parameters with an exquisite accuracy in accordance to unprecedented amount of information in the form of temperature and polarization power spectra [43].

1. Cayley—Klein Spaces and Cayley—Klein Groups

1.1. Dual Numbers

Dual numbers were introduced by Clifford [7] as far back as in the XIX century. They were used by A.P. Kotel'nikov [8] for constructing his theory of screws in three-dimensional spaces of Euclid, Lobachevsky and Riemann, by B.A. Rosenfeld [9] for description of non-Euclidean spaces, by R.I. Pimenov [10, 11] for axiomatic study of spaces with constant curvature. Some applications of dual numbers in kinematics can be found in the work by I.M. Yaglom [12]. The theory of dual numbers as number systems is exposed in monographs by D.N. Seiliger [13] and A.Sh. Bloch [14]. Nevertheless, it is impossible to say that dual numbers are well known, so we start with their description.

Under an associative algebra of rank n over the field of real numbers $\mathbb{R}$, we mean an n-dimensional vector space over this field, on which the operation of multiplication is defined which is associative $a(bc) = (ab)c$, distributive in respect to addition $(a+b)c = ac + bc$, and related with multiplication of elements by real numbers as follows:

$$(ka)b = k(ab) = a(kb), \tag{1.1}$$

where a, b, c are elements of the algebra; k is a real number.

If there is such element e of the algebra that, for any element a of the algebra, the relations $ae = a$, $ea = a$ are valid, then the element e is called unit.

Dual numbers $a = a_0e_0 + a_1e_1$; $a_0, a_1 \in \mathbb{R}$, are elements of an associative algebra of rank 2 with unit and generators satisfying the following conditions: $e_0^2 = e_0$, $e_0e_1 = e_1e_0$, $e_1^2 = 0$. This associative algebra is commutative, and e_0 is its unit. Therefore, further we shall write 1 instead of e_0 and denote generator e_1 by ι_1 (the greek letter "iota") and call it (purely) dual unit. For the sum, product, and quotient of dual numbers a and b, we have

$$a + b = (a_0 + \iota_1 a_1) + (b_0 + \iota_1 b_1) = a_0 + b_0 + \iota_1(a_1 + b_1),$$

$$ab = (a_0 + \iota_1 a_1)(b_0 + \iota_1 b_1) = a_0b_0 + \iota_1(a_1b_0 + a_0b_1),$$

$$\frac{a}{b} = \frac{a_0 + \iota_1 a_1}{c_0 + \iota_1 b_1} = \frac{a_0}{b_0} + \iota_1\left(\frac{a_1}{b_0} - \frac{a_0b_1}{b_0^2}\right). \tag{1.2}$$

Division can be carried out not always. Purely dual numbers $\iota_1 a_1$ do not have inverse element. Dual numbers are equal $(a = b)$, if their real parts are equal $(a_0 = b_0)$ and their purely dual parts are equal $(a_1 = b_1)$. Thus, the equation $a_1\iota_1 = b_1\iota_1$ has the unique solution $a_1 = b_1$ for $a_1, b_1 \neq 0$. This fact can be written formally as $\iota_1/\iota_1 = 1$, and this is how the last relation has to be interpreted, because ι_1^{-1} is not defined.

We denote the real part of a dual number $a = a_0 + \iota_1 a_1$ as $\mathrm{Re}a = a_0$, and purely dual – $\mathrm{Du}a = a_1$; modulus of a is modulus of its real part $|a| = |\mathrm{Re}a| = |a_0|$, and argument is the ratio $\mathrm{Du}a/|\mathrm{Re}a| = a_1/|a_0| = \arg a$, so that the trigonometric form of a is as follows:

$$a = |a|(\mathrm{signRe}a + \iota_1 \arg a) = |a_0|(\mathrm{sign}a_0 + \iota_1\frac{a_1}{a_0}), \tag{1.3}$$

$a_0 \neq 0$.

It is convenient also to introduce the notion of the parameter $Pa = a_1/a_0$ of a dual number a and to present this number as

$$a = a_0(1 + \iota_1 Pa) = a_0\left(1 + \iota_1\frac{a_1}{a_0}\right). \tag{1.4}$$

Functions of a dual variable $x = x_0 + \iota_1 x_1$ are defined by their expansion into Taylor series

$$f(x, a, b, \ldots) = f_0 + \iota_1 \delta f_0, \tag{1.5}$$

where all terms with coefficients ι_1^2, ι_1^3,... are omitted, and

$$\begin{aligned} f_0 &= f(x_0, a_0, b_0, \ldots), \\ \delta f_0 &= x_1\frac{\partial f_0}{\partial x_0} + a_1\frac{\partial f_0}{\partial a_0} + b_1\frac{\partial f_0}{\partial b_0} + \ldots . \end{aligned} \tag{1.6}$$

Because $Pf = \delta f_0/f_0 = \delta \ln f_0$, it is possible to write

$$f = f_0(1 + \iota_1 \delta \ln f_0). \tag{1.7}$$

In particular, any dual number a for $a_0 \neq 0$ can be presented as

$$a = a_0 e^{\iota_1 P} = a_0 e^{\iota_1 \frac{a_1}{a_0}}. \tag{1.8}$$

For dual x, we get

$$\begin{aligned} \sin x &= \sin x_0 + \iota_1 x_1 \cos x_0, \quad \sin \iota_1 x_1 = \iota_1 x_1, \\ \cos x &= \cos x_0 - \iota_1 x_1 \sin x_0, \quad \cos \iota_1 x_1 = 1. \end{aligned} \tag{1.9}$$

Properties of the symbol δf_0 can be derived from definition (1.6):

$$\delta f_0 - \delta \varphi_0 = \delta(f_0 - \varphi_0), \quad \frac{\partial}{\partial x_0}(\delta f_0) = \delta \left(\frac{\partial f_0}{\partial x_0} \right), \tag{1.10}$$

i.e. the operations δ and $\partial/\partial x_0$ are commutative. According to (1.5), the difference of two functions of a dual variable can be presented as $f - \varphi = f_0 - \varphi_0 + \iota_1 \delta(f_0 - \varphi_0)$, therefore, if the real parts of the functions f and φ coincide, then the functions f and φ also coincide. Using this fact, D.N. Seiliger showed [13] that, in a domain of dual numbers, all identities of algebra and trigonometry, all theorems of differential and integral calculus remain valid. In particular, the derivative of a function of dual variable can be found as

$$\frac{df(x)}{dx} = \frac{\partial f_0}{\partial x_0} + \iota_1 \delta \left(\frac{\partial f_0}{\partial x_0} \right). \tag{1.11}$$

Let us consider now a more general situation, where n dual units $\iota_1, \iota_2, \ldots, \iota_n$ with properties $\iota_k \iota_p = \iota_p \iota_k \neq 0$, $k \neq p$, $\iota_k^2 = 0$, $p, k = 1, 2, \ldots, n$ are taken as the generators of an associative algebra with unit. Then any element of this algebra $A_n(\iota)$ is a linear combination of monomials $\iota_{k_1} \iota_{k_2} \ldots \iota_{k_r}$, $k_1 < k_2 < \ldots < k_r$, which together with unit element make a basis in the algebra as in a linear space of dimension 2^n, i.e.

$$a = a_0 + \sum_{r=1}^{2^n - 1} \sum_{k_1, \ldots, k_r} a_{k_1 \ldots k_r} \iota_{k_1} \ldots \iota_{k_r}. \tag{1.12}$$

This notation becomes unique, if we put the additional requirement $k_1 < k_2 < \ldots < k_r$ or the condition of symmetry of coefficients $a_{k_1 \ldots k_r}$ in respect to indices $k_1, \ldots, k_r$. Two elements a, b of algebra $A_n(\iota)$ coincide, if their coefficients in expansion (1.12) are equal, i.e. $a_0 = b_0$, $a_{k_1 \ldots k_r} = b_{k_1 \ldots k_r}$. As in the case of dual numbers, this definition of equality of the elements of algebra $A_n(\iota)$ is expressed in the possibility to cancel out equal (with the same index) purely dual units $\iota_k / \iota_k = 1$, $k = 1, 2, \ldots, n$ (but not ι_k / ι_m, $k \neq m$, – such expressions are not defined).

Here it is appropriate to compare algebra $A_n(\iota)$ with Grassmannian algebra $A_{2n}(\xi)$, i.e. the associative algebra with unit, where the set of nilpotent generators $\xi_1, \xi_2, \ldots, \xi_{2n}$, $\xi_k^2 = 0$ exhibits the properties of anticommutativity $\xi_k \xi_p = -\xi_p \xi_k \neq 0$, $p \neq k$, p, $k = 1, \ldots, 2n$. Any element f of Grassmannian algebra $\Lambda_{2n}(\xi)$ can be expressed [15] as

$$f(\xi) = f_0 + \sum_{r=1}^{2^{2n}-1} \sum_{k_1, \ldots, k_r} f_{k_1 \ldots k_r} \xi_{k_1} \ldots \xi_{k_r}. \tag{1.13}$$

This representation is unique, if one requires $k_1 < k_2 < \ldots < k_r$ or puts on the condition of skew-symmetry $f_{k_1 \ldots k_r}$ in respect to indices $k_1, \ldots, k_r$. If only terms with even r differ from zero in expansion (1.13), then an element f is called even in respect to the set of canonical generators ξ_k. If, in expansion (1.13), only terms with odd r differ from zero, then f is called odd element. As a linear space, the Grassmannian algebra splits into even ${}^0\Lambda_{2n}$ and odd ${}^1\Lambda_{2n}$ subspaces: $\Lambda_{2n}(\xi) = {}^0\Lambda_{2n} + {}^1\Lambda_{2n}$, where ${}^0\Lambda_{2n}$ is not only a subspace, but also a subalgebra.

Let us consider the nonzero products $\xi_{2k-1}\xi_{2k}$, $k = 1, 2, \ldots, n$ of generators of the Grassmannian algebra $\Lambda_{2n}(\xi)$. It is easy to see that these products possess the same properties as the generators ι_k of algebra $A_n(\iota)$, i.e. purely dual units are simulated by the products of generators of Grassmannian algebra $\iota_k = \xi_{2k-1}\xi_{2k}$, $k = 1, 2, \ldots, n$. Thus, algebra $A_n(\iota)$ is a subalgebra of the even part ${}^0\Lambda_{2n}$ of Grassmannian algebra $\Lambda_{2n}(\xi)$. It is worth mentioning that even products of Grassmannian anticommuting generators are also called para-Grassmannian variables. The latter are employed for classical and quantum descriptions of massive and massless particles with integer spin [16] and in the theory of strings [17].

1.2. Real Cayley—Klein Spaces and Cayley—Klein Special Orthogonal Groups

1.2.1. Three Fundamental Geometries on a Line

Let us introduce an elliptic geometry on a line. Let us consider a circle $S_1^* = \{\mathbf{x}^* \in \mathbb{R}_2 | x_0^{*2} + x_1^{*2} = 1\}$ in Euclidean plane $\mathbb{R}_2$. The rotations $\mathbf{x}^{*'} = g(\varphi^*)\mathbf{x}^*$, i.e.

$$\begin{aligned} x_0^{*\prime} &= x_0^* \cos \varphi^* - x_1^* \sin \varphi^*, \\ x_1^{*\prime} &= x_0^* \sin \varphi^* + x_1^* \cos \varphi^* \end{aligned} \tag{1.14}$$

of group SO(2) bring the circle into itself. Let us identify diametrically opposite points of the circle and introduce the internal coordinate $\xi^{*\prime} = x_1^*/x_0^*$. Then, to rotations (1.14) in $\mathbb{R}_2$ for $\varphi^* \in (-\pi/2, \pi/2)$, there correspond the transformations

$$\xi^{*\prime} = \frac{\xi^* - a}{1 + \xi^* a^*}, \quad a^* = \mathrm{tg}\varphi^*, \tag{1.15}$$

where $a^* \in \mathbb{R}$. These transformations make the group of translations (motions) G_1 of an elliptic line with the rule of composition

$$a^{*\prime} = \frac{a^* + a_1^*}{1 - a^* a_1^*}. \tag{1.16}$$

Let us consider a representation of group SO(2) in the space of differentiable functions on $\mathbb{R}_2$, defined by the left shifts $T(g(\varphi^*))f(\mathbf{x}^*) = f(g^{-1}(\varphi^*)\mathbf{x}^*)$. The generator of the representation $X^* f(\mathbf{x}^*) = \frac{d}{d\varphi^*}(T(g(\varphi^*))f(\mathbf{x}^*))|_{\varphi^*=0}$, corresponding to transformation (1.14), can be easily found:

$$X^*(x_0^*, x_1^*) = x_1^* \frac{\partial}{\partial x_0^*} - x_0^* \frac{\partial}{\partial x_1^*}. \tag{1.17}$$

For the representation of group G_1 by left shifts in the space of differentiable functions on an elliptic line, the generator Z^*, corresponding to transformation (1.15), can be written as

$$Z^*(\xi^*) = (1 + \xi^{*2}) \frac{\partial}{\partial \xi^*}. \tag{1.18}$$

It is worth mentioning that, to rotations $g(\varphi^*) \in \mathrm{SO}(2)$, there corresponds the matrix generator

$$\mathfrak{X}^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.19}$$

Let us consider the transformation of Euclidean plane $\mathbb{R}_2$, consisting of multiplication of Cartesian coordinate x_1 by parameter j_1, namely

$$\psi : \mathbb{R}_2 \to \mathbb{R}_2(j_1), \quad \psi x_0^* = x_0, \quad \psi x_1^* = j_1 x_1, \tag{1.20}$$

where parameter $j_1 = 1, \iota_1, i$.

Mapping (1.20) brings Euclidean plane $\mathbb{R}_2$ into plane $\mathbb{R}_2(j_1)$, the geometry of the latter is defined by the metrics $x^2(j_1) = x_0^2 + j_1^2 x_1^2$. It is easy to see that $\mathbb{R}_2(j_1 = i)$ is a Minkowski plane and $\mathbb{R}_2(j_1 = \iota_1)$ is a Galilean plane.

Our main idea is that a transformation of geometries (1.20) induces a transformation of the corresponding motion groups and their algebras. Let us show how to derive these transformations.

Definition of angle measure in Euclidean plane $\mathbb{R}_2$ is determined by the ratio x_1^*/x_0^*, which turns into $j_1 x_1/x_0$ under transformation (1.20), i.e. angles are transformed according to the rule $\psi\varphi^* = j_1\varphi$. Changing coordinates in (1.14) according to (1.20) and angles according to the derived transformation rule and multiplying both sides of the second equation by j_1^{-1}, we get rotations in plane $\mathbb{R}_2(j_1)$

$$\begin{aligned} x_0{}' &= x_0 \cos j_1\varphi - x_1 j_1 \sin j_1\varphi, \\ x_1{}' &= x_0 j_1^{-1} \sin j_1\varphi + x_1 \cos j_1\varphi, \end{aligned} \tag{1.21}$$

making group $\mathrm{SO}(2, j_1)$. Functions of purely dual quantities are defined by the expansions into series, in particular $\cos \iota_1\varphi = 1$, $\sin \iota_1\varphi = \iota_1\varphi$. Transformations of group $\mathrm{SO}(2;\iota_1)$ are Galilean transformations and elements of group $\mathrm{SO}(2;i)$ are Lorentzian transformations, if x_0 is interpreted as time, and x_1 as spatial coordinate. The domain of definition $\Phi(j_1)$ of the group parameter φ is $\Phi(j_1 = 1) = (-\pi/2, \pi/2)$, $\Phi(\iota_1) = \Phi(i) = \mathbb{R}$.

Rotations (1.21) preserve the circle $S_1(j_1) = \{x \in \mathbb{R}_2(j_1) \,|\, x_0^2 + j_1^2 x_1^2 = 1\}$ (see Fig.1) in plane $\mathbb{R}_2(j_1)$.

Identification of diametrically opposite points gives the upper semicircle (for j_1=1) and connected component of a sphere, passing through the point ($x_0 = 1$, $x_1 = 0$) for $j_1 = \iota_1, i$. The internal coordinate on the circle ξ^* is transformed according to the rule $\psi\xi^* = j_1\xi$. Substituting in (1.15) and cancelling j_1 out of both sides, we get the formula for translations on a line:

$$\xi' = \frac{\xi - a}{1 + j_1^2 \xi_a}, \quad a = \frac{1}{j_1}\mathrm{tg} j_1\varphi, \tag{1.22}$$

where $a \in \mathbb{R}$.

These translations make group $G_1(j_1)$ – group of the motions of elliptic line $S_1(1)$ for $j_1 = 1$; parabolic line $S_1(\iota_1)$ for $j_1 = \iota_1$, and hyperbolic line $S_1(i)$ for $j_1 = i$.

In the space of differentiable functions on $\mathbb{R}_2(j_1)$, the generator $X(\mathbf{x})$ of a representation of group $\mathrm{SO}(2;j_1)$ is defined by the relation

$$X f(\mathbf{x}) = \frac{d}{d\varphi}(T(g(\varphi))f(\mathbf{x})|_{\varphi=0}.$$

Under transformation (1.20), derivative $d/d\varphi^*$ turns into $j_1^{-1}\frac{d}{d\varphi}$, therefore, to obtain derivative $d/d\varphi$, the generator X^* must be multiplied by j_1, i.e. the generators $X^*(\psi\mathbf{x}^*)$ and $X(\mathbf{x})$ are interrelated by the transformation

$$X(\mathbf{x}) = j_1 X^*(\psi\mathbf{x}^*) = j_1^2 x_1 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_1}. \tag{1.23}$$

The generator Z is transformed according to the same rule:

$$Z(\boldsymbol{\xi}) = j_1 Z^*(\psi\boldsymbol{\xi}^*) = (1 + j_1^2\xi^2)\frac{\partial}{\partial \xi}. \tag{1.24}$$

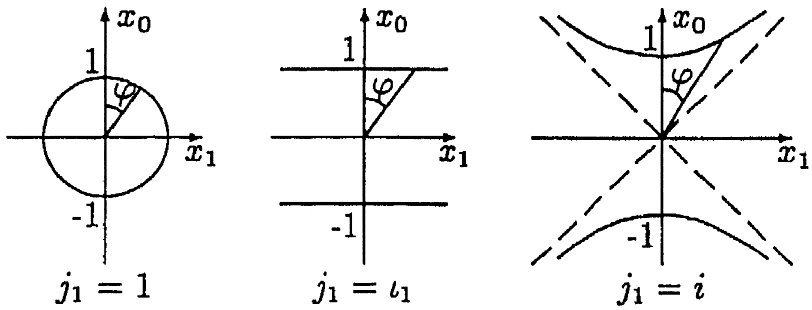

Fig. 1

Transformation rule for the matrix generator of rotations $\mathfrak{X}$: nonzero matrix element -1 over the main diagonal is changed for $-j_1^2$ and the rest matrix elements remain unchanged, i.e.

$$\mathfrak{X} = j_1 \begin{pmatrix} 0 & -j_1 \\ j_1^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -j_1^2 \\ 1 & 0 \end{pmatrix}. \quad (1.25)$$

The group of motions $G_1(j_1)$ of the one-dimensional Cayley—Klein space $S_1(j_1)$ is tightly connected with rotation group $\mathrm{SO}(2;j_1)$ in space $\mathbb{R}_2(j_1)$. Therefore, under Cayley—Klein space, we shall mean further both $S_1(j_1)$ and $\mathbb{R}_2(j_1)$, and under their groups of motions – both $G_1(j_1)$ and $\mathrm{SO}(2;j_1)$. The same rule is taken also in the case of spaces of higher dimensions.

We have studied comprehensively the simplest case of groups $\mathrm{SO}(2;j_1)$, $G_1(j_1)$, because here the main ideas of the method of transitions reveal themselves in the most clear way, not aggravated with mathematical calculations. These ideas are as follows: (a) to define transformation (1.20) from an Euclidean space to an arbitrary Cayley—Klein space; (b) to find the rules of transformations of motions, generators, etc. of the group; (c) using the approach exposed in (b), to derive motions, generators, etc. of the Cayley—Klein group from the corresponding quantities of the classical orthogonal group. The method of transitions, in spite of its simplicity, enables us to describe all Cayley—Klein groups, being aware of only classical orthogonal ones.

1.2.2. Nine Cayley—Klein Groups

Mapping

$$\psi : \mathbb{R}_3 \to \mathbb{R}_3(\mathbf{j}), \quad \psi x_0^* = x_0, \quad \psi x_1^* = j_1 x_1,$$
$$\psi x_2^* = j_1 j_2 x_2, \quad (1.26)$$

where $\mathbf{j} = (j_1, j_2)$; $j_1 = 1, \iota_1, i$; $j_2 = 1, \iota_2, i$, turns the three-dimensional Euclidean space into spaces $\mathbb{R}_3(\mathbf{j})$, on which spheres (or connected components of spheres) $S_2(\mathbf{j}) = \{\mathbf{x} \in \mathbb{R}_3(\mathbf{j}) | x_0^2 + j_1^2 x_1^2 + j_1^2 j_2^2 x_2^2 = 1\}$ for nine geometries of Cayley—Klein planes are realized. The interrelation of geometries and values of parameters $\mathbf{j}$ is clear from Fig.2, where the light cone is shown by dashed lines and the internal coordinates take values $t = x_1/x_0$, $r = x_2/x_0$.

Rotation angle $\varphi_{\mu\nu}^*$ in the coordinate plane $\{x_\nu^*, x_\mu^*\}$, $\mu < \nu$, $\mu,\nu = 0,1,2$, is determined by the ratio x_μ^*/x_ν^* under mapping (1.26) and transformed as $\psi\varphi_{\mu\nu}^* = \varphi_{\mu\nu} \prod_{m=\mu+1}^{\nu} j_m$.

Therefore, for rotations in the plane $\{x_\mu/x_\nu\}$ of space $\mathbb{R}_3(\mathbf{j})$, the following relations are valid:

$$x'_\mu = x_\mu \cos\left(\varphi_{\mu\nu} \prod_{m=\mu+1}^{\nu} j_m\right) -$$
$$-x_\nu \left(\prod_{m=\mu+1}^{\nu} j_m\right) \sin\left(\varphi_{\mu\nu} \prod_{m=\mu+1}^{\nu} j_m\right),$$
$$x'_\nu = x_\mu \left(\prod_{m=\mu+1}^{\nu} j_m^{-1}\right) \sin\left(\varphi_{\mu\nu} \prod_{m=\mu+1}^{\nu} j_m\right) +$$
$$+x_\nu \cos\left(\varphi_{\mu\nu} \prod_{m=\mu+1}^{\nu} j_m\right),$$
$$x'_\lambda = x_\lambda, \quad \lambda \neq \mu,\nu. \quad (1.27)$$

It is easy to find the matrix generators of rotations (1.27):

$$\mathfrak{X}_{01} = \begin{pmatrix} 0 & -j_1^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{X}_{02} = \begin{pmatrix} 0 & 0 & -j_1^2 j_2^2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\mathfrak{X}_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -j_2^2 \\ 0 & 1 & 0 \end{pmatrix}. \quad (1.28)$$

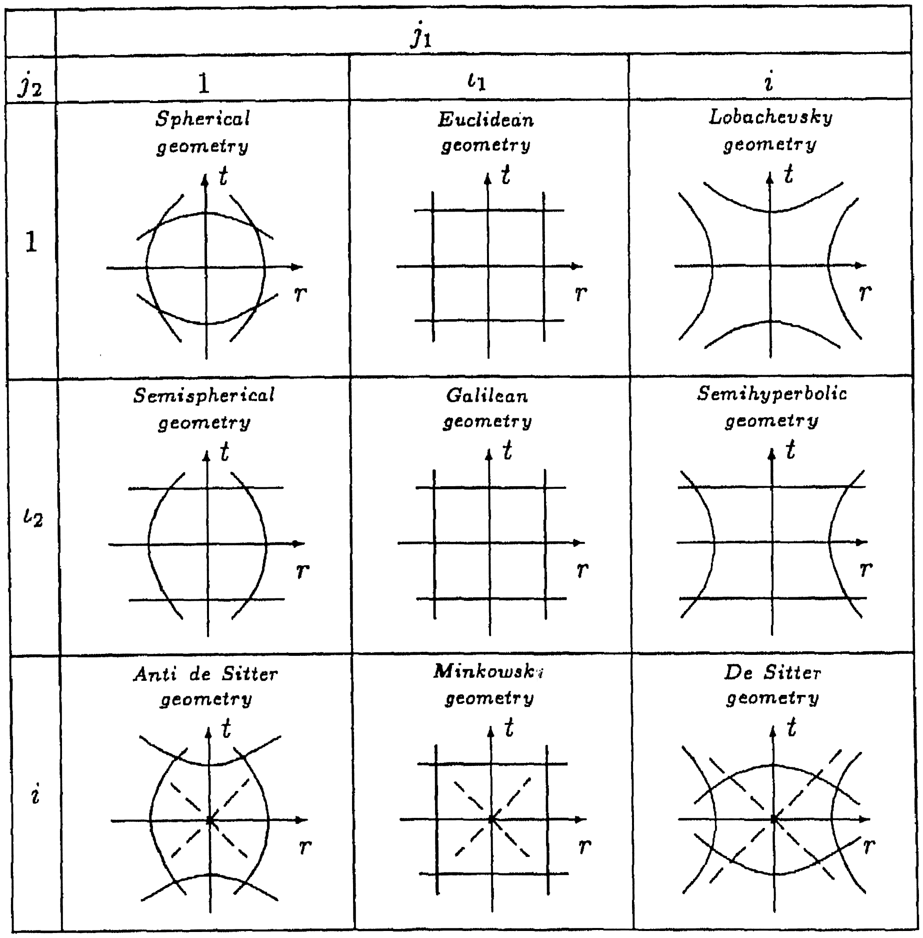

Fig. 2

They make the basis of the Lie algebra of group SO(3;**j**). Rule of transformations for generators of the representation of group SO(3;**j**) in the space of differentiable functions on $\mathbb{R}_3(\mathbf{j})$ by left shifts coincides with the rule of transformation for parameters $\varphi_{\mu\nu}$ and can be written as follows [4–6]:

$$X_{\mu\nu}(\mathbf{x}) = \left(\prod_{m=\mu+1}^{\nu} j_m\right) X^*_{\mu\nu}(\psi\mathbf{x}^*), \tag{1.29}$$

and generators themselves – as

$$X_{\mu\nu}(\mathbf{x}) = \left(\prod_{m=\mu+1}^{\nu} j_m^2\right) x_\nu \frac{\partial}{\partial x_\mu} - x_\mu \frac{\partial}{\partial x_\nu}. \tag{1.30}$$

Knowing generators, one can evaluate their commutators. But we shall derive commutators from the commutation relations of group SO(3). Let us introduce new notations for generators $X^*_{01} = H^*$, $X^*_{02} = P^*$, $X^*_{12} = K^*$. As is well known, commutators of the Lie algebra for group SO(3) can be written as follows:

$$[H^*, P^*] = K^*, \quad [P^*, K^*] = H^*, \quad [H^*, K^*] = -P^*. \tag{1.31}$$

Generators of algebra so(3) are transformed according to the rule $H = j_1 H^*$, $P = j_1 j_2 P^*$, $K = j_2 K^*$, i.e. $H^* = j_1^{-1} H$, $P^* = j_1^{-1} j_2^{-1} P$, $K^* = j_2^{-1} K$. Substituting these expressions in (1.31) and multiplying each commutator by a factor, equal to the denominator on the left side of each equation, i.e. the first – by $j_1^2 j_2$, the second – by $j_1 j_2^2$, the third – by $j_1 j_2$, we get commutators of the Lie algebra for group SO(3;**j**):

$$[H, P] = j_1^2 K, \quad [P, K] = j_2^2 H, \quad [H, K] = -P. \tag{1.32}$$

Cayley—Klein spaces $S_2(\mathbf{j})$ (or spaces of constant curvature) for $j_1 = 1, \iota_1, i$, $j_2 = \iota_2, i$ can serve as models of kinematics. In this case, the internal coordinate $\xi_1 = x_1/x_0$ can be interpreted as a temporal axis, and the internal coordinate $\xi_2 = x_2/x_0$ as a spatial one. Then H is the generator of a temporal shift , P is the generator of a spatial shift, K is the generator of a Galilean transformation for $j_2 = \iota_2$ or Lorentzian transformations for $j_2 = i$.

Final relations should not involve the division by a purely dual number. This requirement suggests the way of finding the rule of transformations for algebraic constructions. Let an algebraic quantity $Q^* = Q^*(A_1^*, \ldots, A_k^*)$ be expressed in terms of quantities $A_1^*, \ldots, A_k^*$ with a known rule of transformation under mapping ψ, for example, $A_1 = a_1 A_1^*, \ldots, A_k = a_k A_k^*$, where coefficients $a_1, \ldots, a_k$ are some products of parameters j_m. Substituting $A_1^* = a_1^{-1} A_1, \ldots, A_k^* = a_k^{-1} A_k$ in the relations for Q^*, we get the formula $Q^*(a_1^{-1} A_1, \ldots, a_k^{-1} A_k)$, involving, in general, indeterminate expressions, when parameters j_m are equal to purely dual units. For this reason, the last formula should be multiplied by such a minimal coefficient q that the final formula would not involve indeterminate expressions:

$$Q = q Q^*(a_1^{-1} A_1, \ldots, a_k^{-1} A_k). \tag{1.33}$$

Then (1.33) is the rule of transformation for quantity Q under mapping ψ. Such a method, stemmed out directly from the definition of coincidence of elements of algebra $A_n(\iota)$, turns out to be very useful and further will be widely employed. Rule of transformation (1.33) for algebraic quantity Q, derived from the requirement of the absence of indeterminate expressions for dual values of parameters j_m, is automatically satisfied for imaginary values of these parameters.

Let us exemplify this rule by a Casimir operator. The only Casimir operator for group SO(3) is

$$C_2^*(H^*, \ldots) = H^{*2} + P^{*2} + K^{*2}. \tag{1.34}$$

Substituting $H^* = j_1^{-1}$, $P^* = j_1^{-1} j_2^{-1} P$, $K^* = j_2^{-1} K$ in (1.34), we get

$$C_2^*(j_1^{-1} H, \ldots) = \frac{1}{j_1^2} H^2 + \frac{1}{j_1^2 j_2^2} P^2 + \frac{1}{j_2^2} K^2. \tag{1.35}$$

The most singular factor for $j_1 = \iota_1$, $j_2 = \iota_2$ is coefficient $j_1^{-2} j_2^{-2}$ of the sum and P^2. Multiplying both sides of Eq. (1.35) by $j_1^2 j_2^2$, we get rid of indeterminate expressions and derive the rule of transformation and the Casimir operator for group SO(3;**j**):

$$C_2(\mathbf{j}; H, \ldots) j_1^2 j_2^2 C_2^*(j_1^{-1} H, \ldots) = j_2^2 H^2 + P^2 + j_1^2 K^2. \tag{1.36}$$

As is known, the Casimir operator for two-dimensional Galilean group $\mathrm{SO}(3; \iota_1, \iota_2)$ is $C_2(\iota_1, \iota_2) = P^2$ (see, for example, [18]), for Poincaré group $\mathrm{SO}(3; \iota_1, i)$ is $C_2(\iota_1, i) = P^2 - H^2$, for group $\mathrm{SO}(3; i, 1) \equiv \mathrm{SO}(2, 1)$ is $C_2(i, 1) = H^2 + P^2 - K^2$ [19]. All these Casimir operators can be obtained from (1.36) for corresponding values of parameters **j**.

Matrix generators (1.28) make a basis of Lie algebra so(3;**j**) of group SO(3;**j**). To the general element

$$\mathfrak{X}(\mathbf{r}, \mathbf{j}) = r_1 \mathfrak{X}_{01} + r_2 \mathfrak{X}_{02} + r_3 \mathfrak{X}_{12} =$$

$$= \begin{pmatrix} 0 & -j_1^2 r_1 & -j_1^2 j_2^2 r_2 \\ r_1 & 0 & -j_2^2 r_3 \\ r_2 & r_3 & 0 \end{pmatrix} \tag{1.37}$$

of algebra so(3;**j**) *via* exponential mapping, one can put in correspondence a finite rotation $g(\mathbf{r}, \mathbf{j}) = \exp \mathfrak{X}(\mathbf{r}, \mathbf{j})$:

$$g(\mathbf{r}, \mathbf{j}) = E \cos r + \mathfrak{X}(\mathbf{r}, \mathbf{j}) \frac{\sin r}{r} + \mathfrak{X}'(\mathbf{r}, \mathbf{j}) \frac{1 - \cos r}{r^2},$$

$$r^2 = j_1^2 r_1^2 + j_1^2 j_2^2 r_2^2 + j_2^2 r_3^2, \tag{1.38}$$

$$\mathfrak{X}'(\mathbf{r}, \mathbf{j}) = \begin{pmatrix} j_2^2 r_3^2 & -j_1^2 j_2^2 r_2 r_3 & j_1^2 j_2^2 r_1 r_3 \\ -j_2^2 r_2 r_3 & j_1^2 j_2^2 r_2^2 & -j_1^2 j_2^2 r_1 r_2 \\ r_1 r_3 & -j_1^2 r_1 r_2 & j_1^2 r_1^2 \end{pmatrix}.$$

Misadvantage of the general parametrization (1.37), (1.38) of group SO(3;**j**) is the complexity of the composition rule for parameters **r** under group multiplication. F.I. Fedorov [20] has proposed a parametrization of rotation group SO(3) for which the composition group is particularly simple. It turns out that it is possible to construct analogues of such parametrization for all groups SO(3;**j**) [4]. The matrix of finite rotation of group SO(3;**j**) can be written as follows:

$$\begin{aligned} g(\mathbf{c}, \mathbf{j}) &= \frac{1 + c^*(\mathbf{j})}{1 - c^*(\mathbf{j})} = 1 + 2\frac{c^*(\mathbf{j}) + c^{*2}(\mathbf{j})}{1 + c^{*2}(\mathbf{j})}, \\ c^2(\mathbf{j}) &= j_2^2 c_1^2 + j_1^2 j_2^2 c_2^2 + j_1^2 c_3^2, \\ c^*(\mathbf{j}) &= \begin{pmatrix} 0 & -j_1^2 c_3 & j_1^2 j_2^2 c_2 \\ c_3 & 0 & -j_2^2 c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}, \end{aligned} \tag{1.39}$$

and, to matrix $g(\mathbf{c}, \mathbf{j}) = g(\mathbf{c}', \mathbf{j}) \cdot g(\mathbf{c}', \mathbf{j})$, there correspond parameters $\mathbf{c}''$, which can be expressed in terms of $\mathbf{c}$ and $\mathbf{c}'$ as follows:

$$\mathbf{c}'' = \frac{\mathbf{c} + \mathbf{c}' + [\mathbf{c}, \mathbf{c}']_j}{1 - (\mathbf{c}, \mathbf{c}')_j}. \tag{1.40}$$

Here, the scalar product of vectors $\mathbf{c}$ and $\mathbf{c}'$ is given by (1.39), and the vector product is given by

$$[\mathbf{c},\mathbf{c}']_j = (j_1^2[\mathbf{c},\mathbf{c}']_1,\ [\mathbf{c},\mathbf{c}']_2,\ j_2^2[\mathbf{c},\mathbf{c}']_3), \tag{1.41}$$

where $[\mathbf{c},\mathbf{c}']_k$ are components of a usual vector product in $\mathbb{R}_3$.

Wigner and Inonu [21] have introduced the operation of contraction (reduction, compression, limit transition) of groups and their representations. Under this operation, the generators of the initial group undergo a transformation, depending on a parameter ε, such that, for $\varepsilon \neq 0$, this transformation is non-singular and, for $\varepsilon \to 0$, it becomes singular. If the limits of transformed generators exist for $\varepsilon \to 0$, then they are generators of a new (contracted) group. It is worth mentioning that transformation (1.29) of generators of group SO(3) for dual values of parameters $\mathbf{j}$ is a Wigner—Inonu contraction. Really, $X^*_{\mu\nu}(\psi\mathbf{x}^*)$ is a singularly transformed generator of initial group SO_3, $\prod_{m=\mu+1}^{\nu} j_m$ plays a role of parameter ε tending to zero, and the resulted generators $X_{\mu\nu}(\mathbf{x})$ are generators of contracted group $SO(3;\mathbf{j})$.

Comparing the rule of transformation for generators (1.29) and expression (1.37) for a general element of algebra so(3), we find that, for imaginary values of parameters $\mathbf{j}$, some of real group parameters r_k become imaginary, i.e. they are analytically continued from the domain of real numbers to the domain of complex numbers. In this case, orthogonal group SO(3) is transformed into pseudo-orthogonal group $SO(p,q)$, $p+q=3$. When parameters $\mathbf{j}$ take dual values, real group parameters r_k become purely dual, i.e. they are continued to the domain of dual numbers. As a result, we get a contraction of group SO(3). Thus, from the point of view of the group transformation under mapping ψ, both at first sight different operations – contraction of groups and analytical continuation of groups – have the same nature: continuation of group parameters from the domain of real numbers to the domain of dual or complex numbers.

1.2.3. Extension to Higher Dimensions

Cayley—Klein geometries of dimension n are realized on spheres

$$S_n(\mathbf{j}) = \left\{\mathbf{x} \in \mathbb{R}_{n+1}(\mathbf{j}) \mid x_0^2 + \sum_{k=1}^{n} x_k^2 \prod_{m=1}^{k} j_m^2 = 1\right\}$$

in the spaces $\mathbb{R}_{n+1}(\mathbf{j})$ resulting from Euclidean space $\mathbb{R}_{n+1}$ under the mapping

$$\psi : \mathbb{R}_{n+1} \to \mathbb{R}_{n+1}(\mathbf{j}) \equiv V(\mathbf{j}),$$

$$\psi x_0^* \to x_0, \quad \psi x_k^* = x_k \prod_{m=1}^{k} j_m, \tag{1.42}$$

where $\mathbf{j} = (j_1,\dots,j_n)$; $\quad j_k = 1, \iota_k, i, \quad k = 1,2,\dots,n$.

The totality of all possible values of parameters $\mathbf{j}$ gives 3^n different real Cayley—Klein spaces $\mathbb{R}_{n+1}(\mathbf{j}) \equiv V(\mathbf{j})$. It is customary to identify the spaces (and their groups of motions), if their metrics have the same signature, i.e., for example, space $\mathbb{R}_3(1,i)$ with metric $x_0^2 + x_1^2 - x_2^2$ and space $\mathbb{R}_3(i,i)$ with metric $x_0^2 - x_1^2 + x_2^2$. But we have fixed Cartesian coordinate axes in $\mathbb{R}_{n+1}(\mathbf{j})$ ascribing fixed numbers to them, and, for this reason in our case, spaces $\mathbb{R}_3(1,i)$ and $\mathbb{R}_3(i,i)$ (and, correspondingly groups $SO(3;1,i)$ and $SO(3;i,i)$) are different. Groups $SO(3;1,i) \equiv SO(2,1)$ and $SO(3;i,1) \equiv SO(1,2)$ are also considered to be different.

Rotations in the two-dimensional plane $\{x_\mu, x_\nu\}$, the rule of transformation for representation generators, and generators themselves are given, correspondingly, by (1.27), (1.29), (1.30), where $\mu,\nu = 0,1,\dots,n$, $\mu<\nu$. For non-zero elements of the matrix generators of rotations, the following relations are valid:

$$(\mathfrak{X}_{\mu\nu})_{\nu\mu} = 1, \quad (\mathfrak{X}_{\mu\nu})_{\mu\nu} = -\prod_{m=\mu+1}^{\nu} j_m^2. \tag{1.43}$$

Commutation relations for Lie algebra $so(n+1;\mathbf{j})$ can be most simply derived from commutators of algebra $so(n+1)$, as has been done in Subsection 1.2.2. The non-zero commutators are

$$[X_{\mu_1\nu_1}, X_{\mu_2\nu_2}] = \\ = \begin{cases} \left(\prod_{m=\mu_1+1}^{\nu_1} j_m^2\right) X_{\nu_1\nu_2}, & \mu_1=\mu_2, \quad \nu_1<\nu_2, \\ \left(\prod_{m=\mu_2+1}^{\nu_2} j_m^2\right) X_{\mu_1\mu_2}, & \mu_1<\mu_2, \quad \nu_1=\nu_2, \\ -X_{\mu_1\nu_2}, & \mu_1<\mu_2=\nu_1<\nu_2. \end{cases} \tag{1.44}$$

Group $SO(n+1)$ has $\left\{\frac{n+1}{2}\right\}$ independent Casimir operators, where $\{x\}$ is the integer part of a number x. As is known [22], for even $n=2p$, Casimir operators are given by

$$\widetilde{C}^*_{2p}(X^*_{\mu\nu}) = \sum_{\alpha_1,\dots,\alpha_p=0}^{n} X^*_{\alpha_1\alpha_2} X^*_{\alpha_2\alpha_3} \dots X^*_{\alpha_{2p}\alpha_1}, \tag{1.45}$$

where $p = 1,2,\dots,k$.

For odd $n = 2k+1$, the operator

$$C^*_n(X^*_{\mu\nu}) = \\ = \sum_{\alpha_1,\dots,\alpha_n=0}^{n} \varepsilon_{\alpha_1\alpha_2\dots\alpha_n} X^*_{\alpha_1\alpha_2} X^*_{\alpha_3\alpha_4} \dots X^*_{\alpha_n\alpha_n+1}, \tag{1.46}$$

where $\varepsilon_{\alpha_1\ldots\alpha_n}$ is the completely antisymmetric unit tensor, must be added to operators (1.45).

Casimir operators $\widetilde{C}^*_{2p}$ can be defined in another way [23] as a sum of principal minors of order $2p$ for an antisymmetric matrix A composed of generators $X^*_{\mu\nu}$, i.e. $(A)_{\mu\nu} = X^*_{\mu\nu}$, $(A)_{\nu\mu} = -X^*_{\mu\nu}$. To obtain Casimir operators of group $\mathrm{SO}(n+1;\mathbf{j})$, we use the method exposed in Subsection 1.2.2. We find $X^*_{\mu\nu} = \left(\prod\limits_{m=\mu+1}^{\nu} j_m^{-1}\right) X_{\mu\nu}$ from (1.29) and substitute $X^*_{\mu\nu}$ in (1.45). The most singular coefficient $\prod\limits_{m=1}^{n} j_m^{-2p}$ is that of the summand $X_{0n}X_{n0}\ldots X_{n0}$ in (1.45). Thus, the rule of transformation for Casimir operators C_{2p} is

$$\widetilde{C}_{2p}(\mathbf{j};X_{\mu\nu}) = \left(\prod_{m=1}^{n} j_m^{2p}\right) C^*_{2p}\left(\left(\prod_{m=\mu+1}^{\nu} j_m^{-1}\right) X_{\mu\nu}\right), \tag{1.47}$$

and Casimir operators themselves turn out to be

$$\widetilde{C}_{2p}(\mathbf{j}) = \sum_{\alpha_1,\ldots,\alpha_{2p}=0}^{n} \left(\prod_{m=1}^{n} j_m^{2p}\right) \prod_{r=1}^{2p}\left(\prod_{l_r=\mu_r+1}^{\nu_r} j_{l_r}^{-1}\right) X_{\alpha_1\alpha_2} \times X_{\alpha_2\alpha_3}\ldots X_{\alpha_{2p}\alpha_{2p+1}}, \tag{1.48}$$

where $\mu_r = \min(\alpha_r,\alpha_{r+1})$; $\nu_r = \max(\alpha_r,\alpha_{r+1})$, $r = 1,2,\ldots,2p-1$; $\mu_{2p} = \min(\alpha_1,\alpha_{2p})$; $\nu_{2p} = \max(\alpha_1,\alpha_{2p})$.

For operators C_{rp} and C'_n, the expression without singular summands can be obtained, multiplying them by factor q, equal to the least common denominator of coefficients of summands, arising after the substitution of generators X for X^*. This least common denominator can be found by induction [4–6]. We restrict ourselves with the final expression for the rule of transformation for these Casimir operators:

$$C_{2p}(\mathbf{j};X_{\mu\nu}) = \left(\prod_{m=1}^{p-1} j_m^{2m} j_{n-m+1}^{2m} \prod_{l=p}^{n-p+1} j_l^{2p}\right) C^*_{2p}\left(X_{\mu\nu}\prod_{l=\mu+1}^{\nu} j_l^{-1}\right), \tag{1.49}$$

$$C'_n(\mathbf{j};X_{\mu\nu}) = \left(j_{\frac{n+1}{2}}^{\frac{n+1}{2}} \prod_{m=1}^{(n-1)/2} j_m^m j_{n-m+1}^m\right) C_n^{*'}\left(X_{\mu\nu}\prod_{l=\mu+1}^{\nu} j_l^{-1}\right). \tag{1.50}$$

Operator $C_{2p}(\mathbf{j})$ (or $C'_n(\mathbf{j})$) commutes with all generators $X_{\mu\nu}$ of group $\mathrm{SO}(n+1;\mathbf{j})$. Really, evaluating zero commutator $[C^*_{2p}, X^*_{\mu\nu}]$, we get the same summands with opposite signs. Under transformations (1.29), (1.47), both summands are multiplied by the same combination of parameters, which is a product of even powers of parameters. Therefore, both summands either change their sign, or vanish, or do not change their sign, but their sum is equal to zero in all cases. Moreover, operators $C_{2p}(\mathbf{j})$ for $p = 1,2,\ldots,k$ are linearly independent because they consist of the different powers of generators $X_{\mu\nu}$.

The next question to be cleared up is as follows: do $\left\{\frac{n+1}{2}\right\}$ Casimir operators (1.49), (1.50) exhaust all invariant operators of group $\mathrm{SO}(n+1;\mathbf{j})$? The answer is given by the following theorem.

THEOREM 1.1. *For any set of values of parameters* $\mathbf{j}$, *the number of invariant operators of group* $SO(n+1;\mathbf{j})$ *is* $\left\{\frac{n+1}{2}\right\}$.

P r o o f. It has been shown in [24] that the number $\tau(G)$ of invariant operators of algebraic group G satisfies the relation

$$\tau(G) = \dim G - r(G), \tag{1.51}$$

where $\dim G$ is the dimension of the group;

$$r(G) = \sup_{(a_1,\ldots,a_N)} \mathrm{rank} M_G. \tag{1.52}$$

Here $(M_G)_{kp} = \sum\limits_s c^s_{kp} a_s$; c^s_{kp} are the structural constants of the group; the supremum is taken over variables $a_k \in \mathbb{R}$, $k = 1,2,\ldots,N$. By Theorem 3 (see §99 in monograph [25]), orthogonal group $\mathrm{SO}(n+1)$ of dimension $n(n+1)/2$ is algebraic. It has $\left\{\frac{n+1}{2}\right\}$ invariant operators, so

$$r(\mathrm{SO}(n+1)) = \frac{n(n+1)}{2} - \left\{\frac{n+1}{2}\right\}, \tag{1.53}$$

according to (1.51). If parameters $\mathbf{j}$ do not take dual values, then the rank of matrix M does not change. But if some of parameters $\mathbf{j}$ are equal to dual units, then some elements of matrix M vanish, and the rank of the matrix can only diminish, i.e. the following inequalities are valid:

$$r(\mathrm{SO}(n+1;\boldsymbol{\iota})) \le r(\mathrm{SO}(n+1;\mathbf{j})) \le r(\mathrm{SO}(n+1)), \tag{1.54}$$

where $\boldsymbol{\iota}$ means that $j_1 = \iota_1$, $j_2 = \iota_2$, …, $j_n = \iota_n$.

In [26], it has been proved the inequality

$$r(\mathrm{SO}(n+1;\iota)) \ge \frac{n(n+1)}{2} - \left\{\frac{n+1}{2}\right\}, \quad i = 1,\ldots,n. \tag{1.55}$$

Taking together (1.53)–(1.55), we get $r(\mathrm{SO}(n+1;\mathbf{j})) = \frac{n(n+1)}{2} - \left\{\frac{n+1}{2}\right\}$ for any set of parameters $\mathbf{j}$. While $\forall\mathbf{j}\ \dim \mathrm{SO}(n+1;\mathbf{j}) = \frac{n(n+1)}{2}$, it follows from (1.51) that $\tau(\mathrm{SO}(n+1;\mathbf{j})) = \left\{\frac{n+1}{2}\right\}$. □

Thus, all invariant operators of group $\mathrm{SO}(n+1;\mathbf{j})$ are polynomial and given by (1.49), (1.50).

1.3. Complex Cayley—Klein Spaces and Cayley—Klein Special Unitary Groups

1.3.1. Definitions, Generators, Commutators

Special unitary groups $\mathrm{SU}(n+1;\mathbf{j})$ are connected with complex Cayley-Klein spaces $\mathbb{C}_{n+1}(\mathbf{j})$ which come out from $(n+1)$-dimensional complex space $\mathbb{C}_{n+1}$ under the mapping

$$\psi : \mathbb{C}_{n+1} \to \mathbb{C}_{n+1}(\mathbf{j}) \equiv V(\mathbf{j}),$$
$$\psi z_0^* = z_0, \quad \psi z_k^* = z_k \prod_{m=1}^{k} j_m, \quad k = 1,2,\ldots,n, \qquad (1.56)$$

where $z_0^*, z_k^* \in \mathbb{C}_{n+1},\ z_0, z_k \in \mathbb{C}_{n+1}(\mathbf{j})$ are complex Cartesian coordinates, $\mathbf{j} = (j_1,\ldots,j_n)$, each of parameters j_k takes three values: $j_k = 1, \iota_k, i$.

The totality of all possible values of parameter $\mathbf{j}$ gives 3^n different complex Cayley—Klein spaces $C_{n+1}(\mathbf{j}) \equiv V(\mathbf{j})$.

Quadratic form $(\mathbf{z}^*,\mathbf{z}^*) = \sum\limits_{m=0}^{n} |z_m^*|^2$ of space $\mathbb{C}_{n+1}$ turns into the quadratic form

$$(\mathbf{z},\mathbf{z}) = |z_0|^2 = \sum_{k=1}^{n} |z_k|^2 \prod_{m=1}^{k} j_m^2 \qquad (1.57)$$

of space $\mathbb{C}_{n+1}(\mathbf{j})$ under mapping (1.56). Here, $|z_k| = (x_k^2+y_k^2)^{1/2}$ is the absolute value (modulus) of a complex number $z_k = x_k + iy_k$, and $\mathbf{z}$ is a complex vector: $\mathbf{z} = (z_0, z_1, \ldots, z_n)$.

Space $\mathbb{C}_{n+1}(\mathbf{j})$ is called non-fiber space, if no one of the parameters $j_1,\ldots,j_n$ takes dual value. Space $\mathbb{C}_{n+1}(\mathbf{j})$ is called $(k_1,k_2\ldots,k_p)$ – fiber space, if $1 \le k_1 < k_2 < \ldots < k_p \le n$ and $j_{k_1} = \iota_{k_1},\ \ldots,\ j_{k_p} = \iota_{k_p}$, and the other $j_k = 1, i$. These fiberings are trivial [27] and can be characterized by the set of consequently nested projections $pr_1, pr_2, \ldots, pr_p$; where, for pr_1, the base is a subspace spanned over the basis vectors $\{e_0, e_1, \ldots, e_{k_1-1}\}$, and the fiber is a subspace spanned over $\{e_{k_1}, e_{k_1+1}, \ldots, e_n\}$; for pr_2, the base is a subspace $\{e_{k_1}, e_{k_1+1}, \ldots, e_{k_2-1}\}$, and the fiber is a subspace $\{e_{k_2}, e_{k_2+1}, \ldots, e_n\}$ and so on.

Definition 1.1. *Group $SU(n+1;\mathbf{j})$ consists of all transformations of space $\mathbb{C}_{n+1}(\mathbf{j})$ with unit determinant, keeping invariant the quadratic form (1.57).*

In the $(k_1,k_2,\ldots,k_p)$-fiber space $\mathbb{C}_{n+1}(\mathbf{j})$, we have a $(p+1)$-quadratic form, which remains invariant under the transformations of group $\mathrm{SU}(n+1;\mathbf{j})$. Under transformations of group $\mathrm{SU}(n+1;\mathbf{j})$, which do not affect coordinates $z_0, z_1, \ldots, z_{k_s-1}$, the form

$$(\mathbf{z},\mathbf{z})_{s+1} = \sum_{\alpha=k_s}^{k_{s+1}-1} |z_\alpha|^2 \prod_{l=k_s+1}^{\alpha} j_l^2, \qquad (1.58)$$

where $s = 0,1,\ldots,p;\ k_0 = 0$, remains invariant.

For $s = p$, the summation over α goes up to n.

Mapping (1.56) induces the transition of classical group $\mathrm{SU}(n+1)$ into group $\mathrm{SU}(n+1;\mathbf{j})$. All $(n+1)^2-1$ generators of group $\mathrm{SU}(n+1)$ are Hermitian matrices. However, because commutators for Hermitian generators are not symmetric, usually one prefers matrix generators A_{km}^*, $k,m = 0,1,2,\ldots,n$, of general linear group $GL(n+1,\mathbb{R})$, such that $(A_{mk}^*)_{km} = 1$ and all other matrix elements vanish. (The asterisk means that A^* is the generator of a classical group.) Commutators of generators A^* satisfy the following relation:

$$[A_{km}^*, A_{pq}^*] = \delta_{mp}A_{kq}^* - \delta_{kq}A_{pm}^*, \qquad (1.59)$$

where δ_{mp} is the Kronecker symbol.

Independent Hermitian generators of group $\mathrm{SU}(n+1)$ are given by the equations

$$Q_{\mu\nu}^* = \frac{i}{2}(A_{\mu\nu}^* + A_{\nu\mu}^*), \quad L_{\mu\nu}^* = \frac{1}{2}(A_{\nu\mu}^* - A_{\mu\nu}^*),$$
$$P_k^* = \frac{i}{2}(A_{k-1,k-1}^* - A_{kk}^*), \qquad (1.60)$$

where $\mu = 0,1,\ldots,n-1;\ \nu = \mu+1, \mu+2,\ldots,n;\ k = 1,2,\ldots,n$.

Matrix generators A^* are transformed under mapping (1.56) as follows:

$$A_{\mu\nu}(\mathbf{j}) = \left(\prod_{m=\mu+1}^{\nu} j_m\right) A_{\mu\nu}^*(\rightarrow) = \left(\prod_{m=\mu+1}^{\nu} j_m^2\right) A_{\mu\nu}^*,$$
$$A_{kk}(\mathbf{j}) = A_{kk}^*,$$
$$A_{\nu\mu}(\mathbf{j}) = \left(\prod_{m=\mu+1}^{\nu} j_m\right) A_{\nu\mu}^*(\rightarrow) = A_{\nu\mu}^*, \quad \mu < \nu, \qquad (1.61)$$

where symbol $A_{\mu\nu}^*(\rightarrow)$ means that non-zero matrix elements of the generator $A_{\mu\nu}^*$ are substituted: $(A_{\mu\nu}^*(\rightarrow))_{\mu\nu} = \left(\prod\limits_{m=\mu+1}^{\nu} j_m\right)(A_{\mu\nu}^*)_{\mu\nu}$, $(A_{\mu\nu}^*(\rightarrow))_{\nu\mu} =$

$\left(\prod_{m=\mu+1}^{\nu} j_m^{-1}\right)(A^*_{\mu\nu})_{\nu\mu}$, $\mu < \nu$. Commutators of generators $A(\mathbf{j})$ can be easily found [5]:

$$[A_{km}, A_{pq}] = \prod_{l=l_1}^{l_2} j_l \prod_{l=l_3}^{l_4} j_l \times$$
$$\times\left(\delta_{mp} A_{kq} \prod_{l=l_5}^{l_6} j_l^{-1} - \delta_{kq} A_{pm} \prod_{l=l_7}^{l_8} j_l^{-1}\right), \tag{1.62}$$

where $l_1 = 1 + \min(k,m)$; $l_2 = \max(k,m)$; $l_3 = 1 + \min(p,q)$; $l_4 = \max(p,q)$; $l_5 = 1 + \min(k,q)$; $l_6 = \max(k,q)$; $l_7 = 1 + \min(m,p)$; $l_8 = \max(m,p)$.

Hermitian generators (1.60) are transformed in the same way under the transition from group SU$(n+1)$ to group SU$(n+1;\mathbf{j})$. This enables to find the matrix generators of group SU$(n+1;\mathbf{j})$:

$$Q_{\mu\nu}(\mathbf{j}) = \left(\prod_{m=\mu+1}^{\nu} j_m\right) Q^*_{\mu\nu}(\to) = \frac{i}{2}[A_{\nu\mu}(\mathbf{j}) + A_{\mu\nu}(\mathbf{j})] =$$
$$= \frac{i}{2}\left(A^*_{\nu\mu} + A^*_{\mu\nu} \prod_{m=\mu+1}^{\nu} j_m^2\right), \tag{1.63}$$

$$L_{\mu\nu}(\mathbf{j}) = \left(\prod_{m=\mu+1}^{\nu} j_m\right) L^*_{\mu\nu}(\to) = \frac{1}{2}[A_{\nu\mu}(\mathbf{j}) - A_{\mu\nu}(\mathbf{j})] =$$
$$= \frac{1}{2}\left(A^*_{\nu\mu} - A^*_{\mu\nu} \prod_{m=\mu+1}^{\nu} j_m^2\right),$$

$$P_k(\mathbf{j}) = P_k^* = \frac{i}{2}(A^*_{k-1\,k-1} - A^*_{kk}), \ k = 1, 2, \ldots, n. \tag{1.64}$$

We do not cite commutation relations for generators $Q_{\mu\nu}(\mathbf{j})$, $L_{\mu\nu}(\mathbf{j})$, $P_k(\mathbf{j})$ because they are cumbersome [5]. They can be found using (1.62).

Let us cite one more realization of generators for a unitary group. If group GL_{n+1} acts via left translations in the space of analytical functions on $\mathbb{C}_{n+1}$, then its generators are $X^*_{\alpha\beta} = z^{*\beta}\partial_\alpha{}^*$, where $\partial_\alpha{}^* \equiv \partial/\partial z^{*\alpha}$. Hermitian generators of group SU$(n+1)$ can be expressed in terms of $X^*_{\alpha\beta}$ using (1.60), in which A^* must be changed for X^*. Under the mapping ψ, they are transformed according to the rule

$$Z_{\alpha\beta} = \left(\prod_{l=1+\min(\alpha,\beta)}^{\max(\alpha,\beta)} j_l\right) Z^*_{\alpha\beta}(\psi z^*), \tag{1.65}$$

where $Z_{\alpha\beta} = Q_{\mu\nu}, L_{\mu\nu}$, $P_k \equiv P_{kk}$.

Generators $X^*_{\alpha\beta}$ are transformed in a similar way, and this gives

$$X_{kk} = z_k\partial_k, \quad X_{\nu\mu} = z_\mu\partial_\nu,$$
$$X_{\mu\nu} = \left(\prod_{l=\mu+1}^{\nu} j_l^2\right) z_\nu\partial_\mu, \tag{1.66}$$

where $k = 1, 2, \ldots, n$; $\mu, \nu = 0, 1, \ldots, n$, $\mu < \nu$.

Matrix generators (1.63) make the basis of Lie algebra $su\,(n+1;\mathbf{j})$. To the general element of the algebra

$$Z(\mathbf{r}, \mathbf{s}, \mathbf{w}, \mathbf{j}) =$$
$$= \sum_{\lambda=1}^{n(n+1)/2} (r_\lambda Q_\lambda(\mathbf{j}) + s_\lambda L_\lambda(\mathbf{j})) + \sum_{k=1}^{n} w_k P_k, \tag{1.67}$$

where index λ is connected with the indices μ, ν, $\mu < \nu$, by the relation

$$\lambda = \nu + \mu(n-1) - \mu(\mu-1)/2, \tag{1.68}$$

and the group parameters r_λ, s_λ, w_k are real, there corresponds a finite group transformation of group SU$(n+1;\mathbf{j})$

$$\Xi(\mathbf{r}, \mathbf{s}, \mathbf{w}, \mathbf{j}) = \exp Z(\mathbf{r}, \mathbf{s}, \mathbf{w}, \mathbf{j}). \tag{1.69}$$

According to the Cayley—Hamilton theorem [28], matrix Ξ can be algebraically expressed in terms of matrices Z^m, $m = 0, 1, 2\ldots, n$.

1.4. Fiber Cayley—Klein Spaces and Cayley—Klein Symplectic Groups

1.4.1. Fiber Space

Before proceed to the symplectic groups Sp$(n;\mathbf{j})$ in Cayley—Klein spaces, let us study space $\mathbb{R}_n(\mathbf{j})$, obtained from n-dimensional Euclidean space $\mathbb{R}_n$ using the mapping

$$\psi : \mathbb{R}_n \to \mathbb{R}_n(\mathbf{j}), \quad \psi x_1^* = x_1,$$
$$\psi x_k^* = x_k \prod_{l=2}^{k} j_l, \quad k = 2, 3, \ldots, n, \tag{1.70}$$

where $\mathbf{x}^* \in \mathbb{R}_n$, $x \in \mathbb{R}_n(\mathbf{j})$, $\mathbf{j} = (j_2, j_3, \ldots, j_n)$, $j_k = 1, \iota_k, i$.

If all $j_k = 1$, then ψ is the identical mapping; if all or some $j_k = i$, and then others are 1, then we get pseudo-Euclidean spaces of different signatures. For $j_{k_1} = \iota_{k_1}, \ldots, j_{k_p} = \iota_{k_p}$ and other $j_k = 1, i$, we obtain a fiber space with zero curvature $k_1, k_2, \ldots, k_p$. This space is characterized by the existence of consequently nested projections $pr_1, pr_2, \ldots, pr_p$, where, for pr_1, subspace $\{x_1, x_2, \ldots, x_{k_1-1}\}$ serves as a base, and space

$\{x_{k_1}, x_{k_1+1}, \dots, x_n\}$ – as a fiber; for pr_2 as a base – subspace $\{x_{k_1}, x_{k_1+1}, \dots, x_{k_2-1}\}$ and as a fiber – subspace $\{x_{k_2}, x_{k_2+1}, \dots, x_n\}$ and so on.

From mathematical point of view, fibering in space $\mathbb{R}_n(\mathbf{j})$ is trivial [32], i.e. globally it has the same structure as locally. From physical point of view, fibering originates absolute physical quantities. For example, the Galilean space, which is realized on sphere $S_4(1, \iota_2, 1, 1)$, can be characterized by the existence of absolute time $t = x_1$ and absolute space $\mathbb{R}_3 = \{x_2, x_3, x_4\}$.

1.4.2. Definitions, Generators, Commutators

Let us define group $\mathrm{Sp}(n;\mathbf{j})$ as the group of transformations of $2n$-dimensional space $\mathbb{R}_n(\mathbf{j}) \times \mathbb{R}_n(\mathbf{j}) \equiv V(\mathbf{j})$ keeping unchanged the bilinear form

$$[\mathbf{x}, \mathbf{y}] = x_1 y_{-1} - x_{-1} y_1 + \sum_{k=2}^{n} \left(\prod_{l=2}^{k} j_l^2 \right) (x_k y_{-k} - x_{-k} y_k). \tag{1.71}$$

Here Cartesian coordinates x_k, y_k, $k = 1, 2 \dots, n$, belong to the first, and x_{-k}, y_{-k} to the second factor in the direct product of spaces.

For the $(k_1, \dots, k_p)$-fiber space, there are $p + 1$ bilinear forms, which are preserved under transformations from group $\mathrm{Sp}(n;\mathbf{j})$, namely: the transformations from $\mathrm{Sp}(n;\mathbf{j})$, which do not affect the coordinates $x_{\pm 1}, \dots, x_{\pm(k_s-1)}$, leave invariant the form

$$[\mathbf{x}, \mathbf{y}]_{s+1} = \sum_{m=k_s}^{k_{s+1}-1} \left(\prod_{l=1+k_s}^{m} j_l^2 \right) (x_k y_{-k} - x_{-k} y_k), \tag{1.72}$$

where $s = 0, 1, 2, \dots, p$, $k_0 = 1$. For $s = p$, the summation over m takes place up to n.

The generators X of group $\mathrm{Sp}(n;\mathbf{j})$ can be obtained from the known [22] generators X^* of classical simplectic group $\mathrm{Sp}(n)$ by the transformation induced by mapping (1.69). Let us consider both matrix generators X connected with transformations in space $\mathbb{R}_n(\mathbf{j}) \times \mathbb{R}_n(\mathbf{j})$, and generators $X(\mathbf{x})$ resulting from the action of group $\mathrm{Sp}(n;\mathbf{j})$ in the space of differentiable functions on $\mathbb{R}_n(\mathbf{j}) \times \mathbb{R}_n(\mathbf{j})$ by left shifts, i.e. $g : f(\mathbf{x}) \to f(g^{-1}\mathbf{x})$. From the definition of a generator $X(\mathbf{x}) = \sum\limits_{k=-n}^{n} \left.\frac{\partial x'_k}{\partial a}\right|_{a=0} \partial_k$, where $\mathbf{x}' = g(a)\mathbf{x}$; $g(0) = 1$; $g \in Sp(n;\mathbf{j})$; $\mathbf{x} \in \mathbb{R}_n(\mathbf{j}) \times \mathbb{R}_n(\mathbf{j})$, we find the rule of transformation for generators under the transition from group $\mathrm{Sp}(n)$ to group $\mathrm{Sp}(n;\mathbf{j})$:

$$X_{\alpha\beta}(\mathbf{x}) = \left(\prod_{l=1+\min\{|\alpha|,|\beta|\}}^{\max\{|\alpha|,|\beta|\}} j_l \right) X^*_{\alpha\beta}(\psi \mathbf{x}^*). \tag{1.73}$$

For $k = -1, -2, \dots$, the upper limit of the product in (1.69) is $|k|$.

Transforming the known generators of the symplectic group, we get generators of group $\mathrm{Sp}(n;\mathbf{j})$ as

$$X_{\alpha\beta}(\mathbf{x}) = \left(\prod_{l=p}^{q} j_l^{1+\mathrm{sign}(|\alpha|-|\beta|)} \right) x_\alpha \partial_\beta -$$
$$- \left(\prod_{l=p}^{q} j_l^{1-\mathrm{sign}(|\alpha|-|\beta|)} \right) \varepsilon_\alpha \varepsilon_\beta x_{-\beta} \partial_{-\alpha}, \tag{1.74}$$

where $p = 1 + \min(|\alpha|, |\beta|)$; $q = \max(|\alpha|, |\beta|)$; $\varepsilon_\alpha = \mathrm{sign}\,\alpha$, i.e. the sign of α is zero ($\varepsilon_\alpha = 0$) for $\alpha = 0$, unit ($\varepsilon_\alpha = 1$) for $\alpha > 0$ and -1 ($\varepsilon_\alpha = -1$) for $\alpha < 0$, $\partial_\alpha \equiv \partial / \partial x_\alpha$, $\alpha, \beta = \pm 1, \dots, \pm n$.

Generators (1.73) are not independent. They are interrelated by the property of symmetry

$$X_{\alpha\beta}(\mathbf{x}) = -\varepsilon_\alpha \varepsilon_\beta X_{-\beta,-\alpha}(\mathbf{x}). \tag{1.75}$$

Dimension of group $\mathrm{Sp}(n;\mathbf{j})$ is $n\,(2n + 1)$ for any set of values of parameters $\mathbf{j}$, and, as independent variables, we choose the following generators:

$$X_{\mu\mu}(\mathbf{x}) = x_\mu \partial_\mu - x_{-\mu} \partial_{-\mu}, \quad \mu = 1, 2, \dots, n,$$
$$X_{\mu,-\mu}(\mathbf{x}) = 2 x_\mu \partial_\mu, \quad \mu = \pm 1, \pm 2, \dots, \pm n,$$
$$X_{\nu\mu}(\mathbf{x}) = \left(\prod_{l=1+|\mu|}^{\nu} j_l^2 \right) x_\nu \partial_\mu - \varepsilon_\mu x_{-\mu} \partial_{-\nu},$$
$$|\mu| < \nu, \quad \nu = 2, 3, \dots, n,$$
$$X_{\mu\nu}(\mathbf{x}) = x_\mu \partial_\nu - \left(\prod_{l=1+|\mu|}^{\nu} j_l^2 \right) \varepsilon_\mu x_{-\nu} \partial_{-\mu}. \tag{1.76}$$

Generators (1.73) satisfy the commutation relations

$$[X_{\alpha\beta}, X_{\alpha'\beta'}] = \left(\prod_{l=p_1}^{q_1} j_l \right) \left(\prod_{l=p'_1}^{q'_1} j_l \right) \times$$
$$\times \left\{ \left(\prod_{l=p_2}^{q_2} j_l^{-1} \right) \delta_{\alpha'\beta} X_{\alpha\beta'} - \left(\prod_{l=p'_2}^{q'_2} j_l^{-1} \right) \delta_{\alpha\beta'} X_{\alpha'\beta} + \right.$$
$$+ \left(\prod_{l=p_3}^{q_3} j_l^{-1} \right) \varepsilon_\alpha \varepsilon_\beta \delta_{-\beta',\beta} X_{\alpha',-\alpha} +$$
$$\left. + \left(\prod_{l=p'_3}^{q'_3} j_l^{-1} \right) \varepsilon_\beta \varepsilon_{\alpha'} \delta_{\alpha',-\alpha} X_{-\beta,\beta'} \right\}, \tag{1.77}$$

where $p_1 = 1+\min(|\alpha|,|\beta|)$; $q_1 = \max(|\alpha|,|\beta|)$; $p_2 = 1+\min(|\alpha|,|\beta'|)$; $q_2 = \max(|\alpha|,|\beta'|)$; $p_3 = 1+\min(|\alpha|,|\alpha'|)$; $q_3 = \max(|\alpha|,|\alpha'|)$; $p'_3 = 1+\min(|\beta|,|\beta'|)$; $q'_3 = \max(|\beta|,|\beta'|)$ and p'_k, q'_k, $k = 1,2$, can be obtained from p_k, q_k substituting α for α', β for β' in the latter.

Matrix generators $X_{\alpha\beta}$ are interrelated with generators $X_{\alpha\beta}(\mathbf{x})$ (1.73) in the space of differentiable functions as follows:

$$X_{\alpha\beta}(\mathbf{x}) = \underline{\partial} X_{\alpha\beta}\mathbf{x}, \tag{1.78}$$

where $\underline{\partial} = (\partial_1,\ldots,\partial_n,\partial_{-1},\ldots,\partial_{-n})$ is a matrix-row, $\mathbf{x} = (x_1,\ldots,x_n,x_{-1},\ldots,x_{-n})$ is a matrix-column, and the product in (1.77) is the usual product of matrices.

The independent generators (1.150) are two-dimensional matrices with the following non-zero elements:

$$(X_{\mu\mu})_{\mu\mu} = 1, \quad (X_{\mu\mu})_{-\mu,-\mu} = -1, \quad \mu = 1,2,\ldots,n,$$
$$(X_{\mu,-\mu})_{-\mu,\mu} = 2, \quad \mu = \pm1,\pm2,\ldots,\pm n,$$
$$(X_{\nu\mu})_{\mu\nu} = \prod_{l=1+|\mu|}^{\nu} j_l^2, \quad (X_{\nu\mu})_{-\nu,-\mu} = -\varepsilon_\mu,$$
$$(X_{\mu\nu})_{\nu\mu} = 1, \quad (X_{\mu\nu})_{-\mu,-\nu} = -\varepsilon_\mu \prod_{l=1+|\mu|}^{\nu} j_l^2,$$
$$|\mu| < \nu, \quad \nu = 2,3,\ldots,n. \tag{1.79}$$

1.5. Classification of Transitions between Cayley—Klein Groups

In the previous subsections, we have found the orthogonal, unitary, and symplectic groups in Cayley—Klein spaces and shown that their generators and other algebraic constructions can be obtained by a transformation of the corresponding constructions for classical groups. Such an approach is natural and justified by the fact that classical groups and their characteristic algebraic constructions are well studied. But is such approach the only one? Is it possible to take, as initial, one of the groups in the Cayley—Klein space? The positive answer to this question is given by the following theorem on the structure of transitions between groups.

Let us define (formally) the transition from space $\mathbb{C}_{n+1}(\mathbf{j})$ and generators $Z_{\alpha\beta}(\mathbf{z},\mathbf{j})$ of unitary group $\mathrm{SU}(n+1;\mathbf{j})$ to space $\mathbb{C}_{n+1}(\mathbf{j}')$ and generators $Z_{\alpha\beta}(\mathbf{z}',\mathbf{j}')$ via transformations, which can be obtained from transformations (1.56) and (1.64), substituting the parameters j_k for $j'_k j_k^{-1}$ in the latter, i.e.

$$\psi' : \mathbb{C}_{n+1}(\mathbf{j}) \to \mathbb{C}_{n+1}(\mathbf{j}'), \quad \psi' z_0 = z'_0,$$
$$\psi' z_k = z'_k \sum_{m=1}^{k} j'_m j_m^{-1}, \quad k = 1,2,\ldots,n,$$
$$Z_{\alpha\beta}(\mathbf{z}',\mathbf{j}') = \left(\prod_{l=1+\min(\alpha,\beta)}^{\max(\alpha,\beta)} j'_l j_l^{-1}\right) Z_{\alpha\beta}(\psi'\mathbf{z},\mathbf{j}). \tag{1.80}$$

The inverse transitions can be obtained from (1.79) by the change of the dashed parameters $\mathbf{j}'$ for the undashed parameters $\mathbf{j}$ and vice versa. Applying (1.79) to quadratic form (1.57) and generators (1.65), we obtain

$$(\mathbf{z}',\mathbf{z}') = |z'_0|^2 + \sum_{k=1}^{n} |z'_k|^2 \prod_{m=1}^{k} j_m'^2, \quad X_{kk} = z'_k\partial'_k,$$
$$X_{\nu\mu} = z'_\mu\partial'_\nu, \quad X_{\mu\nu} = \left(\prod_{l=1+\mu}^{\nu} j_l'^2\right) z'_\nu\partial'_\mu, \tag{1.81}$$

i.e. a quadratic form in space $\mathbb{C}_{n+1}(\mathbf{j}')$ and generators of group $\mathrm{SU}(n+1;\mathbf{j}')$.

However, the constructed transitions make sense not for all groups and spaces, because, for the dual values of parameters $\mathbf{j}$, the expressions ι_k^{-1}, $\iota_m\iota_k^{-1}$ for $k \neq m$ are not defined. We have defined (see Subsection 1.1.1) only expressions $\iota_k\iota_k^{-1}$, $k = 1,2,\ldots,n$. So if for some k, we put $j_k = \iota_k$, then transformations (1.79) will be defined and give us (1.80) only in the case where the dashed parameter with the same number is equal to the same purely dual number, i.e. $j'_k = \iota_k$.

The transitions from space $\mathbb{R}_{n+1}(\mathbf{j})$ to space $\mathbb{R}_{n+1}(\mathbf{j}')$, and from groups $\mathrm{SO}(n+1;\mathbf{j})$, $\mathrm{Sp}(n;\mathbf{j})$ to groups $\mathrm{SO}(n+1;\mathbf{j}')$, $\mathrm{Sp}(n;\mathbf{j}')$ as well, can be, correspondingly, obtained from transitions (1.42), (1.29), (1.69), (1.71) by the same substitution of parameters j_k for $j'_k j_k^{-1}$. Similarly can be justified the permissibility of these transitions. Let us introduce the notations: $G(\mathbf{j}) = \mathrm{SO}(n+1;\mathbf{j})$, $\mathrm{SU}(n+1;\mathbf{j})$, $\mathrm{Sp}(n;\mathbf{j})$, $\mathbb{R}(\mathbf{j}) = \mathbb{R}_{n+1}(\mathbf{j})$, $\mathbb{C}_{n+1}(\mathbf{j})$, $\mathbb{R}_n(\mathbf{j}) \times \mathbb{R}_n(\mathbf{j})$. Let us agree to denote the transformation of group generators by symbol $\Psi G(\mathbf{j}) = G(\mathbf{j}')$. Easy analysis of transformations (1.79) and their inverse transformations from the point of view of admissibility of the transitions [29] implies the following theorem.

THEOREM 1.2. I. *Let $G(\mathbf{j})$ be a group in non-fiber space $\mathbb{R}(\mathbf{j})$ and $G(\mathbf{j}')$ be a group on arbitrary space $\mathbb{R}(\mathbf{j}')$. Then $G(\mathbf{j}') = \Psi G(\mathbf{j})$. If $\mathbb{R}(\mathbf{j}')$ is a non-fiber space, then Ψ is a one-to-one mapping, and $G(\mathbf{j}) = \Psi^{-1}G(\mathbf{j}')$.*

II. *Let $G(\mathbf{j})$ be a group in $(k_1,k_2,\ldots,k_p)$-fiber space $\mathbb{R}(\mathbf{j})$ and $G(\mathbf{j}')$ be a group in $(m_1,m_2,\ldots,m_q)$-fiber space $\mathbb{R}(\mathbf{j}')$. Then $G(\mathbf{j}') = \Psi G(\mathbf{j})$, if the set of* *integers* $(k_1,\ldots,k_p)$ *is involved in the set of numbers* $(m_1,\ldots,m_q)$*;* $G(\mathbf{j}) = \Psi^{-1}G(\mathbf{j}')$ *if and only if* $p = q$, $k_1 = m_1,\ldots,k_p = m_q$.

It follows from Theorem 1.2 that group $G(\mathbf{j})$ for any set of values of the parameters $\mathbf{j}$ can be obtained not only from the classical group, but from a group in an arbitrary non-fiber Cayley—Klein space, i.e. from pseudo-orthogonal, pseudounitary, or pseudosymplectic groups. It is naturally that the transitions between other algebraic constructions, in particular between Casimir operators, are described by this theorem as well.

2. Induced Representations of Cayley—Klein Orthogonal Groups

2.1. Parametrization and Invariant Measure

Special orthogonal group $\mathrm{SO}(n+1;\mathbf{j})$ has been defined in Subsection 1.2. Expansion of algebra $\mathrm{so}(n+1;\mathbf{j})$ as a vector space into the direct sum,

$$\mathrm{so}(n+1;\mathbf{j}) = N_0(\mathfrak{X}_{01},\mathfrak{X}_{02},\ldots,\mathfrak{X}_{0n}) \oplus \mathrm{so}(n;\mathbf{j}'), \qquad (2.1)$$

where $\mathbf{j}' = (j_2, j_3,\ldots,j_n)$; $\mathfrak{X}_{km}$ are matrix generators $(\mathfrak{X}_{\mu\nu})_{\nu\mu} = 1$, $(\mathfrak{X}_{\mu\nu})_{\mu\nu} = -\prod_{m=\mu+1}^{\nu} j_m^2$, is invariant in respect to the adjoint representation of subalgebra $\mathrm{SO}(n;\mathbf{j}') = \{\mathfrak{X}_{km}, k < m, k,m = 1,2,\ldots,n\}$ (special subalgebra in terminology of [30]), because $[N_0, \mathrm{so}(n;\mathbf{j}')] \subset N_0$. Applying expansion (2.1) to subalgebra $\mathrm{SO}(n;\mathbf{j}')$ and setting this process forth, we obtain the complete expansion of algebra $\mathrm{SO}(n+1;\mathbf{j})$:

$$\mathrm{so}(n+1,\mathbf{j}) = N_0 \oplus (N_1 \oplus (N_2 \oplus \ldots \oplus (N_{n-2} \oplus \oplus \mathrm{so}(2;j_n)))\ldots), \qquad (2.2)$$

where $N_k = N_k(\mathfrak{X}_{k,k+1},\ldots,\mathfrak{X}_{kn})$.

General element $X(\mathbf{Q}_k) \in N_k$ is as follows:

$$X(\mathbf{Q}_k) = \sum_{s=k+1}^{n} Q_{ks}\mathfrak{X}_{ks} \qquad (2.3)$$

where $Q_{ks} \in \mathbb{R}$; $\mathbf{Q}_k = (Q_{k,k+1},\ldots,Q_{kn})$, $k = 0,1,\ldots,n-2$, $\mathbf{Q}_{n-1} = Q_{n-1,n}$; $X(\mathbf{Q}_{n-1}) \in so(2;j_n)$.

Exponential mapping brings $X(\mathbf{Q}_k)$ into the element of group $\mathrm{SO}(n+1;\mathbf{j})$ which can be written as follows:

$$s(\mathbf{Q}_k) = e^{X(\mathbf{Q}_k)} = \begin{pmatrix} I_k & 0 & 0 \\ 0 & \cos j_{k+1}Q_k & -\left(\prod_{m=k+2}^{r} j_m^2\right)\frac{Q_{kr}}{Q_k} j_{k+1}\sin j_{k+1}Q_k \\ 0 & \frac{Q_{ks}}{Q_k}\frac{1}{j_{k+1}}\sin j_{k+1}Q_k & \delta_{sr} - \left(\prod_{m=k+2}^{r} j_m^2\right)\frac{Q_{ks}Q_{kr}}{Q_k^2}(1-\cos j_{k+1}Q_k) \end{pmatrix}, \qquad (2.4)$$

where $r,s = k+1, k+2,\ldots,n$; $Q_k = \left(Q_{k,k+1}^2 + \sum_{r=k+2}^{n} Q_{kr}^2 \prod_{m=k+2}^{r} j_m^2\right)^{1/2}$, $k = 0,1,\ldots,n-1$, and, in addition, $Q_{n-1} = Q_{n-1,n}$.

Expansion (2.2) of algebra $\mathrm{so}(n+1;\mathbf{j})$ corresponds to a special expansion of the following group transformations $q \in \mathrm{SO}(n+1;\mathbf{j})$:

$$q \equiv q(\mathbf{Q}_0,\mathbf{Q}_1,\ldots,\mathbf{Q}_{n-1}) = \prod_{k=0}^{n-1} s(\mathbf{Q}_k). \qquad (2.5)$$

Further, we shall use this parametrization of group $\mathrm{SO}(n+1;\mathbf{j})$. To find the geometric sense of special group parameters $\mathbf{Q}_k$, let us consider the connected component of unit sphere $S_n(\mathbf{j})$ in space $\mathbb{R}_{n+1}(\mathbf{j})$ determined by the equation

$$S_n(\mathbf{j}) = \left\{\mathbf{x} \in \mathbb{R}_{n+1}(\mathbf{j}) | x_0^2 + \sum_{k=1}^{n} x_k^2 \prod_{m=1}^{k} j_m^2 = 1\right\}. \qquad (2.6)$$

Group $\mathrm{SO}(n+1;\mathbf{j})$ acts transitively on $S_n(\mathbf{j})$, which enables to realize the connected component of a sphere as factor-space $S_n(\mathbf{j}) = \mathrm{SO}(n+1;\mathbf{j})/\mathrm{SO}(n;\mathbf{j}') = \{s(\mathbf{Q}_0)\}$, the latter takes place in parametrization (2.5). Subgroup $\mathrm{SO}(n;\mathbf{j}')$ is a stationary subgroup for point $\mathbf{F}_0 = (1,0,\ldots,0)$ on the sphere. Acting on $\mathbf{F}_0$ by transformation $s(\mathbf{Q}_0)$, we get point $\mathbf{y}^0 = s(\mathbf{Q}_0)\mathbf{F}_0$ on the sphere with coordinates $x_0 = \cos j_1 Q_0$, $x_r = \frac{Q_{0k}}{Q_0}\frac{1}{j_1}\sin j_1 Q_0$, $r = 1,2,\ldots,n$, i.e. $pr_0\mathbf{y}^0 = \mathbf{Q}_0\frac{\sin j_1 Q_0}{j_1 Q_0}$, where pr_0 is a projection on subspace $\mathbb{R}_n(\mathbf{j}')$ along the axis x_0. We find from here that $s(\mathbf{Q}_0)$ is a rotation in the plane $\{x_0\mathbf{Q}_0\}$, $\mathbf{Q}_0 \in \mathbb{R}_n(\mathbf{j}')$ by angle Q_0. When $\mathbf{y}^0$

runs through all points on the sphere, vector $pr_0\mathbf{y}^0$ fills all projections of the sphere on subspace $\mathbb{R}_n(\mathbf{j}')$. For this reason, the range of definition $D_n(\mathbf{j})$ of parameter $\mathbf{Q}_0$ up to a factor is the projection of the sphere on $\mathbb{R}_n(\mathbf{j}')$:

$$D_n(\mathbf{j}) = \left\{ \begin{array}{ll} B^n_\pi(\mathbf{j}') = \{\mathbf{Q}_0 \in \mathbb{R}_n(\mathbf{j}') \,|\, Q_0^2 \le \pi^2\}, & j_1 = 1, \\ \mathbb{R}_n(\mathbf{j}'), & j_1 = \iota_1, \\ B^n_{i\pi}(\mathbf{j}') = \{\mathbf{Q}_0 \in \mathbb{R}_n(\mathbf{j}') \,|\, Q_0^2 \ge -\pi^2\}, & j_1 = i, \end{array} \right. \tag{2.7}$$

where $B^n_\pi(\mathbf{j}')$ is a solid sphere of real radius π in $\mathbb{R}_n(\mathbf{j}')$; $B^n_{i\pi}(\mathbf{j}')$ is a solid sphere of imaginary radius $i\pi$ in $\mathbb{R}_n(\mathbf{j}')$.

It is necessary to keep in mind that, for some values of parameters $\mathbf{j}'$, the sphere of imaginary radius coincides with the whole $\mathbb{R}_n(\mathbf{j}')$, for example, for $j_1 = i$, $j_2 = j_3 = \ldots = j_n = 1$, $Q_0^2 = \sum_{r=1}^n Q_{0r}^2 \ge -\pi^2$ for any $\mathbf{Q}_0 \in \mathbb{R}_n(\mathbf{j}')$.

Repeating the same considerations for sphere $S_{n-k}(\mathbf{j}^{(k)})$, $k = 1, 2, \ldots, n-2$, $\mathbf{j}^{(k)} = (j_{k+1}, j_{k+2}, \ldots, j_n)$, we find that $s(\mathbf{Q}_k)$ is a rotation in the plane $\{x_k, \mathbf{Q}_k\}$, $\mathbf{Q}_k \in \mathbb{R}_{n-k}(\mathbf{j}^{(k+1)})$ by angle Q_k, where $\mathbf{Q}_k < D_{n-k}(\mathbf{j}^{(k)})$ and

$$D_{n-k}(\mathbf{j}^{(k)}) = \left\{ \begin{array}{ll} B^{n-k}_\pi(\mathbf{j}^{(k+1)}) = \{\mathbf{Q}_k \in \mathbb{R}_{n-k}(\mathbf{j}^{(k+1)}) \,|\, Q_k^2 \le \pi^2\}, & j_{k+1} = 1, \\ \mathbb{R}_{n-k}(\mathbf{j}^{(k+1)}), & j_{k+1} = \iota_{k+1}, \\ B^{n-k}_{i\pi}(\mathbf{j}^{(k+1)}) = \{\mathbf{Q}_k \in \mathbb{R}_{n-k}(\mathbf{j}^{(k+1)}) \,|\, Q_k^2 \ge -\pi^2\}, & j_{k+1} = i. \end{array} \right. \tag{2.8}$$

For $k = n-1$, the transformation $s(\mathbf{Q}_{n-1}) = s(Q_{n-1,n})$ is a rotation in the plane $\{x_{n-1}, x_n\}$ by angle $Q_{n-1,n}$, where $Q_{n-1,n} \in D_1(j_n)$ and

$$D_1(j_n) = \left\{ \begin{array}{ll} [0, 2\pi), & j_n = 1, \\ \mathbb{R}, & j_n = \iota_n, i. \end{array} \right. \tag{2.9}$$

Thus, under the expression of elements of group $\mathrm{SO}(n+1;\mathbf{j})$ into product (2.5), special group parameters $\mathbf{Q}_k$ belong to domains $D_{n-k}(\mathbf{j}^{(k)})$ described by (2.7)–(2.9).

Let us find the expression for invariant measure on group $\mathrm{SO}(n+1;\mathbf{j})$ in parametrization (2.5). Previously let us establish what is invariant measure on spheres. Invariant measure on sphere $S_n(\mathbf{j})$ in Cartesian coordinates is known: $dF_n(\mathbf{x},\mathbf{j}) = d^n x / x_0$, where $x_0 = \left(1 - \sum_{k=1}^n x_k^2 \prod_{m=1}^k j_m^2\right)^{1/2}$; $d^n x = dx_1, \ldots, dx_n$. The relation between Cartesian coordinates and parameters $\mathbf{Q}_0$ can be found from the equation $\mathbf{x} = s(\mathbf{Q}_0)\mathbf{F}_0$ and is as follows: $x_0 = \cos j_1 Q_0$, $x_r = \frac{Q_{0r}}{Q_0}\frac{1}{j_1}\sin Q_0 j_1$, $r = 1, 2, \ldots, n$. Jacobian of the transformation is as follows: $\left|\frac{\partial(\mathbf{x})}{\partial(\mathbf{Q}_0)}\right| = \left(\frac{\sin j_1 Q_0}{j_1 Q_0}\right)^{n-1} |\cos j_1 Q_0|$, and the invariant measure on sphere $S_n(\mathbf{j})$ under parametrization $\mathbf{Q}_0$ can be written as

$$dF_n(\mathbf{Q}_0, j) = \left(\frac{\sin j_1 Q_0}{j_1 Q_0}\right)^{n-1} d^n Q_0, \tag{2.10}$$

where $d^n Q_0 = dQ_{01} \ldots dQ_{0n}$.

Similarly we find the measure on sphere $S_{n-k}(\mathbf{j}^{(k)})$:

$$dF_{n-k}(\mathbf{Q}_k, \mathbf{j}^{(k)}) = \left(\frac{\sin j_{k+1} Q_k}{j_{k+1} Q_k}\right)^{n-k-1} d^{n-k} Q_k, \tag{2.11}$$

where $d^{n-k} Q_k = dQ_{k,k+1} \ldots dQ_{kn}$.

For some values of parameters $\mathbf{j}$, the squared quantity Q_k^2 can be positive, negative, or zero. Let $D^+ = \{\mathbf{Q}_k \in D \,|\, Q_k^2 > 0\}$, $D^- = \{\mathbf{Q}_k \in D \,|\, Q_k^2 < 0\}$, $D^0 = \{\mathbf{Q}_k \in D \,|\, Q_k^2 = 0\}$, then $D_{n-k}(\mathbf{j}^{(k)}) = D^+_{n-k}(\mathbf{j}^{(k)}) \cup D^-_{n-k}(\mathbf{j}^{(k)}) \cup D^0_{n-k}(\mathbf{j}^{(k)})$. In the domain $D^+_{n-k}(\mathbf{j}^{(k)})$, invariant measure $dF^+_{n-k}(\mathbf{Q}_k, \mathbf{j}^{(k)})$ is given by (2.11). In the domain $D^-_{n-k}(\mathbf{j}^{(k)})$, we have $Q_k = i\widetilde{Q}_k$, $\widetilde{Q}_k \in \mathbb{R}$, and, for the measure, we get

$$dF^-_{n-k}(\mathbf{Q}_k, \mathbf{j}^{(k)}) = \left(\frac{\mathrm{sh} j_{k+1}\widetilde{Q}_k}{j_{k+1}\widetilde{Q}_k}\right)^{n-k-1} d^{n-k} Q_k. \tag{2.12}$$

The set $D^0_{n-k}(\mathbf{j}^{(k)})$ has dimension $n-k-1$ and its measure is equal to zero.

Invariant measure on group $\mathrm{SO}(n+1;\mathbf{j})$ in parametrization (2.5) can be written as

$$dq(\mathbf{Q}_0, \mathbf{Q}_1, \ldots, \mathbf{Q}_{n-1}) = \prod_{k=0}^{n-1} \left(\frac{\sin j_{k+1} Q_k}{j_{k+1} Q_k}\right)^{n-k-1} d^{n-k} Q_k. \tag{2.13}$$

It can be shown that this measure is bilaterally invariant.

2.2. *Adjoint Algebra, Adjoint Group, Co-adjoint Representation*

Further we consider groups $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$ isomorphic to groups of motions of a Cayley—Klein space of zero curvature [25]. To this aim, we introduce the notations:

$Q_{0r} \equiv x_r$, $r = 1,2,\dots,n$, $\mathbf{Q}_0 \equiv \mathbf{x}$, $s(\mathbf{Q}_0) \equiv t(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ \mathbf{x} & \mathbf{1}_n \end{pmatrix}$, $x \in \mathbb{R}_n(\mathbf{j}')$. A set of transformations $\{t(\mathbf{x})\} = N(\mathbf{x})$ makes an Abelian group, and group $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$ is the semidirect product

$$\mathrm{SO}(n+1;\iota_1,\mathbf{j}') = N(\mathbf{x}) \circledS \mathrm{SO}(n;\mathbf{j}'). \tag{2.14}$$

Expansion (2.5) becomes $g(\mathbf{x},\mathbf{Q}_1,\dots,\mathbf{Q}_{n-1}) = t(\mathbf{x})\prod_{k=1}^{n-1} s(\mathbf{Q}_k)$; invariant measure on the group comes out of (2.13) for $j_1 = \iota_1$:

$$dg(\mathbf{x},\mathbf{Q}_1,\dots,\mathbf{Q}_{n-1}) = = d^n x \prod_{k=1}^{n-1}\left(\frac{\sin j_{k+1}Q_k}{j_{k+1}Q_k}\right)^{n-k-1} d^{n-k}Q_k. \tag{2.15}$$

Adjoint algebra $\mathrm{ad}\,L$ of a Lie algebra L is defined as $\mathrm{ad}\,X(Y) = [X,Y]$, $X,Y \in L$, or, in matrix form, $(\mathrm{ad}X)_{km} = \sum_\lambda c^k_{\lambda m} y_\lambda$, where $X = \sum_\lambda y_\lambda X_\lambda \in L$, $c^k_{\lambda m}$ are the structure constants of algebra L in a certain basis. If $Y' = \mathrm{ad}X(Y)$, $Y = \sum_\lambda y_\lambda X_\lambda$, $Y' = \sum_\lambda y'_\lambda X_\lambda$, then the relation $y'_\lambda = \sum_m (\mathrm{ad}X)_{\lambda m} y_m$ gives the action of the adjoint algebra on elements of L in matrix form.

Let us find matrices of adjoint algebra $\mathrm{ad}(\mathrm{so}(n+1;\iota_1,\mathbf{j}'))$ in basis $\mathfrak{X}_{km}$. To this aim, we draw up generators $\mathfrak{X}_{km}$, $k < m$, in order of increasing of the number $\lambda = m + k(n-1) - k(k-1)/2$, $\lambda = 1,2,\dots,n(n+1)/2$ for $k<m$, $k,m = 0,1,\dots,n$. From the commutation relations

$$[X_{\mu_1\nu_1}, X_{\mu_2\nu_2}] = = \begin{cases} \left(\prod\limits_{m=\mu_1+1}^{\nu_1} j_m^2\right) X_{\nu_1\nu_2}, & \mu_1 = \mu_2, \quad \nu_1 < \nu_2, \\ \left(\prod\limits_{m=\mu_2+1}^{\nu_2} j_m^2\right) X_{\mu_1\mu_2}, & \mu_1 < \mu_2, \quad \nu_1 = \nu_2, \\ -X_{\mu_1\nu_2}, & \mu_1 < \mu_2 = \nu_1 < \nu_2, \end{cases}$$

it can be established that, for $k \geq 1$, matrices of the adjoint algebra of elements $X(\mathbf{Q}_k)$, given by (2.3), has block diagonal structure $\mathrm{ad}X(\mathbf{Q}_k) = = \begin{pmatrix} A_n & 0 \\ 0 & A_{n(n-1)/2} \end{pmatrix}$. Let us consider only diagonal block A_n, i.e. the part acting on commutative subalgebra $N_0(\mathfrak{X}_{01},\dots,\mathfrak{X}_{0n})$ keeping the notation $\mathrm{ad}X(Q_k)$ for this block. We have $\mathrm{ad}X(\mathbf{x}) = 0$, $\mathrm{ad}X(\mathbf{Q}_k) = \widetilde{X}((\mathbf{Q}_k),\mathbf{j}^{(k)})$, $k = 1,2,\dots,n-1$, where matrix $\widetilde{X}((\mathbf{Q}_k),\mathbf{j}^{(k)})$ comes out of matrix $X(\mathbf{Q}_k)$ by deleting the zero first row and zero first column.

Adjoint group $\mathrm{Ad}(g_0)$ of Lie group $G = \exp L$, $g_0 = \exp X(\mathbf{F}_0)$, is connected with adjoint algebra $\mathrm{ad}X(\mathbf{F}_0)$ by relation $\mathrm{Ad}(g_0) = \exp(\mathrm{ad}X(\mathbf{F}_0))$, and its action on elements $g = \exp X(\mathbf{F})$ of group $G(\mathbf{j})$ is described by the formula

$$\mathrm{Ad}\,(g_0)g \stackrel{\mathrm{def}}{=} g_0 g g_0^{-1} = \exp X(\mathrm{Ad}\,(g_0)\mathbf{F}). \tag{2.16}$$

In the case of group $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$, we consider only that part of the matrix of the adjoint group which acts on subgroup $N(\mathbf{x})$. It is possible due to the block diagonal structure of matrices of the adjoint algebra. Under such a condition, $\mathrm{Ad}(t(\mathbf{x})) \equiv I$, and $\mathrm{Ad}(s(\mathbf{Q}_k)) = \widetilde{s}(\mathbf{Q}_k)$, $k = 1,2,\dots,n-1$, where matrices $\widetilde{s}(\mathbf{Q}_k)$ come out of matrices $s(\mathbf{Q}_k)$, which are described by (2.4), by deleting the first column and the first row, i.e. substituting I_k for I_{k-1}. To the expansion $g = t(\mathbf{x})\prod_{k=1}^{n-1} s(\mathbf{Q}_k)$ of an element of group $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$, there corresponds the expansion of a matrix of the adjoint group:

$$\mathrm{Ad}\,(g) = \mathrm{Ad}\left(t(\mathbf{x})\prod_{k=1}^{n-1} s(\mathbf{Q}_k)\right) = = \prod_{k=1}^{n-1}\mathrm{Ad}\,(s(\mathbf{Q}_k)) = \prod_{k=1}^{n-1}\widetilde{s}(\mathbf{Q}_k). \tag{2.17}$$

Unitary irreducible representations (characters) of the Abelian group of translations $N(\mathbf{x})$ are one-dimensional. Each character can be written as follows:

$$H(t(\mathbf{x})) = \exp(i\langle\mathbf{h},\mathbf{x}\rangle) = \exp\left(i\sum_{k=1}^n h_k x_k\right), \tag{2.18}$$

where $h_k \in \mathbb{R}$, and the group of characters $\widehat{N}(\mathbf{h})$ is isomorphic to $N(\mathbf{x})$. The action of subgroup $\mathrm{SO}(n;\mathbf{j}')$ on $\widehat{N}(\mathbf{h})$ is given by

$$kH(t(\mathbf{x})) \stackrel{\mathrm{def}}{=} H(k^{-1}t(\mathbf{x})k) = \exp(i,\langle\mathbf{h},\mathrm{Ad}\,(k^{-1})\mathbf{x}\rangle), \tag{2.19}$$

where $k \in \mathrm{SO}_n(\mathbf{j}')$.

If adjoint group $\mathrm{Ad}(k)$ acts in $N(\mathbf{x})$ according to the rule $\mathbf{x}' = \mathrm{Ad}(k)\,\mathbf{x}$, then co-adjoint group (co-adjoint representation) $\mathrm{Ad}^*(k)$ acts in the space of characters $\widehat{N}(\mathbf{h})$ according to the rule $\mathbf{h}' = \mathrm{Ad}^*(k)\,\mathbf{h}$, and this action can be found from the relation $\langle\mathrm{Ad}^*(k)\,\mathbf{h},\mathbf{x}\rangle = \langle\mathbf{h},\mathrm{Ad}(k^{-1})\mathbf{x}\rangle$. In a matrix realization, the last requirement gives $\mathrm{Ad}^*(k) = [\mathrm{Ad}(k^{-1})]^T$. Because $s^{-1}(\mathbf{Q}_k) = s(-\mathbf{Q}_k)$, we have $\mathrm{Ad}^*(s(\mathbf{Q}_k)) = [\widetilde{s}(-\mathbf{Q}_k)]^T$ or, in explicit form,

$$\mathrm{Ad}^*(s(\mathbf{Q}_k)) = \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & \cos j_{k+1}Q_k & -\frac{Q_{ks}}{Q_k}\frac{1}{j_{k+1}}\sin j_{k+1}Q_k \\ 0 & \left(\prod\limits_{m=k+2}^{r} j_m^2\right)\frac{Q_{kr}}{Q_k} j_{k+1}\sin j_{k+1}Q_k & \delta_{rs} - \left(\prod\limits_{m=k+2}^{r} j_m^2\right)\frac{Q_{kr}Q_{ks}}{Q_k^2}(1-\cos j_{k+1}Q_k) \end{pmatrix}. \quad (2.20)$$

To expansion (2.5) of elements of group $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$, there corresponds the expansion of the co-adjoint representation

$$\mathrm{Ad}^*(g) = \mathrm{Ad}^*\left(t(x)\prod_{k=1}^{n-1} s(\mathbf{Q}_k)\right) =$$

$$= \prod_{k=1}^{n-1} \mathrm{Ad}^*(s(\mathbf{Q}_k)). \quad (2.21)$$

2.3. Orbits in the Space of Characters and Their Stationary Subgroups

A set of all $\mathbf{h} = \mathrm{Ad}^*(k)\,\mathbf{h}_0$, when k runs over all transformations of $\mathrm{SO}(n;\mathbf{j}')$, is called orbit $O(\mathbf{h}_0)$ of character $\mathbf{h}_0$ in respect to subgroup $\mathrm{SO}(n;\mathbf{j}')$. It is known [31, 32] that, in the case of semidirect products, the space of characters $\widehat{N}(\mathbf{h})$ is splitted into disjoint orbits, i.e. surfaces in $\widehat{N}(\mathbf{h})$ are invariant in respect to the action of the co-adjoint representation of group $Ad^*(\mathrm{SO}(n;\mathbf{j}'))$. For group $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$, the equation for orbits can be obtained, substituting generators $\mathfrak{X}_{0k}$, $k = 1,2,\ldots,n$, in the Casimir operator of the second order $C_2(\iota_1,\mathbf{j}')$ for Cartesian coordinate functions h_k in space $\widehat{N}(\mathbf{h})$. Casimir operator $C_2(\mathbf{j})$ of group $\mathrm{SO}(n+1;\mathbf{j})$ is as follows:

$$C_2(\mathbf{j}) = \prod_{r=1}^{n}\left(\prod_{m=r+1}^{n} j_m^2\right)\mathfrak{X}_{0r}^2 +$$

$$+ \sum_{\alpha_2>\alpha_1=1}^{n}\left(\prod_{m=1}^{\alpha_1} j_m^2\right)\left(\prod_{l=1+\alpha_2}^{n} j_l^2\right)\mathfrak{X}_{\alpha_1\alpha_2}^2. \quad (2.22)$$

For $j_1 = \iota_1$, the second sum in (2.22) vanishes, and the Casimir operator of group $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$ is reduced to the first summand in (2.22). Substituting then $\mathfrak{X}_{0r}$ for h_r, we obtain the equation of orbits

$$\sum_{r=1}^{n-1}\left(\prod_{m=r+1}^{n} j_m^2\right)h_r^2 + h_n^2 = \mathrm{inv}. \quad (2.23)$$

The invariant on the right side of (2.23) can be positive, negative, or zero. Positive values $\mathrm{inv} = R^2 > 0$ can be taken for any values of parameters $\mathbf{j}'$, negative values $\mathrm{inv} = -\rho^2 < 0$ – for all values of $\mathbf{j}'$ except for $\mathbf{j}' = \mathbf{1}$ and $\mathbf{j}'_k = (j_2,\ldots,j_{n-k-1},\iota_{n-k},1,\ldots,1)$, $k = 0,1,\ldots,n-2$. Zero values $\mathrm{inv} = 0$ are possible for any values of parameters $\mathbf{j}'$. However, for $\mathbf{j}' = \mathbf{1}$, the orbit degenerates into point $\mathbf{o} = (0,\ldots,0)$; for $\mathbf{j}'_k$, $k = 0,1,\ldots,n-2$, the orbit is entirely situated in subspace $\mathbb{R}_{n-k-1} \subset \mathbb{R}_n$, and, consequently, its dimension is less than $n-1$. In all other cases, the dimension of the orbit is $n-1$.

Substituting, in Euclidean space $\mathbb{R}_n$, Cartesian coordinates h_r for $h_r\prod_{m=r+1}^{n} j_m$, $r = 1,2,\ldots,n-1$, we obtain space $\widetilde{\mathbb{R}}_n(\mathbf{j}')$ with scalar product $(\mathbf{h},\mathbf{h})_{\mathbf{j}'} = \sum_{r=1}^{n-1}\left(\prod_{m=r+1}^{n} j_m^2\right)h_r^2 + h_n^2$. Then the space of characters $\widehat{N}(\mathbf{h}) = \widetilde{\mathbb{R}}_n(\mathbf{j}')$, and the orbits are spheres in space $\widetilde{\mathbb{R}}_n(\mathbf{j}')$ of real $R > 0$, imaginary $i\rho$, $\rho > 0$, and zero radius. Depending on values of parameters $\mathbf{j}'$, spheres of nonzero radius are either connected or consist of two connected components. Let us consider the cases of real and imaginary radius separately.

Let $\mathrm{inv} = R^2$, $R > 0$. We denote generators of rotations in planes $\{h_k, h_r\}$ of space $\mathbb{R}_n$ by Y_{kr}. These generators are compact if, under the transition from $\mathbb{R}_n$ to $\widetilde{\mathbb{R}}_n(\mathbf{j}')$, they are multiplied by a real number, and noncompact if they are multiplied by an imaginary or dual number. Let us consider generators Y_{rn}, $r = 1,2,\ldots,n-1$, of rotations in planes $\{h_r, h_n\}$. Under the above-mentioned transition, they are multiplied by $\prod\limits_{m=r+1}^{n} j_m$. If all these generators are noncompact, then the orbit consists of two connected components, differing in the sign of h_k. But if at least one of these generators is compact, then the orbit is connected. Positive part of the coordinate axis intersects each connected orbit at the point $M^+ = (0,\ldots,0,h_n = R)$, and when the orbit consists of two components, the whole axis intersects one component at the point M^+, and the other – at the point $M^- = (0,\ldots,0,h_n = -R)$. Analysis of products $\prod\limits_{m=r+1}^{n} j_m$ enables us to state the following proposition.

PROPOSITION 2.1. *Orbits (2.23) of positive radius make one family, which is characterized by points M^+ for $\mathbf{j}' = (j_2,\ldots,j_k,j_{k+1} = i, j_{k+2} = 1,\ldots,j_{n-1} = 1, j_n = i)$, $k = 1,2,\ldots,n-2$. In other cases, orbits make two subfamilies, one of which is characterized by points M^+ and the other – by points M^-.*

Let inv $= -\rho^2$, $\rho > 0$. If $j_{m+1} = i$ and parameters $j_{m+2} = \ldots = j_n = 1$, then points $P_m^+ = (0, \ldots, 0, h_m = \rho, 0, \ldots, 0)$ are the intersection points of axis h_m with orbits of imaginary radius. Generators Y_{rm}, $r = 1, 2, \ldots, m-1$, are multiplied by $\prod_{l=r+1}^{m} j_l$. Similarly to the previous case, the following proposition is valid.

PROPOSITION 2.2. *Orbits (2.23) of imaginary radius make one family, characterized by points* P_m^+ *for* $\mathbf{j}' = (j_2, \ldots, j_{m-1}, j_m = 1, j_{m+1} = i, j_{m+2} = 1, \ldots, j_n = 1)$, $m = 2, 3, \ldots, n-1$ *and* $\mathbf{j}' = (j_2, \ldots, j_r, j_{r+1} = i, j_{r+2} = 1, \ldots, j_{m-1} = 1, j_m = i, j_{m+1} = i, j_{m+2} = 1, \ldots, j_n = 1)$, $r = 1, 2, \ldots, m-2$. *In other cases, orbits make two subfamilies, one of which is characterized by points* P_m^+, *and the other – by points* $P_m^- = (0, \ldots, 0, h_m = -\rho, 0, \ldots, 0)$, $\rho > 0$.

Equation (2.23) for inv $= 0$ gives one orbit and not their family. For $\mathbf{j}' = (j_2, \ldots, j_{n-k-1}, j_{n-k} = \iota_{n-k}, j'_{n-k+1}, \ldots, j'_s, j_{s+1} = i, j_{s+2} = 1, \ldots, j_n = 1)$, where $j'_{n-k+1}, \ldots, j'_s = 1, i, s = n-k, n-k+1, \ldots, n-1$, $k = 0, 1, \ldots, n-2$, the orbit of zero radius can be characterized by points $\widetilde{O} = (0, \ldots, 0, h_s = \pm 1, 0, \ldots, 0, h_n = \pm 1)$.

DEFINITION 2.1. *Group* $K_{\mathbf{h}_0} = \{s \in \mathrm{SO}(n; \mathbf{j}') |$ $\mathrm{Ad}^*(s)\mathbf{h}_0 = \mathbf{h}_0\}$ *is called stationary subgroup* $K_{\mathbf{h}_0}$ *of orbit* $O(\mathbf{h}_0)$.

Using (2.20), (2.21), we find for the family of orbits of positive radius that the stationary subgroup of points $M^\pm$ is a subgroup of group $\mathrm{SO}(n; \mathbf{j}')$ consisting of transformations leaving invariant axis x_n, i.e. $K_{M^\pm} = \mathrm{SO}(n-1; j_2, \ldots, j_{n-1})$; moreover, transformations $s(n) \in \mathrm{SO}(n-1; j_2, \ldots, j_{n-1})$ can be written as follows:

$$s(n) = \prod_{k=1}^{n-2} s(\mathbf{Q}_k(Q_{kn} = 0)). \tag{2.24}$$

For the family of orbits of imaginary radius, the stationary subgroup of points $P_m^\pm$ is a subgroup of group $\mathrm{SO}(n; \mathbf{j}')$ leaving invariant the axis x_m, i.e. $K_{P_m^\pm} = \mathrm{SO}_{n-1}^m(j_2, \ldots, j_n)$, where transformations $s(m) \in \mathrm{SO}^m(n-1; j_2, \ldots, j_n)$ are

$$s(m) = \prod_{k=1}^{m-1} s(\mathbf{Q}_k(Q_{km} = 0)) \prod_{r=m+1}^{n-1} s(\mathbf{Q}_r). \tag{2.25}$$

2.4. Irreducible Unitary Representations

Group $K_{\mathbf{h}_0}$ is a stabilizer of character $\mathbf{h}_0$ in respect to the action of a co-adjoint representation of group $\mathrm{SO}(n; \mathbf{j}')$. Let $T_{\mathbf{h}_0}$ be an irreducible unitary representation of group $K_{\mathbf{h}_0}$ in some Hilbert space $\mathcal{H}_T$. Then irreducible unitary representation $e^{i\mathbf{h}_0} \otimes T_{\mathbf{h}_0}$ of subgroup $N(\mathbf{x}) \otimes K_{\mathbf{h}_0}$ is realized in $\mathcal{H}_T$ by the relation

$$e^{i\mathbf{h}_0} \oplus T_{h_0}(t(\mathbf{x})s(\mathbf{h}_0)) = e^{i\langle \mathbf{h}_0, \mathbf{x}\rangle} T_{\mathbf{h}_0}(s(\mathbf{h}_0)), \tag{2.26}$$

where $t(\mathbf{x}) \in N(\mathbf{x})$; $s(\mathbf{h}_0) \in K_{\mathbf{h}_0}$.

If inv $= R^2$, then $\mathbf{h}_0 = M^\pm$, $\langle \mathbf{h}_0, \mathbf{x}\rangle = \pm R x_n$, and the irreducible unitary representation of subgroup $N(\mathbf{x}) \otimes \mathrm{SO}(n-1; j_2, \ldots, j_{n-1})$ is $e^{\pm i R x_n} T_{M^\pm}(s(n))$. If inv $= -\rho^2$, then $\mathbf{h}_0 = P_m^\pm$, $\langle \mathbf{h}_0, \mathbf{x}\rangle = \pm \rho x_m$ and the irreducible unitary representation of subgroup $N(\mathbf{x}) \otimes \mathrm{SO}^m(n-1; \mathbf{j}')$ is $e^{i\rho x_m} T_{P_m^\pm}(s(m))$.

Each irreducible unitary representation of group $\mathrm{SO}(n+1; \iota_1, \mathbf{j}') = N(\mathbf{x}) \otimes \mathrm{SO}(n; \mathbf{j}')$ is induced by the irreducible unitary representation $e^{i\mathbf{h}_0} \otimes T_{\mathbf{h}_0}$ of its subgroup $N(\mathbf{x}) \otimes K_{\mathbf{h}_0}$. Let us denote these representations of group $\mathrm{SO}(n+1; \iota_1, \mathbf{j}')$ by symbol $\omega_{\mathbf{h}_0, T}$. Operators $\omega_{\mathbf{h}_0, T}$ act in the Hilbert space of square-integrable functions on $\mathrm{SO}(n; \mathbf{j}')$

$$\begin{aligned} &\mathcal{H}_{\mathbf{h}_0, T} = \{f \mid f(k(\mathbf{Q})) \in L^2(\mathrm{SO}(n; \mathbf{j}'))\& \\ &\& T_{\mathbf{h}_0}(s(\mathbf{h}_0)) f(k(Q)s(\mathbf{h}_0)) = \\ &= f(k(\mathbf{Q})) \forall s(\mathbf{h}_0) \in K_{\mathbf{h}_0}, \quad k(\mathbf{Q}) \in \mathrm{SO}(n; \mathbf{j}')\} \end{aligned} \tag{2.27}$$

according to the rule

$$\begin{aligned} &\omega_{\mathbf{h}_0, T}(t(\mathbf{x})k(\mathbf{Q})) f(k(\mathbf{Q}^0)) = \\ &= e^{i\langle \mathbf{h}_0, \mathrm{Ad}(k^{-1}(\mathbf{Q}^0))\mathbf{x}\rangle} f(k^{-1}(\mathbf{Q})k(\mathbf{Q}^0)). \end{aligned} \tag{2.28}$$

Let us evaluate vector $\mathbf{x}^{(n-1)} = \mathrm{Ad}(k^{-1}(\mathbf{Q}))\mathbf{x}$. Group element $k(\mathbf{Q}) \in \mathrm{SO}(n; \mathbf{j}')$ in parametrization (2.5) can be written as a product $k(\mathbf{Q}) = \prod_{k=1}^{n-1} s(\mathbf{Q}_k)$. Then $k^{-1}(\mathbf{Q}) = \prod_{k=n-1}^{1} s(-\mathbf{Q}_k)$ and, correspondingly, $\mathrm{Ad}(k^{-1}(\mathbf{Q})) = \prod_{k=n-1}^{1} \mathrm{Ad}(s(-\mathbf{Q}_k))$. Let us introduce notations:

$$\mathbf{x}^{(k)} = \prod_{r=k}^{1} \mathrm{Ad}(s(-\mathbf{Q}_r))\mathbf{x}, \tag{2.29}$$

$$X_k = x_k \frac{1}{j_{k+1}} \sin j_{k+1} Q_k + \frac{(\mathbf{Q}_k, \mathbf{x})}{Q_k}(1 - \cos j_{k+1} Q_k), \tag{2.30}$$

$$\begin{aligned} A_{pk} = &-\frac{Q_{pk}}{Q_p} \frac{1}{j_{k+1}} \sin j_{k+1} Q_k + \\ &+ \frac{(\mathbf{Q}_p, \mathbf{Q}_k)}{Q_p Q_k}(1 - \cos j_{k+1} Q_k), \quad p < k, \end{aligned} \tag{2.31}$$

$$Y_k = x_k \cos j_{k+1} Q_k + \frac{(\mathbf{Q}_k, \mathbf{x})}{Q_k} j_{k+1} \sin j_{k+1} Q_k, \qquad (2.32)$$

$$B_{pk} = -\frac{Q_{pk}}{Q_p} \cos j_{k+1} Q_k + \\ + \frac{(\mathbf{Q}_p, \mathbf{Q}_k)}{Q_p Q_k} j_{k+1} \sin j_{k+1} Q_k, \quad p < k, \qquad (2.33)$$

$$(\mathbf{Q}_k, \mathbf{x}) = \sum_{r=k+1}^{n} \left(\prod_{l=k+2}^{r} j_l^2 \right) Q_{kr} x_r,$$
$$(\mathbf{Q}_p, \mathbf{Q}_k) = \sum_{r=k+1}^{n} \left(\prod_{l=k+2}^{r} j_l^2 \right) Q_{pr} Q_{kr}, \qquad (2.34)$$

where $\prod_{l=k}^{r} j_l^2 \equiv 1$ for $r < k$. Rather cumbersome calculations give:

$$x_r^{(k)} = x_r^{(r)} = Y_r - \sum_{p=1}^{r-1} D_p B_{pr}, \quad r = 1, 2, \ldots, k,$$
$$x_r^{(k)} = x_r - \sum_{p=1}^{k-1} D_p \frac{Q_{pr}}{Q_p}, \quad r = k+1, k+2, \ldots, n, \ (2.35)$$

where coefficients D_p satisfy the recurrent relation

$$D_p = X_p - \sum_{s=1}^{p-1} D_s A_{sp}, \quad D_1 = X_1. \qquad (2.36)$$

To find D_p, we proceed to the set of linear equations

$$D_p + \sum_{s=1}^{p-1} D_s A_{sp} \qquad (2.37)$$

or $A\mathbf{D} = \mathbf{X}$ in matrix form, where A is a triangle matrix of dimension p with units on the main diagonal; nonzero matrix elements are $(A)_{rs} = A_{sr}$, $s < r$, $s = 1, 2, \ldots, p-1$, $r = 2, 3, \ldots, p$. It is obvious that $\det A = 1$. By the Cramer rule, we find

$$D_p = \det \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 & X_1 \\ A_{12} & 1 & 0 & \ldots & 0 & X_2 \\ A_{13} & A_{23} & 1 & \ldots & 0 & X_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{1,p-1} & A_{2,p-1} & A_{3,p-1} & \ldots & 1 & X_{p-1} \\ A_{1,p} & A_{2,p} & A_{3,p} & \ldots & A_{p-1,p} & X_p \end{vmatrix}. \qquad (2.38)$$

Thus, coefficients D_p are determined up to the evaluation of a determinant of p-th order. Knowing D_p, one can find from (2.35) that

$$x_n^{(n-1)} = (\mathrm{Ad}(k^{-1}(\mathbf{Q}^0)\mathbf{x})_n = x_n - \sum_{p=1}^{n-1} D_p^0 \frac{Q_{pn}^0}{Q_p^0}, \qquad (2.39)$$

$$x_m^{(n-1)} = x_m^{(m)} = Y_m^0 - \sum_{p=1}^{m-1} D_p^0 B_{pm}^0. \qquad (2.40)$$

Hence, to the family of orbits of real radius $R > 0$, there corresponds a series of irreducible unitary representations of group $\mathrm{SO}(n+1; \iota_1, \mathbf{j}')$, which is realized by the operators

$$\omega_{R,T_M}^{\pm}(t(\mathbf{x})k(\mathbf{Q}))f(k(\mathbf{Q}^0)) = \qquad (2.41)$$
$$= \exp\left\{ \pm iR \left(x_n - \sum_{p=1}^{n-1} D_p^0 \frac{Q_{pn}^0}{Q_p^0} \right) \right\} f(k^{-1}(\mathbf{Q})k(\mathbf{Q}^0))$$

in Hilbert space $\mathcal{H}_{R,T_M}$.

The family of orbits of imaginary radius $i\rho$, $\rho > 0$, gives a series of irreducible unitary representations of group $\mathrm{SO}(n+1; \iota_1, \mathbf{j}')$, whose operators

$$\sigma_{\rho,T_{P_m}}^{\pm}(t(\mathbf{x})k(\mathbf{Q}))f(k(\mathbf{Q}^0)) = \qquad (2.42)$$
$$= \exp\left\{ \pm i\rho \left(Y_m^0 - \sum_{p=1}^{m-1} D_p^0 B_{pm}^0 \right) \right\} f(k^{-1}(\mathbf{Q})k(\mathbf{Q}^0))$$

act in Hilbert space $\mathcal{H}_{\rho,T_{P_m}}$. The signs plus and minus in (2.41), (2.42) are chosen in accordance with Propositions 2.1 and 2.2.

Let us consider the simplest case $j_i' = i_i'$, $j_2 = \iota_2, \ldots, j_n = \iota_n$. Group $\mathrm{SO}(n+1; \iota)$ is group $Z\,(n+1)$ of lower triangular matrices of the $n+1$-th order with units on the main diagonal and all elements above the main diagonal are equal to zero. For $n = 2$, group $\mathrm{SO}(3; \iota_1, \iota_2)$ is the Heisenberg group, for which we, using (2.30), (2.31), (2.38), (2.41), obtain irreducible representations

$$\omega_R^{\pm}(t(\mathbf{x})s(Q_{12}))f(s(Q_{12}^0)) =$$
$$= e^{\pm iR(x_2 - x_1 Q_{12}^0)} f(s(Q_{12}^0 - Q_{12})), \qquad (2.43)$$

coinciding with the known representation of this group [22]. For $n \geq 3$, we derive irreducible unitary representations of group $Z\,(n+1)$ from (2.41) [33]. Comparing, it is necessary to keep in mind that, in the handbook by D.P. Zhelobenko, A.I. Stern [33], operators of the representation act on functions depending on other variables that occurring in (2.27).

2.5. Some Transitions between Irreducible Representations of Cayley—Klein Groups

Using the method of inducing, we have constructed irreducible unitary representations of groups $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$. Can all these representation be obtained from irreducible representations of group $\mathrm{SO}(n+1;\iota_1,\mathbf{1}) = N(\mathbf{x}) \otimes \mathrm{SO}(n;\mathbf{1})$ of motions of the n-dimensional Euclidean space? (Representations of the Euclidean group are described by (2.27), (2.41), when all parameters take unit values: $j_k = 1$.) One can, first, derive only representations ω, corresponding to orbits of real radius. Second, parameter $R > 0$, which numbers irreducible representations of the Euclidean group, has to be substituted for $\pm R$ in accordance with Proposition 2.1, in order to obtain both series of irreducible representations, when they exist.

Under the transition from the Euclidean group to $\mathrm{SO}(n+1;\iota_1,\mathbf{j}')$, subgroup $\mathrm{SO}(n)$ turns into $\mathrm{SO}(n;\mathbf{j}')$. This changes the space of functions (2.27), namely: the range of definition for functions changes, therefore the requirement of square-integrability changes the forms of functions, the structural condition imposed on functions is modified as well, because the stabilizer of character $\mathrm{SO}(n-1)$ goes into $\mathrm{SO}(n-1;j_2,\dots,j_{n-1})$.

Operators of representation (2.41) change less radically. Here, elements g^* of the Euclidean group are replaced by elements $g \in \mathrm{SO}(n+1;\iota_1,\mathbf{j}')$, and the exponent $R^*(\mathbf{x}^*,\mathbf{Q}^*)$ under the transition ψ: $\mathrm{SO}(n+1;\iota_1,\mathbf{1}) \to \mathrm{SO}(n+1;\iota_1,\mathbf{j}')$, $\psi x_1^* = x_1$, $\psi x_k^* = x_k \prod_{l=2}^{k} j_l$, $k = 2,3,\dots,n$, $\psi Q_{pk}^* = Q_{pk} \prod_{l=p+1}^{k} j_l$, $p < k$, $p,k = 1,2,\dots,n$, acquires the factor $\prod_{l=2}^{n} j_l$. For this reason, it must be transformed according to the rule $R(\mathbf{x},\mathbf{Q}) = \left(\prod_{l=2}^{n} j_l^{-1}\right) R^*(\psi\mathbf{x}^*,\psi\mathbf{Q}^*)$. (All quantities referring to the Euclidean group are marked with asterisk.)

Because orbits of imaginary radius do not emerge under pure contractions, we, transforming irreducible representations of the Euclidean group, obtain irreducible representations of the contracted group. Moreover, if $j_r = \iota_r, j_{r+1} = \dots = j_n = 1$, and parameters $j_m = 1, \iota_m, i$, $m = 2,3,\dots,r-1$, then such groups do not have orbits of imaginary radius, and all their irreducible representations can be obtained, as described above, transforming irreducible representations of the Euclidean group.

Only transitions from the Euclidean group similar to analytical continuation and combination of contractions and analytical continuation, different from ones mentioned in the previous section, give irreducible representations of contracted groups, corresponding to orbits of real radius, and do not allow to obtain the representations corresponding to orbits of imaginary radius. Let us recall that we have not considered representations generated by orbits of zero radius. Let us notice as well that Kirillov's method of orbits [32] proves to be relevant for study of the behavior of irreducible representations under transitions between groups.

3. Cayley—Klein Classical Hermitian Categories

T h e C a y l e y — K l e i n u n i t a r y (CK**U**) c a t e g o r y. An object of the category CK**U** is a finite-dimensional complex linear Cayley—Klein space $V(\mathbf{j})$ endowed with a non-degenerate Hermitian form $\langle\cdot,\cdot\rangle$ (which, in general, is indefinite). Let $p_{V(\mathbf{j})}$ and $q_{V(\mathbf{j})}$ be the positive and negative indices of inertia of the form $\langle\cdot,\cdot\rangle$.

Let $V(\mathbf{j})$ and $W(\mathbf{j})$ be objects of the category CK**U**. As always, we endow the direct sum $V(\mathbf{j}) \oplus W(\mathbf{j})$ with the difference of the forms $\langle\cdot,\cdot\rangle_{V(\mathbf{j})}$ and $\langle\cdot,\cdot\rangle_{W(\mathbf{j})}$, that is, with the form

$$\langle(v,w),(v',w')\rangle_{V(\mathbf{j})\oplus W(\mathbf{j})} = \langle v,v'\rangle_{V(\mathbf{j})} - \langle w,w'\rangle_{W(\mathbf{j})}. \quad (3.1)$$

By a morphism in $V(\mathbf{j}) \to W(\mathbf{j})$, we mean a subspace of $P(\mathbf{j}) \subset V(\mathbf{j}) \oplus W(\mathbf{j})$ satisfying the conditions:

(a) the form $\langle\cdot,\cdot\rangle_{V(\mathbf{j})\oplus W(\mathbf{j})}$ on $P(\mathbf{j})$ is non-negative-definite, moreover, $P(\mathbf{j})$ is a maximal subspace satisfying this property (so that $\dim P(\mathbf{j}) = p_{V(\mathbf{j})} + q_{W(\mathbf{j})}$);

(b) the form $\langle\cdot,\cdot\rangle_{V(\mathbf{j})}$ is strictly positive-definite on $\ker P(\mathbf{j})$, while the form $\langle\cdot,\cdot\rangle_{W(\mathbf{j})}$ is strictly negative-definite on $\mathrm{Indef} P(\mathbf{j})$.

The morphisms are multiplied as linear relations.

Note that condition (a) can be interpreted as meaning that the linear relation $P(\mathbf{j})$ 'contracts' the form $\langle\cdot,\cdot\rangle_{V(\mathbf{j})}$; in fact, $(v,w) \in P(\mathbf{j})$ implies that $\langle v,v\rangle_{V(\mathbf{j})} \geq \langle w,w\rangle_{W(\mathbf{j})}$.

We recall the following definition from operator theory. Let $H(\mathbf{j})$ be a linear space endowed with a (possibly indefinite) Hermitian form $\mathcal{T}(\cdot,\cdot)$. An operator $T : H(\mathbf{j}) \to H(\mathbf{j})$ is called a *$\mathcal{T}$-contraction* if $(\mathcal{T}v,\mathcal{T}v) \geq \mathcal{T}(v,v)$ for all $v \in H(\mathbf{j})$.

Note also that the non-negativeness of the form $\langle\cdot,\cdot\rangle_{V(\mathbf{j})}$ on $\ker V(\mathbf{j})$ and the non-positiveness of the form $\langle\cdot,\cdot\rangle_{W(\mathbf{j})}$ on $\mathrm{Indef} W(\mathbf{j})$ follow from condition (a). Condition (b) is merely a slight strengthening of condition (a).

PROPOSITION 3.1. *Let* $P(\mathbf{j}) \in \mathrm{Mor}_{\mathbf{CK}}(V(\mathbf{j}), W(\mathbf{j}))$ *and* $Q(\mathbf{j}) \in \mathrm{Mor}_{\mathbf{CK}}(W(\mathbf{j}), Y(\mathbf{j}))$, *and suppose that* $Q(\mathbf{j})P(\mathbf{j}) \neq \mathrm{null}$. *Then*

$$\dim Q(\mathbf{j})P(\mathbf{j}) = \dim P(\mathbf{j}) + \dim Q(\mathbf{j}) - \dim(W(\mathbf{j})). \quad (3.2)$$

P r o o f. Let $H(\mathbf{j}) = V(\mathbf{j}) \oplus W(\mathbf{j}) \oplus Y(\mathbf{j})$, and let $Z(\mathbf{j})$ be the subspace of vectors of the form (v, w, w, y). We define the subspace $T(\mathbf{j}) = P(\mathbf{j}) \oplus Q(\mathbf{j})$ as the set of all vectors of the form (v, w, w', y), where $(v, w) \in Q$ and $(w', y) \in P$. From $\mathrm{im}P(\mathbf{j}) + D(Q(\mathbf{j})) = W(\mathbf{j})$, we have $T(\mathbf{j}) + Z(\mathbf{j}) = H(\mathbf{j})$. Thus, $T(\mathbf{j}) \cap Z(\mathbf{j})$ has dimension $\dim Z(\mathbf{j}) + \dim T(\mathbf{j}) - \dim H(\mathbf{j}) = \dim P(\mathbf{j}) + \dim Q(\mathbf{j}) - \dim W(\mathbf{j})$. Next we denote the projection of $H(\mathbf{j})$ onto $V(\mathbf{j}) \oplus Y(\mathbf{j})$ along $W(\mathbf{j}) \oplus W(\mathbf{j})$ by π. Then, as is easily seen, $\pi\,(T(\mathbf{j})) \cap Z(\mathbf{j}))$ is the product $Q(\mathbf{j})P(\mathbf{j})$. Furthermore, π is injective on $T(\mathbf{j}) \cap Z(\mathbf{j})$. In fact, $\pi(v, w, w'y) = 0$ implies that $v = 0$ and $y = 0$, while $(v, w, w', y) \in Z(\mathbf{j})$ implies that $w = w'$. Finally, $(v, w, w, 0) \in T$ implies that w lies in $\ker Q(\mathbf{j}) \cap \mathrm{Indef}P(\mathbf{j})$, which, in accordance to the equality $\ker Q(\mathbf{j}) \cap \mathrm{Indef}P(\mathbf{j}) = 0$, consists merely of the origin. This completes the proof of (3.2). □

PROPOSITION 3.2. *The category* CK**U** *is well defined, that is, a product of morphisms of the category* CK**U** *is again a morphism of the category* CK**U**.

P r o o f. Let $P(\mathbf{j}) \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}), W(\mathbf{j}))$ and $Q \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(W(\mathbf{j}), Y(\mathbf{j}))$. All we need to show is that $Q(\mathbf{j})P(\mathbf{j})$ is a maximal subspace on which the form $\langle\cdot,\cdot\rangle_{V(\mathbf{j})\oplus W(\mathbf{j})}$ is non-negative-definite. We claim that $Q(\mathbf{j})P(\mathbf{j}) \neq \mathrm{null}$. In fact, condition (b) implies that $\mathrm{Indef}P(\mathbf{j}) \cap \ker Q(\mathbf{j}) = 0$. Next we show that $\mathrm{im}P(\mathbf{j}) + D(Q(\mathbf{j})) = W(\mathbf{j})$. For this we consider a non-zero vector $w \in D(Q(\mathbf{j}))^{\perp}$ in $W(\mathbf{j})$. Note that $(w, 0)$ is orthogonal to $Q(\mathbf{j})$. If $\langle w, w\rangle_{W(\mathbf{j})} \geq 0$, then the vector $(w, 0)$ can be added to the subspace $Q(\mathbf{j})$, as a result of which the form $\langle\cdot,\cdot\rangle_{V(\mathbf{j})\oplus W(\mathbf{j})}$ would be non-negative-definite on the linear span of $Q(\mathbf{j})$ and the vector w. By maximality of $Q(\mathbf{j})$, we would then have $(w, 0) \in Q(\mathbf{j})$. But this is possible (that is, $(w, 0) \in Q(\mathbf{j}) \cap Q^{\perp}(\mathbf{j})$) only when $\langle w, w\rangle = 0$. We now have a contradiction with the strict positive-definiteness of the form $\langle\cdot,\cdot\rangle$ on $\ker Q(\mathbf{j})$.

Thus, the form $\langle\cdot,\cdot\rangle_{W(\mathbf{j})}$ is strictly negative-definite on $D(Q(\mathbf{j}))^{\perp}$. In exactly the same way, it can be shown that $\langle\cdot,\cdot\rangle_{W(\mathbf{j})}$ is strictly positive-definite on $\mathrm{im}(P(\mathbf{j}))^{\perp}$. Therefore $D(Q(\mathbf{j}))^{\perp} \cap \mathrm{im}(P(\mathbf{j}))^{\perp} = 0$, and hence $D(Q(\mathbf{j})) + \mathrm{im}(P(\mathbf{j})) = W(\mathbf{j})$.

Thus, $QP \neq \mathrm{null}$, and consequently (see Proposition 3.1) $\dim Q(\mathbf{j})P(\mathbf{j}) = \dim Q(\mathbf{j}) + \dim P(\mathbf{j}) - \dim W(\mathbf{j}) = p_{V(\mathbf{j})} + q_{Y(\mathbf{j})}$, as required. □

LEMMA 3.1. *The following conditions on a subspace* $P(\mathbf{j}) \subset V(\mathbf{j}) \oplus V(\mathbf{j})$ *are equivalent:*

(a) $P(\mathbf{j}) \in \mathrm{Aut}(V(\mathbf{j}))$;

(b) $P(\mathbf{j})$ *is the graph of an operator contained in the Cayley—Klein group* $\mathrm{CK}(V(\mathbf{j}))$.

P r o o f. Let $g \in \mathrm{CK}(V(\mathbf{j}))$. Then for any $v, v' \in V(\mathbf{j})$ we have

$$\{gv, gv'\} = \{v, v'\} \quad (3.3)$$

that is, the graph P_g of g is an isotropic subspace $V(\mathbf{j}) \oplus V(\mathbf{j})$. Furthermore, the dimension of $P_g(\mathbf{j})$ is half that of $V(\mathbf{j}) \oplus V(\mathbf{j})$, which means that it is a maximal isotropic subspace.

Conversely, suppose that $P(\mathbf{j}) \in \mathrm{Aut}(V(\mathbf{j}))$. Let $Q(\mathbf{j})$ be the inverse of $P(\mathbf{j})$. Then

$$\ker P(\mathbf{j}) \subset \ker Q(\mathbf{j})P(\mathbf{j}) = \ker E(\mathbf{j}) = 0,$$
$$D\,(P(\mathbf{j})) \supset D\,(Q(\mathbf{j})P(\mathbf{j})) = D(E(\mathbf{j})) = V(\mathbf{j}),$$
$$\mathrm{Indef}P(\mathbf{j}) \subset \mathrm{Indef}P(\mathbf{j})Q(\mathbf{j}) = \mathrm{Indef}E(\mathbf{j}) = 0.$$

Hence P is the graph of an operator; in particular, the dimension of P is half the dimension. Furthermore, (3.3) holds in view of the isotropy of $P(\mathbf{j})$. □

Note that the group $\mathrm{Aut}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}))$ is none other than the Cayley—Klein pseudo-unitary group $U(p_{V(\mathbf{j})}, q_{V(\mathbf{j})})$. In fact, an automorphism of the object $V(\mathbf{j})$ must be the graph of an invertible operator $A : V(\mathbf{j}) \to V(\mathbf{j})$ (see Lemma 3.1), where, by condition (a), we have $\langle Av, Av\rangle \leq \langle v, v\rangle$, $\langle A^{-1}v, A^{-1}v\rangle \leq \langle v, v\rangle$ for all v, and hence $\langle Av, Av\rangle = \langle v, v\rangle$.

On the other hand, the semigroup $\mathrm{End}(V(\mathbf{j}))$ contains many operators (formally speaking, graphs of operators). Namely, any operator $T : V(\mathbf{j}) \to V(\mathbf{j})$ satisfying the condition

$$\langle Tv, Tv\rangle \leq \langle v, v\rangle \quad (v \in V(\mathbf{j})) \quad (3.4)$$

is contained in $\mathrm{End}(V(\mathbf{j}))$.

P o t a p o v t r a n s f o r m a t i o n. Conditions (a) and (b) in themselves do not have great intuitive appeal, but there is a method of making them more understandable. First we introduce suitable coordinates on the set $\mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$.

For each $V(\mathbf{j}) \in \mathrm{Ob}(CK\mathbf{U})$, we fix a decomposition $V(\mathbf{j}) = V_+(\mathbf{j}) \oplus V_-(\mathbf{j})$ into a direct sum of orthogonal subspaces $V_+(\mathbf{j}), V_-(\mathbf{j})$ for which the form $\langle\cdot,\cdot\rangle_{V(\mathbf{j})}$ is positive-definite on $V_+(\mathbf{j})$ and negative-definite on $V_-(\mathbf{j})$.

THEOREM 3.1. *The space* $P(\mathbf{j}) \subset V(\mathbf{j}) \oplus W(\mathbf{j})$ *is a morphism of the category* CK**U** *if and only if* $P(\mathbf{j})$ *is the graph of an operator*

$$S(P(\mathbf{j})) = \begin{pmatrix} K & L \\ M & N \end{pmatrix} : W_-(\mathbf{j}) \oplus V_+(\mathbf{j}) \to$$
$$\to W_+(\mathbf{j}) \oplus V_-(\mathbf{j})$$

satisfying the conditions

(i) $||S|| \le 1$;
(ii) $||K|| < 1, \quad ||N|| < 1$.

Before giving the proof, we note that the space $W_-(\mathbf{j}) \oplus V_+(\mathbf{j})$ is Euclidean, when all parameters take unit values: $j_k = 1$, while, on the space $W_+(\mathbf{j}) \oplus V_-(\mathbf{j})$, the form $\langle \cdot, \cdot \rangle_{V(\mathbf{j}) \oplus W(\mathbf{j})}$ is negative-definite. Therefore the form $\langle \cdot, \cdot \rangle_{V(\mathbf{j}) \oplus W(\mathbf{j})}$ defines a scalar product in $V_+(\mathbf{j}) \oplus W_-(\mathbf{j})$. Here the norm of the operator is, as always, the Euclidean norm. In the proof which follows, the norm of a vector refers to the norm in one of the spaces $V_+(\mathbf{j}) \oplus W_-(\mathbf{j})$ or $V_-(\mathbf{j}) \oplus W_+(\mathbf{j})$.

P r o o f. Let $P(\mathbf{j}) \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}), W(\mathbf{j}))$. Since the form $\langle \cdot, \cdot \rangle_{V(\mathbf{j}) \oplus W(\mathbf{j})}$ is negative-definite on $W_+(\mathbf{j}) \oplus V_-(\mathbf{j})$ and non-negative-definite on $P(\mathbf{j})$, we have $P(\mathbf{j}) \cap (W_+(\mathbf{j}) \oplus V_-(\mathbf{j})) = 0$. Since $\dim P(\mathbf{j}) = \dim(V_+(\mathbf{j}) \oplus W_-(\mathbf{j}))$, the subspace $P(\mathbf{j})$ is the graph of the operator $S : W_-(\mathbf{j}) \oplus V_+(\mathbf{j}) \to W_+(\mathbf{j}) \oplus V_-(\mathbf{j})$. Next, for any $h \in W_-(\mathbf{j}) \oplus V_+(\mathbf{j})$, we have $0 \le \langle (h, Sh), (h, Sh) \rangle = ||h||^2_{W_-(\mathbf{j}) \oplus V_+(\mathbf{j})} - ||Sh||^2_{W_+(\mathbf{j}) \oplus V_-(\mathbf{j})}$, which means that $||S|| \le 1$.

Suppose that $||K|| = 1$, and let w_- be a vector at which this norm is attained:

$$||Kw_-|| = ||w_-||. \tag{3.5}$$

Then $S(w_-, 0) = (Kw_-, Mw_-)$, and hence

$$||S(w_-, 0)||^2 = ||Kw_-||^2 + ||Mw_-||^2.$$

Bearing in mind that $||S|| \le 1$, we see that $Mw_+ = 0$. Hence, we find that the vector $r = (w_-, Kw_-) \in W_-(\mathbf{j}) \oplus W_+(\mathbf{j})$ lies in $P(\mathbf{j})$, that is, $r \in \ker P(\mathbf{j})$. On the other hand, it follows from (3.6) that $\langle r, r \rangle = 0$, which contradicts condition (b) and the definition of a morphism. Thus, we have proved the 'only if' half of the theorem. It is easily seen that the above argument is reversible, so that the converse holds as well. □

We call the matrix $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ corresponding to the subspace P the *Potapov transform* of the morphism P, and we denote it by $S(P)$.

Let $P \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}), V'(\mathbf{j}))$ and $Q \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V'(\mathbf{j}), V''(\mathbf{j}))$. Let

$$S(P) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}, \qquad S(Q) = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Then $S(QP)$ is equal to

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} * \begin{pmatrix} K & L \\ M & N \end{pmatrix} := \tag{3.6}$$
$$:= \begin{pmatrix} K + LX(1 - NX)^{-1}M & L(1 - XN)^{-1}Y \\ Z(1 - NX)^{-1}M & W + Z(1 - NX)^{-1}Y \end{pmatrix}.$$

T h e s e m i g r o u p $\mathrm{End}V(\mathbf{j})$. We now discuss what the Potapov transformations look like on the semigroup $\mathrm{End}(V(\mathbf{j}))$. To avoid confusion, we shall suppose that an element $P(\mathbf{j}) \in \mathrm{End}(V(\mathbf{j}))$, rather than being a subspace of $V(\mathbf{j}) \oplus V(\mathbf{j})$, is a subspace of $V(\mathbf{j}) \oplus W(\mathbf{j})$, where $W(\mathbf{j})$ is a second copy of $V(\mathbf{j})$.

PROPOSITION 3.3. *Let* $Y(\mathbf{j}) = Y_+(\mathbf{j}) \oplus Y_-(\mathbf{j})$ *and* $Z(\mathbf{j}) = Z_+(\mathbf{j}) \oplus Z_-(\mathbf{j})$ *be linear spaces. Let* $z_\pm \in Z_\pm(\mathbf{j})$, $y_\pm \in Y_\pm(\mathbf{j})$, *and*

$$\begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix}. \tag{3.7}$$

Suppose that the matrix D *is invertible. Then*

$$\begin{pmatrix} z_+ \\ y_- \end{pmatrix} = \begin{pmatrix} BD^{-1} & A - BD^{-1}C \\ D^{-1} & -D^{-1}C \end{pmatrix} \begin{pmatrix} z_- \\ y_+ \end{pmatrix}. \tag{3.8}$$

P r o o f. In view of (3.8), we have

$$z_+ = Ay_+ + By_-, \quad z_- = Cy_+ + Dy_-.$$

Eliminating y_-, we obtain

$$y_- = D^{-1}z_- - D^{-1}Cy_+,$$
$$z_+ = BD^{-1}z_{-1} + \left(A - BD^{-1}C\right)y_+,$$

as required. □

PROPOSITION 3.4. *(a) Let* $P(\mathbf{j})$ *be the graph of an invertible operator* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *from* $V(\mathbf{j})$ *to* $W(\mathbf{j}) = V(\mathbf{j})$. *Then*

$$S(P) = \begin{pmatrix} BD^{-1} & A - BD^{-1}C \\ D^{-1} & -D^{-1}C \end{pmatrix}. \tag{3.9}$$

(b) Let $P \in \mathrm{End}(V(\mathbf{j}))$, *and let* $S(P) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$. *Suppose that the block* M *is invertible. Then* P *is the graph of an operator* $V(\mathbf{j}) \to W(\mathbf{j}) = V(\mathbf{j})$ *with the matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} L - KM^{-1}N & KM^{-1} \\ -M^{-1}N & M^{-1} \end{pmatrix}. \tag{3.10}$$

(c) $P \in \mathrm{Mor}(V(\mathbf{j}))$ is the graph of an operator element of $CK\mathbf{U}(p_{V(\mathbf{j})}, q_{V(\mathbf{j})})$ if and only $S(P)$ is unitary.

P r o o f. (a) If we apply the operator $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to the vector $(0, v_-)$, then we obtain the vector (Bv_-, Dv_-). By condition (3.4), we have $||Cv_-||^2 - ||Dv_-||^2 \leq -||v_-||^2$. Hence $||Bv_-|| \leq ||v_-||$ and, consequently, D is invertible. Finally (see Proposition 3.3), we simply express v_- and w_+ in terms of v_+ and w_- from the formula

$$\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} w_+ \\ w_- \end{pmatrix}.$$

The proofs of (b) and (c) are obvious. □

T h e C a y l e y — K l e i n s i m p l e c t i c c a t e g o r y CK**Sp**. Let $V_{\mathbb{R}}(\mathbf{j})$ be a real linear space endowed with a non-degenerate skew-symmetric bilinear form $\{\cdot,\cdot\}_{V(\mathbf{j})}$.

Consider the complexification $V(\mathbf{j})$ of the space $V_{\mathbb{R}}(\mathbf{j})$. Then the skew-symmetric bilinear form $L_{Y(\mathbf{j})}(\cdot,\cdot)$ is canonically defined on $V(\mathbf{j})$ as the complexification on the form $\{\cdot,\cdot\}_{V(\mathbf{j})}$. But the form $\{\cdot,\cdot\}_{V(\mathbf{j})}$ can be extended to $V(\mathbf{j})$ also by sesquilinearity. Hence the following sesquilinear form is canonically defined on $V(\mathbf{j})$:

$$M_{V(\mathbf{j})}(v, w) = \frac{1}{i} L(v, \bar{w}). \tag{3.11}$$

It is indefinite and its negative and positive indices of inertia are the same.

An object of the category CK**Sp** is a space $V(\mathbf{j})$ endowed with the above three structures (that is, the structure of the complexification of $V_{\mathbb{R}}(\mathbf{j})$, the skew-symmetric bilinear form $L_{Y(\mathbf{j})}$, and the Hermitian form $M_{V(\mathbf{j})}$. Thus, $V(\mathbf{j})$ is simultaneously an object of the category CK**U** and an object of the Cayley—Klein complex category **CK** (see [34–36]). By definition, a morphism $V(\mathbf{j}) \to W(\mathbf{j})$ is a linear subspace $P(\mathbf{j}) \subset V(\mathbf{j}) \oplus W(\mathbf{j})$ that is simultaneously a morphism of the category CK**U** and a morphism of the Cayley—Klein complex category **CK**.

Lemma 3.2. *The group* $\mathrm{Aut}(V(\mathbf{j}))$ *is the real Cayley—Klein symplectic group* $\mathrm{Sp}(V_{\mathbb{R}}(\mathbf{j}))$.

P r o o f. Let $P(\mathbf{j}) \in \mathrm{Aut}(V(\mathbf{j}))$. Then $P(\mathbf{j})$ is the graph of an invertible operator A. Since $P(\mathbf{j})$ is an automorphism of the Cayley—Klein complex category CK, it follows that $L(Av, Av) \simeq L(v, v)$ for all v. Since $P(\mathbf{j})$ is an automorphism of the category CK**U**, it follows that $M(Av, Av) = M(v, v)$.

We claim that this implies the equality $\overline{Aw} = A\bar{w}$. In fact, for any v, w, we have $L(v, \overline{Aw}) = iM(v, Aw) = iM(A^{-1}v, w) = L(A^{-1}v, \bar{w}) = L(v, A\bar{w})$, and hence $\overline{Aw} = A\bar{w}$.

Thus, A preserves the real subspace $V_{\mathbb{R}}(\mathbf{j}) \in V(\mathbf{j})$ and the form $\{\cdot,\cdot\}$ in this space. □

We decompose $V_{\mathbb{C}}(\mathbf{j})$ into a direct sum of two subspaces: $V_{\mathbb{C}}(\mathbf{j}) = V_+(\mathbf{j}) \oplus V_-(\mathbf{j})$, which are isotropic with respect to the skew-symmetric form L and orthogonal with respect to the Hermitian form M, where M is positive-definite on $V_+(\mathbf{j})$ and negative-definite on $V_-(\mathbf{j})$. We further require that the subspaces $V_+(\mathbf{j})$ and $V_-(\mathbf{j})$ be complex conjugates of each other.

Such a decomposition is indeed possible: let $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ be a canonical basis in $V_{\mathbb{R}}(\mathbf{j})$, that is, $\{p_\alpha, p_\beta\} = \{q_\alpha, q_\beta\} = 0$, $\{p_\alpha, q_\beta\} = \delta_{\alpha\beta}$. Then we can choose as $V_+(\mathbf{j})$ the subspace spanned by the vectors $e_\alpha = (1/\sqrt{2})(p_\alpha - iq_\alpha)$, and as $V_-(\mathbf{j})$ the subspace spanned by the vectors $f_\alpha = (1/\sqrt{2})(p_\alpha + iq_\alpha)$. It follows now that

$$\begin{aligned} L(e_\alpha, e_\beta) &= L(f_\alpha, f_\beta) = 0, \quad L(e_\alpha, f_\beta) = \delta_{\alpha\beta}, \\ M(e_\alpha, e_\beta) &= \delta_{\alpha\beta}, \\ M(f_\alpha, f_\beta) &= -\delta_{\alpha\beta}, \quad M(e_\alpha, f_\beta) = 0. \end{aligned} \tag{3.12}$$

It is customary to think of $V(\mathbf{j})$ as having a fixed basis $\{e_\alpha, f_\alpha\}$ satisfying (3.13) at the very outset.

Let $Q \in \mathrm{Aut}(V(\mathbf{j}))$. As we have seen, Q is the graph of an operator in $V(\mathbf{j})$ which commutes with the conjugation operation $v \mapsto \bar{v}$. Hence it follows that the matrix of the operator $Q : V_+(\mathbf{j}) \oplus V_-(\mathbf{j}) \to V_+(\mathbf{j}) \oplus V_-(\mathbf{j})$ must have the form

$$\begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix},$$

and, of course, it must also be symplectic, that is, it must preserve the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The Potapov transformation $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ of a morphism P is defined in CK**U** in the same way as in the category CK**Sp**.

Theorem 3.2. *A subspace* $P(\mathbf{j}) \subset V(\mathbf{j}) \oplus W(\mathbf{j})$ *is a morphism of the category* CK**Sp** *if and only if its*

Potapov transform $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ *satisfies the conditions*

$$\begin{array}{ll} \text{(i)} & \|S\| \le 1, \\ \text{(ii)} & \|K\| < 1, \quad \|N\| < 1, \\ \text{(iii)} & S = S^t. \end{array} \tag{3.13}$$

P r o o f. It follows from Proposition 3.4. □

As in the case of the category CK**U**, the condition $P \in \mathrm{Aut}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j}))$ is equivalent to the requirement that the matrix $S(P)$ be unitary.

T h e C a y l e y — K l e i n o r t h o g o n a l c a t e g o r y CK**SO***. As an object of this category, CK**SO*** is a finite-dimensional quaternionic space $V(\mathbf{j})$ endowed with a non-degenerate skew-Hermitian form $H(\cdot,\cdot)$.

Let $V^{\mathbb{C}}(\mathbf{j})$ be the same space $V(\mathbf{j})$ but regarded as a linear space over $\mathbb{C}(\mathbf{j})$. We represent our skew-Hermitian form H as a sum: $H(v,w) = IM(v,w) + JL(v,w)$, where $M(n,m) \in \mathbb{C}(\mathbf{j})$ and $L(v,w) \in \mathbb{C}(\mathbf{j})$. It is easy to see that M is a Hermitian form on $V_{\mathbb{C}}(\mathbf{j})$ and L is a symmetric bilinear form. Thus, $V_{\mathbb{C}}(\mathbf{j})$ is, at the same time, an object of the category CK**U** and an object of the Cayley—Klein complex category CK**GD** [36]. A morphism from $V(\mathbf{j})$ to $W(\mathbf{j})$ is a (complex) subspace of $V^{\mathbb{C}}(\mathbf{j}) \oplus W^{\mathbb{C}}(\mathbf{j})$ which is simultaneously a morphism of the category CK**U** and Cayley—Klein complex category CK**GD**.

The group of automorphisms of an n-dimensional object $V(\mathbf{j})$ (in the quaternionic sense) is isomorphic to the group $\mathrm{SO}^*(2n;\mathbf{j})$. As in the case of the category CK**Sp**, we decompose $V^{\mathbb{C}}(\mathbf{j})$ into a direct sum of subspaces $V^{\mathbb{C}}(\mathbf{j}) = V_+(\mathbf{j}) \oplus V_-(\mathbf{j})$, where $V_+(\mathbf{j})$ and $V_-(\mathbf{j})$ are maximal isotropic subspaces with respect to the form L which are orthogonal with respect to M, the form M being positive-definite on $V_+(\mathbf{j})$ and negative-definite on $V_-(\mathbf{j})$. It is also convenient to assume that the operation of multiplication by J in $V^{\mathbb{C}}(\mathbf{j})$ interchanges $V_+(\mathbf{j})$ and $V_-(\mathbf{j})$.

The Potapov transform

$$S(P) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$$

of a morphism P is defined in the same way as before. It satisfies the conditions

$$\begin{array}{ll} \text{(i)} & \|S\| \le 1; \\ \text{(ii)} & \|K\| < 1, \quad \|N\| < 1, \\ \text{(iii)} & K = -K^t, \quad N = -N^t, \quad M = L^t. \end{array}$$

I n v o l u t i o n i n H e r m i t i a n c a t e g o r i e s. Let $P \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}), W(\mathbf{j}))$. We denote by P^* the element of the space $\mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(W(\mathbf{j}), V(\mathbf{j}))$ such that the subspace $P^* \subset V(\mathbf{j}) \oplus W(\mathbf{j})$ is the orthocomplement with respect to P.

Proposition 3.5. *If*

$$S(P) = \begin{pmatrix} K & L \\ M & N \end{pmatrix},$$

then

$$S(P^*) = S^\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \\ = \begin{pmatrix} N^* & M^* \\ L^* & K^* \end{pmatrix}. \tag{3.14}$$

□

It is not difficult to verify that

$$(QP)^* = P^*Q^*,$$

that is, $P \mapsto P^*$ really is an involution in the category CK**U**.

Suppose further that $P \in \mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j}), W(\mathbf{j}))$. We denote by P^* the element of $\mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(W(\mathbf{j}), V(\mathbf{j}))$ which is the orthocomplement of P with respect to the Hermitian form $M_{V(\mathbf{j})\oplus W(\mathbf{j})}$ in $V(\mathbf{j}) \oplus W(\mathbf{j})$. It is not immediately obvious that the same orthocomplement is an isotropic subspace with respect to the skew-symmetric form $L_{V(\mathbf{j})\oplus W(\mathbf{j})}$. But it is easy to see from (3.15) that, if S is the Potapov transformation of the morphism $P \in \mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j}), W(\mathbf{j}))$ (that is, it satisfies conditions (3.14)), then S^σ is the Potapov transformation of some element of $\mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(W(\mathbf{j}), V(\mathbf{j}))$. The involution in the category CK**SO*** is defined in a similar fashion.

C a n o n i c a l f o r m s. Let CK be one of the three categories CK**Sp**, CK**U**, or CK**SO***. We are interested in two closely related questions;

1. How to describe the orbits of the group $\mathrm{Aut}(V(\mathbf{j})) \times \mathrm{Aut}(W(\mathbf{j}))$ on $\mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$.

2. How to describe the orbits of the group $\mathrm{Aut}(V(\mathbf{j}))$ on the set of self-adjoint elements of $\mathrm{End}(V(\mathbf{j}))$.

We begin with the following elementary problem of linear algebra. Let $V(\mathbf{j})$ be a space endowed with a non-degenerate (generally indefinite) Hermitian form $M_{V(\mathbf{j})}$ and let A be a self-adjoint operator in $V(\mathbf{j})$ with respect to the form $M_{V(\mathbf{j})}$ (that is, $M_{V(\mathbf{j})}(Av_1, v_2) = M_{V(\mathbf{j})}(v_1, Av_2)$). We then say that the self-adjoint operators A_1 and A_2 are equivalent if there exists an operator B preserving the form $M_{V(\mathbf{j})}$ such that

$B_1B^{-1} = A_2$. We wish to describe the equivalence classes.

This problem is equivalent to the following. Let $V(\mathbf{j})$ and A be as above. We say that the triple $\{V(\mathbf{j}), M_{V(\mathbf{j})}, A\}$ is decomposable if there exist A-invariant subspaces $V_1(\mathbf{j}), V_2(\mathbf{j}) \subset V(\mathbf{j})$ such that $V(\mathbf{j}) = V_1(\mathbf{j}) \oplus V_2(\mathbf{j})$, $V_2(\mathbf{j}) = V_1^{\perp}(\mathbf{j})$. Our problem is to describe all the indecomposable triples.

It is easy to show that all the indecomposable triples are accounted for by the following two types:

(a) The type $\mathrm{I}^{\pm}(\lambda, n)$, where $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$. A basis in $V(\mathbf{j})$ consists of elements $e_1, \ldots, e_n$, where (for $k \neq n$)

$$M(e_i, e_j) = \pm\delta_{i+j,n+1}, \quad e_k = \lambda s_k | s_{k+1}, \quad A_{e_n} = \lambda_{e_n}.$$

(b) The type $\mathrm{II}^{\pm}(\lambda, n)$, where $\mathrm{Im}\lambda > 0$, $n \in \mathbb{N}$. A basis in $V(\mathbf{j})$ consists of elements $e_1, \ldots, e_n$, $f_1, \ldots, f_n$, where (for $k \neq n$)

$$M(e_i, e_j) = M(f_i, f)j) = 0, \quad M(e_i, f_j) = \delta_{i+j,n+1},$$
$$Ae_k = \lambda e_k + e_{k+1}, \quad Ae_n = \lambda e_n,$$
$$Af_k = \bar{\lambda} f_k + f_{k+1}, \quad Af_n = \bar{\lambda} f_n.$$

From these types, it is not difficult to select those in which the operator A is $M_{V(\mathbf{j})}$-contraction. It turns out that the indecomposable $M_{V(\mathbf{j})}$-contractions are the operators in a one-dimensional space and one of the two-dimensional Jordan boxes $\mathrm{I}^{\pm}(\mp 1, 2)$.

The case of an arbitrary self-adjoint element $P \in \mathrm{End}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}))$ reduces immediately to the case of the operator

$$V(\mathbf{j}) = \mathrm{Ker}P \oplus \mathrm{Indef}P \oplus (\mathrm{Ker}P \oplus \mathrm{Indef}P)^{\perp}.$$

This solves the problem of describing the orbits of $\mathrm{Aut}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}))$ on the set of self-adjoint elements of $\mathrm{End}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}))$.

We now turn to the problem of the orbits of the group $\mathrm{Aut}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j})) \times \mathrm{Aut}_{\mathrm{CK}\mathbf{U}}(W(\mathbf{j}))$ on $\mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}), W(\mathbf{j}))$. It reduces to the previous problem *via* transformation $P \mapsto P^*P$. Again we say that an element $P \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$ is decomposable if there exist $V_1(\mathbf{j}), V_2, (\mathbf{j}) W_1(\mathbf{j}), W_2(\mathbf{j}) \in \mathrm{Ob}(CK\mathbf{U})$ and $P_1 \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V_1(\mathbf{j}), W_1(\mathbf{j}))$, $P_2 \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V_2(\mathbf{j}), W_2(\mathbf{j}))$, such that $V(\mathbf{j}) = V_1(\mathbf{j}) \oplus V_2(\mathbf{j})$, $W(\mathbf{j}) = W_1(\mathbf{j}) \oplus W_2(\mathbf{j})$ (orthogonal direct sums) and $P = P_1 \oplus P_2$.

D i s s i p a t i v e   o p e r a t o r s. Let $V(\mathbf{j})$ be an object of the category CK**U**. An operator A in $V(\mathbf{j})$ is said to be $M_{V(\mathbf{j})}$-dissipative if $M_{V(\mathbf{j})}(Ax, x) \leq 0$ for all $x \in V(\mathbf{j})$.

Finally, we point out that the $M_{V(\mathbf{j})}$-dissipative operators form a convex cone $\tilde{C}$ which is invariant with respect to the action $g : A \mapsto g^{-1}Ag$ of the group $\mathrm{Aut}(V(\mathbf{j}))$. Furthermore $\tilde{C}$ decomposes into a direct sum of the Cayley—Klein algebra CKA of $\mathrm{Aut}(V(\mathbf{j}))$ and the invariant convex cone C of all $M_{V(\mathbf{j})}$-self-adjoint $M_{V(\mathbf{j})}$-dissipative operators.

4. The Krein—Shmul'yan Functor

Let **CK** be an arbitrary Cayley—Klein category. From each object Z° of **CK**, we construct a functor (I, i) from **CK** to the category of sets as follows. Let $V(\mathbf{j}) \in \mathrm{Ob}(\mathbf{CK})$. Then $I(V(\mathbf{j})) = \mathrm{Mor}(Z^{\circ}, V(\mathbf{j}))$, while each morphism $P : V_1(\mathbf{j}) \to V_2(\mathbf{j})$ is associated with the map $i(P) : Q \mapsto PQ$ from $I(V_1(\mathbf{j})) = \mathrm{Mor}(Z^{\circ}, V_1(\mathbf{j}))$ to $I(V_2(\mathbf{j})) = \mathrm{Mor}(Z^{\circ}, V_2(\mathbf{j}))$.

In many particular cases, this generally trivial construction proves to be of great interest. In this Section, we shall apply the construction to the Cayley—Klein categories CK**U** and CK**Sp**, where, in all cases, we shall take Z° to be the zero-dimensional object of the appropriate category (which we shall denote by 0).

T h e   K r e i n — S h m u l ' y a n   f u n c t o r   o f   t h e   c a t e g o r y   CK**U**. Let $V(\mathbf{j}) \in \mathrm{Ob}(\mathrm{CK}\mathbf{U})$.

PROPOSITION 4.1. *The following conditions on a subspace $H(\mathbf{j}) \subset V(\mathbf{j})$ are equivalent:*

(a) $H(\mathbf{j}) \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(0, V(\mathbf{j}))$;

(b) $H(\mathbf{j})$ is the graph of an operator $S = S(H) : V_-(\mathbf{j}) \to V_+(\mathbf{j})$, with $||S|| < 1$.

P r o o f. This proposition is a particular case of Theorem 3.1. □

The operator $S = S(H) : V_-(\mathbf{j}) \to V_+(\mathbf{j})$ is called the *Krein angular operator* of the subspace $H(\mathbf{j})$.

Thus the set $Z(V(\mathbf{j})) = \mathrm{Mor}(0, V(\mathbf{j}))$ can be identified with the set of operators $V_-(\mathbf{j}) \to V_+(\mathbf{j})$ with norm < 1 (or, in other words, with the set $Z_{p,q}$ of $q \times p$ matrices, where $p = p_{V(\mathbf{j})}$ and $q = q_{V(\mathbf{j})}$ are the indices of inertia of the Hermitian form in $V(\mathbf{j})$). We call these sets *operator* (or *matrix) balls.*

We now describe the action of the category CK**U** on the matrix balls.

PROPOSITION 4.2. *Let $P \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$, and let $H \in Z(V(\mathbf{j})) = \mathrm{Mor}(0, V(\mathbf{j}))$. Let*

$$S(P) = \begin{pmatrix} K & L \\ M & N \end{pmatrix}, \qquad S(H) = X.$$

Then the transformation

$\tau(P) : H \mapsto PH$

from $Z(V(\mathbf{j}))$ *to* $Z(W(\mathbf{j}))$ *is given by the formula*

$$\tau(P)X = K + LX(1 - NX)^{-1}M. \qquad (4.1)$$

P r o o f. This is a particular case of formula (3.7). □

Since τ is a functor, it follows that

$$\tau(Q)\tau(P) = \tau(QP) \qquad (4.2)$$

for any $P \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$ and $Q \in \mathrm{Mor}(W(\mathbf{j}), Y(\mathbf{j}))$.

Let $p = p_{V(\mathbf{j})}$, $q = q_{V(\mathbf{j})}$ be the indices of inertia of the Hermitian form $M_{V(\mathbf{j})}$ on $V(\mathbf{j})$. Then, clearly, the group $\mathrm{Aut}(V(\mathbf{j})) = \mathbf{U}(p, q, \mathbf{j})$ acts on $Z(v(\mathbf{j})) \simeq Z_{p,q}$ by invertible transformations.

Proposition 4.3. *The space* $Z(V(\mathbf{j}))$ *is a homogeneous* $U(p, q, \mathbf{j})$*-space, the stabilizer of the point* $0 \in Z(V(\mathbf{j}))$ *being* $U(p, \mathbf{j}) \times U(q, \mathbf{j})$.

The proof appeals to Witt's theorem, which we recall without proof.

Theorem 4.1. (Witt; see [37]) *Let* $V(\mathbf{j})$ *be a finite-dimensional vector space over* $\mathbb{K}$ *(where* $\mathbb{K}$ *is either* $\mathbb{R}, \mathbb{C}$*, or* $\mathbb{H}$*) endowed with a non-degenerate Hermitian or skew-Hermitian form* B*. Let* $G(\mathbf{j})$ *denote the Cayley—Klein group of operators in* $V(\mathbf{j})$ *preserving* B*. Let* $W_1(\mathbf{j})$ *and* $W_2(\mathbf{j})$ *be subspaces of* $V(\mathbf{j})$ *of equal dimension. If* $A : W_1(\mathbf{j}) \to W_2(\mathbf{j})$ *is an invertible operator such that*

$$B(Ax, Ay) = B(x, y) \qquad \text{for all}\, x, y \in W_1(\mathbf{j}),$$

then there exists an operator $Q \in G$ *such that* $Qx = Ax$ *for all* $x \in W_1(\mathbf{j})$.

P r o o f o f P r o p o s i t i o n 4.3. The transitivity of the action of $U(p, q, \mathbf{j})$ follows from Witt's theorem. Next, suppose that 0 stays under the action of $g \in U(p, q, \mathbf{j})$. Then $gV_-(\mathbf{j}) = V_-(\mathbf{j})$ and $V_+(\mathbf{j}) = V^\perp(\mathbf{j})$, and hence $gV_+(\mathbf{j}) = V_+(\mathbf{j})$. □

It is helpful to have a convenient formula for this action, which we now give in a slightly more general setting.

Proposition 4.4. *Let* $P \in \mathrm{Mor}_{\mathrm{CK}\mathbf{U}}(V(\mathbf{j}), W(\mathbf{j}))$ *be the graph of the linear operator*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : V_+(\mathbf{j}) \oplus V_-(\mathbf{j}) \to W_+(\mathbf{j}) \oplus W_-(\mathbf{j}).$$

Then

$$\tau(P)X = (AX + B)(CX + D)^{-1}. \qquad (4.3)$$

P r o o f. Consider the subspace $Q(\mathbf{j})$ with the angular operator X. It consists of vectors of the form $(Xv_-, v_-) \in V_+(\mathbf{j}) \oplus V_-(\mathbf{j}) = V(\mathbf{j})$. Then the subspace $QP \subset W(\mathbf{j})$ consists of vectors of the form

$$\tau = ((AX + B)v_-, (CX + D)v_-)\,. \qquad (4.4)$$

Denoting $(CX + D)v_-$ by w_-, we see that QP consists of vectors of the form $((AX + B)(CX + D)^{-1}w_-, w_-)$, that is, the angular operator of the subspace QP is given by (4.3).

However, we still need to prove that the operator $CX + D$ is invertible. In fact, suppose that $(CX + D)v_- = 0$ for some v_-. Then $\langle r, r\rangle_{V(\mathbf{j})\oplus W(\mathbf{j})} = ||(AX + B)v||^2$, which is non-negative. This contradicts the fact that $\tau \in QP$ (in fact, the form $\langle\cdot,\cdot\rangle_M$ is strictly negative-definite on QP). □

Formula (4.3) is certainly more familiar to us than (4.1). Maps of the form (4.3) are called *fractional-linear*, while maps of the form (4.1) are called *generalized fractional-linear*.

T h e K r e i n — S h m u l ' y a n f u n c t o r f o r t h e c a t e g o r y CK**Sp**. Here we associate the set $L(V(\mathbf{j})) = \mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(0, V(\mathbf{j}))$ with each $V(\mathbf{j}) \in \mathrm{Ob}(\mathrm{CK}\mathbf{Sp})$. Its elements are subspaces of $V(\mathbf{j})$ which are maximal isotropic with respect to the form $L_{V(\mathbf{j})}$ and the form $M_{V(\mathbf{j})}$ is negative-definite on them. The Potapov transformation identifies $L(V(\mathbf{j}))$ with the set $Z(V(\mathbf{j}))$ of symmetric operators $V_-(\mathbf{j}) \to V_+(\mathbf{j})$ with norm < 1 (we call the set $Z(V(\mathbf{j}))$ the matrix ball). Correspondingly to each morphism $V(\mathbf{j}) \to W(\mathbf{j})$, we have a generalized fractional-linear map $L(V(\mathbf{j})) \to L(W(\mathbf{j}))$ defined by (4.1).

We see that the group $\mathrm{Aut}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j})) = \mathrm{Sp}(V_{\mathbb{R}}(\mathbf{j}))$ consists of matrices of the form

$$\begin{pmatrix} \Phi & \Psi \\ \bar\Psi & \bar\Phi \end{pmatrix} : V_+(\mathbf{j}) \oplus V_-(\mathbf{j}) \to V_+(\mathbf{j}) \oplus V_-(\mathbf{j})$$

preserving the skew-symmetric bilinear form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (see Lemma 3.2). This group acts on $Z(V(\mathbf{j}))$ by transformations of the form

$$\begin{pmatrix} \Phi & \Psi \\ \bar\Psi & \bar\Phi \end{pmatrix} : Z \to (\Phi Z + \Psi)(\bar\Psi Z + \bar\Phi)^{-1} \text{ (see(4.2))}.$$

Lemma 4.1. *The group* $\mathrm{Aut}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j})) \simeq \mathrm{Sp}(V_{\mathbb{R}}(\mathbf{j}))$ *acts transitively on* $Z(V_{\mathbb{C}}(\mathbf{j}))$.

P r o o f. The fractional-linear transformation of the region $Z(V(\mathbf{j}))$ corresponding to the operator

$$\begin{pmatrix} (1-Z\bar{Z})^{-1/2} & Z(1-Z\bar{Z})^{-1/2} \\ \bar{Z}(1-Z\bar{Z})^{-1/2} & (1-\bar{Z}Z) \end{pmatrix} \in \mathrm{Aut}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j}))$$

takes the point 0 into Z. □

It is easily seen that the stabilizer of 0 consists of all matrices of the form $\begin{pmatrix} \Phi & 0 \\ 0 & \bar{\Phi} \end{pmatrix}$. Since such matrices are symplectic, the matrix Φ must be unitary.

Thus, if $\dim V(\mathbf{j}) = 2n$, then $Z(V(\mathbf{j}))$ is the homogeneous space $\mathrm{Sp}(2n, \mathbb{R}(\mathbf{j}))\mathrm{U}(n, \mathbf{j})$.

5. Weil Representations of the Cayley—Klein Symplectic Category CKSp

O p e r a t o r s w i t h G a u s s i a n k e r n e l s. Let $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be a symmetric block matrix of dimension $(m+n)\times(m+n)$. We denote by $B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ the operator F acting in $\mathbb{C}^{n+m}$ with the kernel

$$\exp\left\{\frac{1}{2}(z\bar{u})\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\begin{pmatrix} z^t \\ \bar{u}^t \end{pmatrix}\right\}, \tag{5.1}$$

where $z = (z_1 \cdots z_m)$ and $\bar{u} = (\bar{u}_1 \cdots \bar{u}_n)$ are row matrices.

R e p r o d u c i n g p r o p e r t y. Let $u \in H$. We set

$$\phi_u(z) = \exp(\langle z, u\rangle). \tag{5.2}$$

LEMMA 5.1. *The function* $\phi_u(z)$ *lies in* $F(H)$. *Furthermore,*

$$\langle \phi_u, \phi_v\rangle = \exp(v, u). \tag{5.3}$$

P r o o f. Expanding (5.2) in a series, we obtain

$$\phi_u(z) = \sum \frac{(z_j\bar{u}_j)^k}{k!} = \sum \frac{\bar{u}_1^{i_1}\cdots\bar{u}_n^{i_n}}{\sqrt{i_1!}\cdots\sqrt{i_n!}}\frac{z_1^{i_1}\cdots z_n^{i_n}}{\sqrt{i_1!}\cdots\sqrt{i_n!}} = \sum \frac{\bar{u}_1^{i_1}\cdots\bar{u}_n^{i_n}}{\sqrt{i_1!}\cdots\sqrt{i_n!}}e_{i_1\cdots i_n}(z). \tag{5.4}$$

The assertion of the lemma is now obvious. □

THEOREM 5.1. *For any* $f \subset F(H)$ *and any* $u \in H$, *we have*

$$\langle f, \phi_u\rangle = f(u). \tag{5.5}$$

P r o o f. Let

$$f(z) = \sum a_{j_1\cdots j_n}\frac{z_1^{j_1}\cdots z_n^{j_n}}{\sqrt{j_1!}\cdots\sqrt{j_n!}} = \sum a_{j_1\cdots j_n}e_{j_1\cdots j_n}(z).$$

Then, by (5.4), we have

$$\langle f, \phi_u\rangle = \sum a_{j_1\cdots j_n}\frac{u_1^{j_1}\cdots u_n^{j_n}}{\sqrt{j_1!}\cdots\sqrt{j_n!}},$$

as required. □

Note that (5.5) means, in particular, that the linear functional $\delta_u(f) = f(u)$ is continuous on $F(H)$.

K e r n e l s o f o p e r a t o r s. Let A be a bounded operator from $F(H_1)$ to $F(H_2)$. Consider the function

$$K_A(u, \bar{v}) = \langle A\phi_v, \phi_u\rangle, \tag{5.6}$$

where ϕ_u is defined by (5.2). We shall show presently that $K_A(u,\bar{v})$ is none other than the kernel of A in the usual sense of the word.

Note that when we write the expression $K(u,\bar{v})$, it can be understood in two ways. We can suppose that $K_A(u,\bar{v})$ is a function of two variables $u \in H$, $v \in H$, where the bar over v signifies that $K_A(u,\bar{v})$ is antiholomorphic in v. Alternatively, the variable $\bar{v}$ can be regarded as the element of the dual space H' corresponding to v under the identification of H and H'. We introduce the functions

$$\kappa_A^{\bar{u}}(z) = K(z,\bar{u}), \qquad \kappa_A^{z}(u) = \overline{K(z,\bar{u})}. \tag{5.7}$$

Then, as is easily seen,

$$\kappa_A^{\bar{u}}(z) = K(z,\bar{u}) = \langle A\phi_u, \phi_z\rangle = (A\phi_u)(z), \tag{5.8}$$

$$\kappa_A^{z}(u) = \overline{K(z,\bar{u})} = \langle \phi_z, A\phi_u\rangle = \langle A^*\phi_z, \phi_u\rangle = (A^*\phi_z)(u). \tag{5.9}$$

In particular,

$$\kappa_A^{\bar{u}} \in F(H_1) \quad \text{and} \quad \kappa_A^{z} \in F(H_2). \tag{5.10}$$

W e a k c o n v e r g e n c e o f o p e r a t o r s.

PROPOSITION 5.1. *Let* A_n *and* A *be operators from* $F(H)$ *to* $F(K)$, *and let* $K_n(z,\bar{u})$ *and* $K(z,\bar{u})$ *be their respective kernels.*

(a) *If the sequence* $K_n(z,\bar{u})$ *is pointwise convergent to* $K(z,\bar{u})$ *and the quantities* $||A_n||$ *are bounded above by some constant* C, *then* A *is bounded and* $A_n \to A$ *weakly.*

(b) *If $A_n \to A$ weakly, then the sequence $K_n(z,\bar{u})$ is pointwise convergent to $K(z,\bar{u})$.*

P r o o f. Assertion (b) follows immediately from the explicit formula (5.6) for the kernel of an operator. Assertion (a) follows from the criterion for weak convergence and the fact that the linear combinations of the functions $\phi_n(z)$ are dense in the Fock space.

PROPOSITION 5.2. *(cf. Theorem 3.2) Suppose that the operator* $B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ *is bounded. Then:*

(a) $||S|| \le 1$;
(b) $||K|| < 1, \quad ||M|| < 1$.

P r o o f. Let A be a bounded operator, and let K_A be its kernel. Then, by (5.6) and (5.3), we have

$$K_A(z,\bar{u}) = \langle A\phi_u, \phi_z\rangle \le ||A|| ||\phi_u|| ||\phi_z|| =$$
$$= ||A|| \exp\left(\frac{1}{2}||u||^2\right) \exp\left(\frac{1}{2}||v||^2\right).$$

Hence assertion (a) follows. Now $B[S]\cdot 1 = \exp\{\frac{1}{2}zA^*z^*\} \in F(\mathbb{C}^n)$ and $B[S]^*1 = \exp\{\frac{1}{2}zMz^t\} \in F(\mathbb{C}^m)$, and hence (b) follows; see (5.13). □

T h e W e i l r e p r e s e n t a t i o n (We, we) o f t h e c a t e g o r y **CKSp**. We associate the boson Fock space $F(V_-(\mathbf{j}))$ with each object $V(\mathbf{j})$ of the category **CKSp**. We assume that $\mathrm{We}(X) = F(V(\mathbf{j}))$. Let $P \in \mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j}), W(\mathbf{j}))$, and let $S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be its Potapov transform. We then assume that

$$\mathrm{we}(P) = B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}. \tag{5.11}$$

Note that the simplest thing is to assume that the canonical bases are fixed in all the spaces $V(\mathbf{j}) \in \mathrm{Ob}(\mathrm{CK}\mathbf{Sp})$. Then (5.1) has a perfectly clear meaning. Sometimes this point of view is not all that convenient, so that we need to describe the invariant meaning of expression (5.1). In this case, one must not pay attention to the transposition sign over the vector (that is, over z, $\bar{u}$). Then $z^t \in W_-(\mathbf{j})$, $Kz^t \in W_+(\mathbf{j})$, and $zKz^t := L(z, Kz^t)$. The quantities $\bar{u}M\bar{u}^t$ and $zL\bar{u}^t$ are defined similarly.

THEOREM 5.2. *The conditions $||S|| \le 1$, $||K|| < 1$, and $||M|| < 1$ suffice for the boundlessness of the operator $B[S]$.*

THEOREM 5.3. (a) (We, we) *is a projective representation of the category* CK**Sp**. *More precisely, let $P \in \mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j}), W(\mathbf{j}))$ and $Q \in \mathrm{Mor}_{\mathrm{CK}\mathbf{Sp}}(W(\mathbf{j}), Y(\mathbf{j}))$. Let $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ and $\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ be their Potapov transforms. Then*

$$\mathrm{we}(Q)\mathrm{we}(P) = \det\left[(1-KC)^{-1/2}\right]\mathrm{we}(QP). \tag{5.12}$$

(b) (We, we) *is a $*$-representation of the category* CK**Sp**.

(c) *Let $P \in \mathrm{Aut}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j})) \simeq \mathrm{Sp}(V_{\mathbb{R}}(\mathbf{j}))$. Then, to within multiplication by a constant, the operator* $\mathrm{we}(P)$ *is unitary.*

Note that the matrix $(1-KC)^{-1/2}$ is well defined. In fact, $||KC|| < 1$, and therefore

$$(1-KC)^\alpha = 1 + \alpha KC + \frac{\alpha(\alpha-1)}{2!}(KC)^2 + \cdots.$$

The rest of this section is taken up with the proofs of Theorems 5.2 and 5.3. Among the auxiliary results, the upper estimate of the norm of $B[S]$ is of interest in its own right (see (5.30)), as is the fixed-point method used in obtaining this estimate.

G a u s s i n t e g r a l s.

PROPOSITION 5.3. *Let $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be a symmetric $((n+n)\times(n+n))$-matrix, let $||S|| < 1$, and let α, β be row matrices of length n. Then*

$$\int\int \exp\left\{\frac{1}{2}(z\bar{z})\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\begin{pmatrix} z^t \\ \bar{z}^t \end{pmatrix} +\right.$$
$$\left. +\alpha z^t + \beta\bar{z}^t\right\} d\mu(z) = \det\left[\begin{pmatrix} E-L & -K \\ -M & E-L^t \end{pmatrix}^{-1/2}\right] \times$$
$$\times \exp\left\{\frac{1}{2}(\alpha\beta)\begin{pmatrix} -M & E-L \\ E-L^t & -N \end{pmatrix}^{-1}\begin{pmatrix} \alpha^t \\ \beta^t \end{pmatrix}\right\}, \tag{5.13}$$

where the integral is absolutely convergent.

P r o o f. Everything reduces to the following standard formula from elementary analysis

$$\int_{\mathbb{R}^k} \exp\left(-\frac{1}{2}xAx^t + bx^t\right) dx =$$
$$= \frac{(\pi/2)^{n/2}}{\sqrt{\det A}} \exp\left(-\frac{1}{2}bA^{-1}b^t\right), \tag{5.14}$$

where A is a symmetric matrix whose real part is positive-definite. On making the substitution $z = x+iy$, $\bar{z} = x - iy$, we obtain the following formula for the left-hand side of (5.13):

$$\left(\frac{2}{\pi}\right)^{n/2} \int\int \exp\left\{\frac{1}{2}(xy)\begin{pmatrix} E & E \\ iE & -iE \end{pmatrix} \times\right.$$
$$\times \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\begin{pmatrix} E & iE \\ E & -iE \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} +$$
$$\left. + (\alpha\beta)\begin{pmatrix} E & iE \\ E & -iE \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - x^2 - y^2\right\} dxdy,$$

that is, in the notation of (5.13),

$$A = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} - \frac{1}{2}\begin{pmatrix} E & E \\ iE & -iE \end{pmatrix} \times$$
$$\times \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\begin{pmatrix} E & iE \\ E & -iE \end{pmatrix},$$

$$b = (\alpha\beta)\begin{pmatrix} E & iE \\ E & -iE \end{pmatrix}. \tag{5.15}$$

These matrices A and b must then be substituted into the right-hand side of formula (5.14). The only thing that is not completely trivial is the calculation of det$A$. To do this, in equality (5.15), we multiply by $\theta_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} E & E \\ iE & -iE \end{pmatrix}$ on the left and by $\theta_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} E & iE \\ E & -iE \end{pmatrix}$ on the right. Then

$$\det A = \det(\theta_1\theta_2)\det\left(\theta_1^{-1}\theta_2^{-1} - \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\right) =$$
$$= \det\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \times$$
$$\times \det\left(\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} - \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\right) =$$
$$= (-1)^n \det\begin{pmatrix} -K & 1-L \\ 1-L^t & -M \end{pmatrix}$$

and, interchanging the columns, we obtain the required expression. □

We are interested mainly in the case where $L = 0$. In this case, expression (5.13) can be written in a more convenient form as follows:

$$\det\begin{pmatrix} 1 & -M \\ -N & 1 \end{pmatrix} = \det(1 - MN), \tag{5.16}$$

$$\begin{pmatrix} -M & 1 \\ 1 & -N \end{pmatrix}^{-1} =$$
$$= \begin{pmatrix} N(1-MN)^{-1} & (1-NM)^{-1} \\ (1-MN)^{-1} & M(1-NM)^{-1} \end{pmatrix} \tag{5.17}$$

((5.17) can be verified by direct multiplication).

We therefore have the following formula

$$\int\int \exp\left\{\frac{1}{2}zMz^t + \frac{1}{2}\bar{z}N\bar{z}^t + \alpha\bar{z}^t + \beta\bar{z}^t\right\} d\mu(z) =$$
$$= \det(1-MN)^{-1/2} \times$$
$$\times \exp\left\{\frac{1}{2}(\alpha\beta)\begin{pmatrix} N(1-MN)^{-1} & (1-NM)^{-1} \\ (1-MN)^{-1} & M(1-NM)^{-1} \end{pmatrix} \times\right.$$
$$\left.\times \begin{pmatrix} \alpha^t \\ \beta^t \end{pmatrix}\right\}. \tag{5.18}$$

T h e v e c t o r s $b[T|l]$. Let T be an operator from $V_-(\mathbf{j})$ to $V_+(\mathbf{j})$ with norm less than 1, and let $l^t \in V_+(\mathbf{j})$. We define a function $b[T|l] \in F(V_-(\mathbf{j}))$ by the formula

$$b[T|l](z) = \exp\left\{\frac{1}{2}zTz^t + lz^t\right\}.$$

Note that if $V(\mathbf{j})$ is a coordinate space, then $zl^t = lz^t$ is simply the quantity $\sum z_j l_j$. In invariant language, zl^t is the result of pairing $l \in V_+(\mathbf{j})$ and $z \in V_-(\mathbf{j})$.

LEMMA 5.2. *We have* $b[T|l^t] \in F(V_-(\mathbf{j}))$.

P r o o f. The result follows from Proposition 5.3. □

PROPOSITION 5.4. *Let*

$$B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : F(V_-(\mathbf{j})) \to F(W_-(\mathbf{j}))$$

be an operator satisfying the conditions of Theorem 5.2. Then

$$B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} b[T|l^t] = \det\left[(1-MT)^{-1/2}\right] b\left[K+\right.$$
$$\left. + LT(1-MT)^{-1}L^t | L(1-TM)^{-1}l^t\right]. \tag{5.19}$$

Furthermore, the integral

$$\int\int K(z,\bar{u}) \exp\left\{\frac{1}{2}uT^t u^t + lu^t\right\} d\mu(u) =$$
$$= \langle k^{\bar{u}}, b[T|l^t](u)\rangle \tag{5.20}$$

is absolutely convergent, and the vector in $[\cdot|\cdot]$ *on the right-hand side of (5.19) in fact lies in* $F(W_-(\mathbf{j}))$, *that is,*

$$||K + LT(1-MT)^{-1}L^t|| < 1. \tag{5.21}$$

P r o o f. Expression (5.20) is equal to

$$\exp\left\{\frac{1}{2}zKz^t\right\}\times$$
$$\times\int\int\exp\left\{\frac{1}{2}(u\bar{u})\begin{pmatrix}M & 0\\ 0 & T\end{pmatrix}\begin{pmatrix}u^t\\ \bar{u}^t\end{pmatrix}+uL^tz^t\right\}d\mu(u)$$

and an application of (5.18) now proves (5.19).

To prove (5.21) 'by a frontal attack' is fairly hard, but fortunately we have already proved it, for (5.21) means that the generalized fractional-linear map takes a matrix ball to itself (see Proposition 4.2). □

Note that we still do not know whether the operators $B[S]$ are bounded.

M u l t i p l i c a t i o n o f t h e o p e r a t o r $B[S]$. We denote by $F_0(H)\subset F(H)$ the set of finite linear combinations of vectors of the form $b[T|l^t]$.

We now consider an arbitrary operator

$$B\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}: F(V_-(\mathbf{j}))\to F(W_-(\mathbf{j})).$$

In view of Proposition 5.2, the operator $B\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}$ takes $F_0(V_-(\mathbf{j}))$ to $F_0(W_-(\mathbf{j}))$. Thus, although we do not know whether our operator is bounded, at least we know that it is well defined as an operator from $F_0(V_-(\mathbf{j}))$ to $F_0(W_-(\mathbf{j}))$. It is also clear from what has already been said that the product of such an operator is also well defined.

THEOREM 5.4. *Suppose that we possess the operators*

$$B\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}: F(V_-(\mathbf{j}))\ \to\ F(W_-(\mathbf{j})),$$
$$B\begin{pmatrix}P & Q\\ Q^t & R\end{pmatrix}: F(W_-(\mathbf{j}))\ \to\ F(Y_-(\mathbf{j})).$$

Then, for any f and for any $F_0(V_-(\mathbf{j}))$, we have

$$B\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}B\begin{pmatrix}P & Q\\ Q^t & R\end{pmatrix}f=\det(1-MP)^{-1/2}B\begin{pmatrix}K+LP(1-MP)^{-1}L^t & L(1-PM)^{-1}Q\\ Q^t(1-MP)^{-1}L^t & R+Q^t(1-MP)^{-1}MQ\end{pmatrix}f.\quad(5.22)$$

As a corollary, we obtain assertion (a) of Theorem 5.3, that is, $P\mapsto \mathrm{we}(P)$ really is a representation of the Cayley—Klein symplectic category CK**Sp** (see (3.7)).

We now give a formal discussion. Calculating the formal convolution of the kernels in accordance with (5.5), we see that product (5.22) must have a kernel of the form

$$\int\int\exp\left\{\frac{1}{2}(z\bar{u})\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\begin{pmatrix}z^t\\ \bar{u}^t\end{pmatrix}+\right.$$
$$\left.+\frac{1}{2}(u\bar{w})\begin{pmatrix}P & Q\\ Q^t & R\end{pmatrix}\begin{pmatrix}u^t\\ \bar{w}^t\end{pmatrix}\right\}d\mu(u)=$$
$$=\exp\left\{\frac{1}{2}zKz^t+\frac{1}{2}\bar{w}R\bar{w}^t\right\}\times$$
$$\times\int\int\exp\left\{\frac{1}{2}(u\bar{u})\begin{pmatrix}P & 0\\ 0 & M\end{pmatrix}\begin{pmatrix}u^t\\ \bar{u}^t\end{pmatrix}+\right.$$
$$\left.+(u\bar{u})\begin{pmatrix}Q & \bar{w}^t\\ L^t & z^t\end{pmatrix}\right\}d\mu(u)=$$
$$=\det\left[\begin{pmatrix}1 & -M\\ -P & 1\end{pmatrix}^{-1/2}\right]\exp\left\{\frac{1}{2}zKz^t+\frac{1}{2}\bar{w}R\bar{w}\right\}\times$$
$$\times\exp\left\{(zLQ^t\bar{w})\begin{pmatrix}-P & 1\\ 1 & -M\end{pmatrix}^{-1}\begin{pmatrix}Q^t & z^t\\ Q & \bar{w}^t\end{pmatrix}\right\},\quad(5.23)$$

the last line being obtained from Proposition 5.3. After this, an application of formulae (5.16)–(5.17) gives the required expression.

Unfortunately, we do not know whether our operators are bounded. We therefore need a further argument justify this.

P r o o f o f T h e o r e m 5.4. We have

$$B\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\left(B\begin{pmatrix}P & Q\\ Q^t & R\end{pmatrix}b[T|l](z)\right)=$$
$$=\int\int\left[\int\int\exp\left\{\frac{1}{2}(z\bar{u})\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\begin{pmatrix}z^t\\ \bar{u}^t\end{pmatrix}+\right.\right.$$
$$+\frac{1}{2}(u\bar{w})\begin{pmatrix}P & Q\\ Q^t & R\end{pmatrix}\begin{pmatrix}u^t\\ \bar{w}^t\end{pmatrix}+$$
$$\left.\left.+\ \frac{1}{2}wTw^t+lw^t\right\}d\mu(w)\right]d\mu(u).\quad(5.24)$$

To see this, we need to justify the possibility of changing the order of integration from $d\mu(w)d\mu(u)$ to $d\mu(u)d\mu(w)$ in the 'iterated' integral. This can be done certainly if (5.24) converges absolutely as a double integral. The latter, at any rate, is true in the case where

$$\left\| \begin{matrix} P & Q \\ Q^t & R \end{matrix} \right\| < 1 \quad \text{and} \quad \left\| \begin{matrix} K & L \\ L^t & M \end{matrix} \right\| < 1.$$

Considerations of continuity now complete the proof. In fact, if we denote the operator with kernel (5.23) by U, then $Ub[T|l^t]$ has the form $\det(A)^{1/2}b[Z|m^t]$, while (5.24) has the form $\det(B)^{1/2}b[Y|n^t]$, where A, Z, B, Y, m, n depend rationally on K, L, M, P, Q, R, T, l. Hence it follows from continuity considerations that $A = B$, $Z = Y$, and $m = n$. □

E q u i v a r i a n t m a p f r o m t h e m a t r i x b a l l $Z(V(\mathbf{j}))$ t o t h e p r o j e c t i v e s p a c e $\mathbb{P}F(V_-(\mathbf{j}))$. We associate the vector $b[T]$ defined as $b[T|0]$ with each point T of the domain $Z(V(\mathbf{j})) \simeq \mathrm{Mor}_{\mathrm{CKSp}}(0, V(\mathbf{j}))$.

Let $Q \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j}))$, and let $S(Q) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be its Potapov transform. Then, by (5.19), we have

$$\mathrm{we}(Q)b[T] = \det(1 - MT)^{-1/2}b]\tau(Q)T], \tag{5.25}$$

where $\tau(\cdot)$ denotes the Krein—Shmul'yan functor. We point out also that (5.25) is a particular case of formula (5.22); that is to say, if $R \in \mathrm{Mor}(0, V(\mathbf{j}))$ is the subspace with angular operator T, then $b[T]$ is none other than $\mathrm{we}(R)\cdot 1$, where 1 denotes the vacuum vector in $F(\mathbb{C}^0) \simeq \mathbb{C}^1$.

P r o o f o f a s s e r t i o n (b) o f T h e o r e m 5.3. Let $P \in \mathrm{Mor}_{\mathrm{CKSp}}(V(\mathbf{j}), W(\mathbf{j}))$ and let $S \simeq \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be its Potapov transform. Then the Potapov transform of the morphism P^* is given by the matrix

$$S^\sigma = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}^\sigma := \begin{pmatrix} \overline{M} & \overline{L}^t \\ \overline{L} & \overline{K} \end{pmatrix}. \tag{5.26}$$

Calculating formally the kernel of the conjugate operator to $B[S] = \mathrm{we}(P)$, we obtain precisely the kernel of the operator $B[S^\sigma] = \mathrm{we}(P^*)$. Unfortunately, however, we still do not know whether we are dealing here with bounded operators, and until we do, the formula $B[S]^* = B[S^\sigma]$ is hazardous. We must observe rigour here.

LEMMA 5.3. *For any* $f_1 \in F_0(V_-(\mathbf{j}))$, $f_2 \in F_0(W_-(\mathbf{j}))$, *we have*

$$\langle B[S]f_1, f_2\rangle = \langle f_1, B[S^\sigma]f_2\rangle.$$

P r o o f. If $||S|| < 1$, then, on both sides of the equation, we have the same absolutely convergent integral which differs only in the order of integration. In the case $||S|| = 1$, we use continuity considerations, as in the case of Theorem 5.4. Then, in particular, if $S = S^\sigma$, then $B[S]$ is symmetric on F_0. □

LEMMA 5.4. *Let τ be a projective $*$-representation of the category* **CK** *. Then, for any $P \in \mathrm{Aut}^*_{\mathbf{CK}}(V(\mathbf{j}))$, the operator $\tau(P)$ is unitary to within multiplication by a constant.*

P r o o f. In fact,

$$\tau(P)^*\tau(P) = \lambda\tau(P^*P) = \lambda\tau(1_{V(\mathbf{j})}) = \lambda E,$$

$$\tau(P)\tau(P^*) = \mu\tau(PP^*) = \mu\tau(1_{V(\mathbf{j})}) = \mu E$$

for some $\lambda, \mu \in \mathbb{C}(\mathbf{j})$. We then observe that $\tau(P)^*\tau(P)$ and $\tau(P)\tau(P^*)$ are positive self-adjoint operators, from which it follows that $\lambda > 0$ and $\mu > 0$. But

$$\lambda\tau(P) = \tau(P)\left(\tau(P)^*\tau(P)\right) = \tau(P)\tau(P^*)\tau(P) = \mu\tau(P).$$

Consequently $\lambda = \mu$ and therefore the operator $\lambda^{-1/2}\tau(P)$ is unitary. □

P r o o f o f a s s e r t i o n (c) o f T h e o r e m 5.3. This is an immediate consequence of Lemma 5.4.

R e d u c t i o n t o t h e s y m m e t r i c c a s e. Our next aim is to prove that the operators $\mathrm{we}(P)$ are bounded, which is the most difficult part of the theorem. Let us recall that the usual method of calculating the norm of an operator is to use the formula $||A||^2 = ||A^*A||$.

LEMMA 5.5. *An operator $B[S]$ is bounded if and only if the operator $B[S^\sigma]B[S]$ is bounded.*

P r o o f. Suppose that $B[S^\sigma]B[S]$ is bounded. In particular, the function $q(f) = \langle B[S^\sigma]B[S]f, f\rangle$ is bounded on the unit ball D of the space F_0. By Lemma 5.3, we have $q(f) = \langle B[S^\sigma]f, B[S]f\rangle = ||B[S]f||^2$. The boundedness of the latter expression on D now implies that $B[S]$ is bounded. □

The operator $B[S^\sigma]B[S]$ is symmetric and has the form $B[\cdot]$. Thus, it suffices for us to verify the boundedness of the operator $B[S]$ in the case where it is symmetric.

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F i x e d p o i n t p r i n c i p l e. We denote the vector $b[T|0]$ by $b[T]$. Recall that

$$B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} b[T] = \det(1-MT)^{-1/2} b[K + +LT(1-MT)^{-1}L^t]. \qquad (5.27)$$

PROPOSITION 5.5. *The following assertions are equivalent:*

(a) $b[T]$ is an eigenvector of the operator $B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$.

(b) T is a fixed point of the transformation

$$\tau\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : T \mapsto K + LT(1-MT)^{-1}L^t. \qquad (5.28)$$

□

We emphasize once more that map (5.27) takes the matrix ball into itself (see Proposition 4.2).

Suppose now that the operator

$$B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : F(V_-(\mathbf{j})) \to F(V_-(\mathbf{j}))$$

is symmetric, that is, $K = \overline{M}$ and $L = L^*$.

THEOREM 5.5. *Suppose that the operator* $B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ *is symmetric, and let T be a fixed point of transformation (5.27) with $||T|| < 1$. Then,*

$$\left|\left|B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\right|\right| = \det[(1-MT)^{-1/2}], \qquad (5.29)$$

where the norm is attained at the vector $b[T]$.

P r o o f. The case where $T = 0$ is particularly simple. It is then clear from (5.27) that $K = 0$, and hence $M = 0$ as well, that is, our operator has the form $B\begin{pmatrix} 0 & L \\ L^t & 0 \end{pmatrix}$. So,

$$B\begin{pmatrix} 0 & L \\ L^t & 0 \end{pmatrix} f(z) = f(Lz).$$

As before, let $F^{(k)}(V_-(\mathbf{j})) \subset F(V_-(\mathbf{j}))$ be the space of homogeneous polynomials of degree k. The bounded operators $f(z) \mapsto f(Lz)$ on $F^{(k)}(V(\mathbf{j}))$ are the kth symmetric powers S^kL of L. Taking into account that $||L|| \le 1$, we find that $||S^kL|| = ||L||^k \le 1$. We see that

$$\left|\left|B\begin{pmatrix} 0 & L \\ L^t & 0 \end{pmatrix}\right|\right| = 1,$$

where the norm is attained at the vector $f(s) = 1$.

Next let T be arbitrary. We have seen (see Lemma 4.1) that the Cayley—Klein symplectic group $\mathrm{Sp}(V_{\mathbb{R}}(\mathbf{j}))$ acts transitively on the matrix ball $Z(V(\mathbf{j}))$. Let $g \in \mathrm{Sp}(V_{\mathbb{R}}(\mathbf{j}))$, and suppose that the corresponding transformation $\tau(g)$ of the matrix ball takes 0 to T. Then the operator $\mathrm{we}(g)$ takes the vector $b[0]$ to $b[T]$. Therefore the vector $b[0]$ is an eigenvector for the operator

$$A = \mathrm{we}(g)^{-1} B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \mathrm{we}(g),$$

which, as before, is symmetric (we recall that $\mathrm{we}(g)$ is unitary to within multiplication by a constant; see Lemma 5.4) and has the form $\lambda B[H]$, where λ is a scalar. But we have just proved that the norm of the operator A is attained at the vector $b[0]$. Therefore the norm of the operator

$$B\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} = \mathrm{we}(g) A \mathrm{we}(g)^{-1}$$

is attained at the eigenvector $b[T]$. Formula (5.29) now follows immediately from equation (5.27). □

E x i s t e n c e o f a f i x e d p o i n t. The previous analysis implies the following lemma.

LEMMA 5.6. *If* $\left|\left|\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\right|\right| < 1$, *then* $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ *takes $\overline{Z(V(\mathbf{j}))}$ to $Z(V(\mathbf{j}))$.*

PROPOSITION 5.6. *Let*

$$\left|\left|\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\right|\right| < 1.$$

Then the generalized fractional-linear map $\tau\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ *of the matrix ball to itself has a fixed point.*

P r o o f. We denote by $Z(V(\mathbf{j}))$ the set of symmetric operators from $V_+(\mathbf{j})$ to $V_-(\mathbf{j})$ with norm < 1. We denote by $\overline{Z(V(\mathbf{j}))}$ the set of symmetric operators from $V_+(\mathbf{j})$ to $V_-(\mathbf{j})$ with norm ≤ 1. Clearly, map (4.1) extends to a continuous map

$$\tau\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : \overline{Z(V(\mathbf{j}))} \to \overline{Z(V(\mathbf{j}))}.$$

From the topological point of view, $\overline{Z(V(\mathbf{j}))}$ is a ball. By Browser's theorem, the map $\tau\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ has a fixed point in $\overline{Z(V(\mathbf{j}))}$ and, by the above lemma, this fixed point lies in $Z(V(\mathbf{j}))$. □

E s t i m a t e o f t h e n o r m o f t h e o p e r a t o r $\mathrm{we}(P)$

PROPOSITION 5.7. *Let* $P \in \mathrm{End}_{\mathrm{CK}\mathbf{Sp}}(V(\mathbf{j}), V(\mathbf{j}))$, *where* $P = P^*$. *Let*

$$S\begin{pmatrix} \overline{M} & L \\ L^t & M \end{pmatrix}$$

be its Potapov transform. Then

$$||\mathrm{we}(P)|| \leq \det(1 - |M|)^{-1/2}. \tag{5.30}$$

P r o o f. We start with the case where $||S|| < 1$. Then Theorem 5.5 and Proposition 5.6 are available. It therefore suffices for us to establish the following lemma.

LEMMA 5.7. *Let* M *and* X *be* $n \times n$ *matrices, with* $||M|| < 1$ *and* $||X|| < 1$. *Then*

$$|\det(1 - XM)| > \det(1 - |M|).$$

P r o o f. We can suppose without loss of generality that the matrix M is positive self-adjoint. We wish to prove that

$$1 \leq \det(1 - M)^{-1}|\det(1 - XM)|. \tag{5.31}$$

Transforming the right-hand side, we obtain

$$|\det(1 + (1 - X)M(1 - M)^{-1})|. \tag{5.32}$$

Let $\Lambda = \sqrt{M(1 - M)^{-1}}$. Then (5.32) is equal to

$$|\det(1 + \Lambda(1 - X)\Lambda)|, \tag{5.33}$$

and in order to prove (5.31), it suffices for us to verify that the eigenvalues of the matrix $\Lambda(1 - X)\Lambda$ have non-negative real parts. For this, it suffices to verify that

$$\mathrm{Re}\langle\Lambda(1 - X)\Lambda v, v\rangle \geq 0$$

for all v. Transforming this expression, we obtain

$$\mathrm{Re}\langle\Lambda(1 - X)\Lambda v, v\rangle = \mathrm{Re}\langle(1 - X)\Lambda v, \Lambda v\rangle =$$
$$= ||\Lambda v||^2 - \mathrm{Re}\langle X\Lambda v, \Lambda v\rangle = ||\Lambda v||^2 - ||X||||\Lambda v||^2 \geq 0,$$

as required. This completes the proof of the lemma. □

Thus, Proposition 5.7 and the boundedness of the operator $\mathrm{we}(P)$ in the case $||S|| < 1$ are proved. Suppose now that $||S|| = 1$, and consider the sequence of linear relations (P_n) with the Potapov transform

$$S_n = \begin{pmatrix} \overline{M} & (1 - \frac{1}{n})\overline{L} \\ (1 - \frac{1}{n})L^t & M \end{pmatrix}.$$

Let $K_n(z, \bar{u})$ be the kernel of $\mathrm{we}(P_n)$, and let $K(z, \bar{u})$ be the kernel of $\mathrm{we}(P)$. Then, clearly, $\{K_n(z, \bar{u})\}$ converges pointwise to $K(z, u)$; furthermore, the norms of the operators $\mathrm{we}(P_n)$ are bounded by the constant $\det(1 - |M|)^{-1/2}$. It therefore follows from Proposition (5.1) that $\mathrm{we}(P_n)$ converges weakly to $\mathrm{we}(P)$ and

$$||\mathrm{we}(P)|| \leq \det(1 - |M|)^{-1/2}.$$

□

H o w e d u a l i t y f o r CK**Sp**. Let **CK** be a Cayley—Klein category, and let $H(\mathbf{j})$ be a Cayley—Klein group. Then we define the product $\mathbf{CK} \times H(\mathbf{j})$ to be the category whose objects are the same as those in **CK**, and

$$\mathrm{Mor}_{\mathbf{CK} \times H(\mathbf{j})}(V(\mathbf{j}), W(\mathbf{j})) := \mathrm{Mor}_{\mathbf{CK}}(V(\mathbf{j}), W(\mathbf{j})) \times H(\mathbf{j}).$$

Multiplication of morphisms is defined in obvious fashion:

$$(p_1, h_1)(p_2, h_2) = (p_1p_2, h_1h_2).$$

Let R be a representation of $\mathbf{CK} \times H(\mathbf{j})$. We say that **CK** and $H(\mathbf{j})$ are dual in R if

$$R = \bigoplus_{\sigma} (T_\sigma \otimes \pi_\sigma),$$

where the T_σ are irreducible representations of **CK**, the π_σ are irreducible representations of $H(\mathbf{j})$, and the index σ runs through some set; here $\sigma \neq \sigma'$ implies that $T_\sigma \neq T_{\sigma'}$ and $\pi_\sigma \neq \pi_{\sigma'}$.

We now construct the canonical functor τ_n : CK**Sp** $\times$ $O(n, \mathbf{j}) \to$ CK**Sp**. Consider the space $\mathbb{R}^n(\mathbf{j})$ endowed with the standard scalar product. Let $V(\mathbf{j}) \in \mathrm{Ob}(\mathrm{CK}\mathbf{Sp} \times O(n, \mathbf{j})) = \mathrm{Ob}(\mathrm{CK}\mathbf{Sp})$. Then $\tau_n(V(\mathbf{j})) = V_{\mathbb{R}}(\mathbf{j}) \otimes \mathbb{R}^n(\mathbf{j})$. Further, the pair $(L, h) \in \mathrm{Mor}(V(\mathbf{j}), W(\mathbf{j})) \times O(n, (\mathbf{j}))$ is associated with the subspace $\tau_n(L, h) \subset (V(\mathbf{j}) \otimes_{\mathbb{R}(\mathbf{j})} \mathbb{R}^n(\mathbf{j})) \oplus (W(\mathbf{j}) \otimes_{\mathbb{R}(\mathbf{j})} \mathbb{R}^n(\mathbf{j}))$ generated by vectors of the form

$$(v \otimes x) \oplus (w \otimes hx),$$

where $(v, w) \in L$, $x \in \mathbb{R}^n(\mathbf{j})$. We now consider the representation $W^{(n)} = \mathrm{We} \circ \tau_n$ of the category CK**Sp** $\times$ $O(n, (\mathbf{j}))$, where We is the Weil representation.

HOWE'S DUALITY THEOREM 5.6

(a) CK**Sp** *and* $O(n, \mathbf{j})$ *are dual in* $W^{(n)}$.

(b) *In the decomposition*

$$W^{(n)} = \bigoplus \left(T_\lambda^{(n)} \otimes \pi_\lambda^{(n)}\right)$$

of the representation $W^{(n)}$ *into irreducible subrepresentations, the representation* $\pi_\lambda^{(n)}$ *runs through all the irreducible representations of* $O(n, \mathbf{j})$.

(c) *If* $n \neq m$, *then* $T_\lambda^{(n)}$ *is not equivalent to* $T_\mu^{(m)}$.

(d) *Each holomorphic projective* $*$*-representation of* CK**Sp** *has the form* $T_\lambda^{(n)}$.

Thus, the holomorphic projective $*$-representations of CK**Sp** are indexed by pairs (n, π), where $n \in \mathbb{Z}_+$ and π is an irreducible representation of $O(n+1, \iota; \mathbf{j})$ (2.41), (2.42).

Any projective *-representation of CK**Sp** is the product of a holomorphic projective *-representation and an antiholomorphic projective *-representation [3].

Conclusions

We suppose that the true cosmological model belongs to the class of models based on Cayley-Klein geometries with permanent curvatures. This is one of the articles [35, 36] which would be sufficient for an categorical expression of the basic concepts of the theory of the Cosmic Microwave Background. It makes possible in this way to express the Cosmic Microwave Background data in terms of monoidal Cayley-Klein categories.

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ПРЕДСТАВЛЕННЯ ВЕЙЛЯ ЕРМІТОВОЇ СИМПЛЕКТИЧНОЇ КАТЕГОРІЇ КЕЛІ—КЛЕЙНА

С.С. Москалюк

Р е з ю м е

Метод категорійних продовжень застосовано в теорії груп Келі—Клейна. Цей метод використовує простори Келі—Клейна як об'єкти категорій Келі—Клейна з морфізмами у вигляді всеможливих лінійних перетворень або білінійних форм. Побудовані індуковані представлення груп Келі—Клейна шляхом категоріфікації переносяться на конструкції представлень Вейля класичних ермітових категорій Келі—Клейна. В роботі отримано в явному вигляді представлення Вейля ермітової симплектичної категорії Келі—Клейна.

ПРЕДСТАВЛЕНИЕ ВЕЙЛЯ ЭРМИТОВОЙ СИМПЛЕКТИЧЕСКОЙ КАТЕГОРИИ КЕЛИ—КЛЕЙНА

С.С. Москалюк

Р е з ю м е

Метод категорийных продолжений применяется в теории групп Кели—Клейна. Этот метод использует пространство Кели—Клейна в качестве объектов категорий Кели—Клейна с морфизмами в виде всевозможных линейных преобразований или билинейных форм. Построенные индуцированные представления групп Кели—Клейна путем категорификации переносятся на конструкции представлений Вейля классических эрмитовых категорий Кели—Клейна. Получены в явном виде представления Вейля эрмитовой симплектической категории Кели—Клейна.