DIFFERENTIAL GEOMETRIC MECHANISMS IN OSTROHRADSKYJ RELATIVISTIC SPHERICAL TOP DYNAMICS [1]

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Applications of the higher-order variational calculus to some classical models of a relativistic particle motion began in 1937 and continue till now. Differential geometry of Ostrohrads'kyj's mechanics has been an object of renewed interest among contemporary mathematicians for last three decades. In the present article, we demonstrate the work of some intrinsic tools of the formal theory of variational equations in application to one specific example of the third-order evolution equation of a free relativistic top in three-dimensional space-time. The main goal is to introduce a combined approach of simultaneous utilization of symmetry principles and inverse variational problem considerations in terms of vector-valued differential forms. Next, some simple algorithm of transition between the autonomous variational problem and the variational problem in parametric form is established. The example definitely solved shows the no-existence of a globally and intrinsically defined Lagrangian for the Poincaré-invariant and well-defined unique variational equation in the case in hand. The Hamiltonian counterpart in terms of Poisson bracket is discussed too. The model appears to provide a generalized canonical description of a quasi-classical spinning particle governed by the Mathisson—Papapetrou equations in flat space-time.

Introduction

During past decades, the subject of Ostrohrads'kyj's mechanics was revisited by many authors from the point of view of global analysis including certain features of the intrinsic differential geometry ideology (see monographs [1—3], preceded and followed by the large number of other reviews and articles). The more intriguing is that the investigations on the application of Ostrohrads'kyj's mechanics to real physical models haven't been abandoned since the pioneer works by Chrapływyj, Mathisson, Bopp, Weyssenhoff, Raabe, and Hönl (see references). Most of the applications consider models of test particles endowed with inner degrees of freedom [9—16] or models which put the notion of acceleration onto the framework of a general differential geometric structure of the extended configuration space of a particle [17]. One interesting example of how the derivatives of the third order appear in the equations of motion of a test particle is provided by the Mathisson—Papapetrou equations

$$\frac{D}{d\zeta}\left(m_0\frac{u^\alpha}{\|\boldsymbol{u}\|}+\frac{u_\gamma}{\|\boldsymbol{u}\|^2}\frac{D}{d\zeta}S^{\alpha\gamma}\right)=\mathcal{F}^\alpha, \qquad (1)$$

$$\frac{D}{d\zeta}S^{\alpha\beta}=\frac{1}{\|\boldsymbol{u}\|^2}\left(u^\beta u_\gamma\frac{D}{d\zeta}S^{\alpha\gamma}-u^\alpha u_\gamma\frac{D}{d\zeta}S^{\beta\gamma}\right) \qquad (2)$$

together with the supplementary condition

$$u_\gamma S^{\alpha\gamma}=0. \qquad (3)$$

It is immediately clear that the second term in (1) may produce the derivatives of the third order of space-time variables x^α as soon as one dares to substitute $u_\gamma\dfrac{DS^{\alpha\gamma}}{d\zeta}$ by $-S^{\alpha\gamma}\dfrac{Du^\gamma}{d\zeta}$ in virtue of (3). Such substitution in fact means differentiating Eq. (3). However, the system of equations thus obtained will not possess any additional solutions comparing to that of (1)—(3) as far as one does not forget the original constraint (3). System (1)—(3) was recently a subject of discussion in [18]. In (1), the right-hand side vanishes if there is no gravitation.

It is a matter of common consent that the relativistic motion of simple particles in gravitational field may be described mathematically via the notion of geodesic paths. Because less simple particles obey higher-order equations of motion, it seems worthwhile to investigate the appropriate geometries. But, in the same way as pseudo-Riemannian geometry descends down to the natural representation of the Lorentz group, a more complicated geometry should break out first from some symmetry considerations of global character.

We intend to present some tools from the arsenal of intrinsic analysis on manifolds that may appear helpful

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in solving the invariant inverse problem of the calculus of variations. In a special case of three-dimensional space-time, we shall successfully follow some prescriptions for obtaining third-order Poincaré-invariant variational equations up to the very final solution thus discovering the unique possible one, which will then be identified with the motion of a free relativistic top by means of comparing it to (1)—(3) when $R^{\alpha}{}_{\beta\delta\gamma} = 0$. This case of two-dimensional motion in space makes quite a good sense from the viewpoint of the general theory as well [19]. On the other hand, one can show directly that even in the four-dimensional special relativity case, the world line of a particle obeying the system of equations (1)—(3) has the third curvature equal to zero (see also [20]). Thus, even in this case, the particle actually propagates in two-dimensional space. Another feature of this limited case is that the spin four-vector

$$\sigma_\alpha = \frac{1}{2\|\boldsymbol{u}\|}\epsilon_{\alpha\beta\gamma\delta}u^\beta S^{\gamma\delta} \tag{4}$$

keeps constant under the condition for the motion to be free. So, knowing a Lagrange function for some third-order equation equivalent to (1)—(3) allows offering a generalized Hamiltonian description in terms of Poisson brackets that might be considered as a canonical equivalent to (1), (3). Our example exposes some typical features of variational calculus:

– the nonexistence (in our case) of a well-defined invariant Lagrangian along with intrinsically very well defined equation of motion with the Poincaré symmetry produced by each of a family of degenerate Lagrangians which transform into one another by renumbering the axes of a Lorentz frame;

– all handled Lagrangians give rise to the same system of canonical equations;

– each Lagrangian includes a different set of second order derivatives, thus their sum is not a Lagrangian of minimal order.

1. Homogeneous Form and Parametric Invariance

Presentation of the equation of motion in the so-called "manifestly covariant form" stipulates introducing the space of Ehresmann's velocities of the configuration manifold M of a particle, $T^kM = \{x^\alpha, \dot{x}^\alpha, \ddot{x}^\alpha \dots x^\alpha_{(k)}\}$. In future, the notations u^α, $\dot{u}^\alpha$, $\ddot{u}^\alpha$, $u^\alpha_{(r)}$ will frequently be used in place of $\dot{x}^\alpha$, $\ddot{x}^\alpha$, $x^\alpha_{(3)}$, $x^\alpha_{(r+1)}$, and $x^\alpha_{(0)}$ sometimes will merely denote x^α. We call some mapping $\zeta \mapsto x^\alpha(\zeta)$ the *parametrized* (by means of ζ) *world line* and its image in M will be called the *non-parametrized world line.* As far as we are interested in a variational equation (of order s) that would describe the non-parametrized world lines of the particle,

$$\mathcal{E}_\alpha\left(x^\alpha, u^\alpha, \dot{u}^\alpha, \ddot{u}^\alpha, \dots, u^\alpha_{(s-1)}\right) = 0, \tag{5}$$

the Lagrange function $\mathcal{L}$ has to satisfy the Zermelo conditions, which read in our case of at most the second order derivatives in $\mathcal{L}$

$$u^\beta\frac{\partial}{\partial u^\beta}\mathcal{L} + 2\dot{u}^\beta\frac{\partial}{\partial \dot{u}^\beta}\mathcal{L} = \mathcal{L},$$

$$u^\beta\frac{\partial}{\partial \dot{u}^\beta}\mathcal{L} = 0.$$

In this approach, the independent variable ζ (called the *parameter along the world line*) is not included into the configuration manifold M. Thus, the space T^kM is an appropriate candidate for the role of the underlying manifold on which the variational problem in the autonomous form should be posed. We may include the parameter ζ into the configuration manifold by introducing the trivial fibre manifold $\mathbb{R}\times M \to \mathbb{R}$, $\zeta \in \mathbb{R}$, and putting into consideration its k^{th}-order prolongation, $J^k(\mathbb{R}, M)$, i.e., the space constituted by the k^{th}-order jets of local cross-sections of $Y = \mathbb{R}\times M$ over $\mathbb{R}$. Each such cross-section of Y is nothing but the graph in $\mathbb{R}\times M$ of some local curve $x^\alpha(\zeta)$ in M. For each $r \in \mathbb{N}$, there exists an obvious projection

$$p^r_0 : J^r(\mathbb{R}, M) \to T^rM. \tag{6}$$

The manifold T^rM consists of the derivatives up to the rth-order of curves $x^\alpha(\zeta)$ in M evaluated at $0 \in \mathbb{R}$. If, for every $\tau \in \mathbb{R}$, we denote the mapping $\zeta \mapsto \zeta + \tau$ of $\mathbb{R}$ onto itself by same character τ, then the projection reads

$$p^r_0 : \left(\tau; x^\alpha(\tau), \frac{d}{d\zeta}x^\alpha(\tau), \frac{d^2}{d\zeta^2}x^\alpha(\tau), \dots\right.$$

$$\left.\dots,\ fracd^r d\zeta^r x^\alpha(\tau)\right) \mapsto$$

$$\mapsto \left((x^\alpha\circ\tau)(0), \frac{d}{d\zeta}(x^\alpha\circ\tau)(0),\right.$$

$$\frac{d^2}{d\zeta^2}\left(x^\alpha \circ \tau\right)(0), \ldots, \frac{d^r}{d\zeta^r}\left(x^\alpha \circ \tau\right)(0)\Big). \quad (7)$$

By means of projection (6), (7), every Lagrange function $\mathcal{L}$ initially defined on T^kM may be pulled back to the manifold $J^k(\mathbb{R}, M)$ and defines there some function $\mathcal{L}_0$ by the obvious formula $\mathcal{L}_0 = \mathcal{L} \circ p_0^k$. We say that the differential form

$$\lambda = \mathcal{L}_0 d\zeta \quad (8)$$

constitutes a variational problem in extended parametric form because the independent variable ζ was artificially doubled in the construction of the new configuration manifold $\mathbb{R} \times M$. But we shall need this construction later.

Let us return to the variational problem set on the manifold T^kM by a given Lagrange function $\mathcal{L}$. At the very first moment we impose the Zermelo conditions, the problem becomes degenerate. There exists one way to avoid degeneracy by reducing the number of velocities, of course, at the cost of losing the "homogeneity" property of Eq. (5). Consider some way of segregating the variables $x^\alpha \in M$ into $t \in \mathbb{R}$ and $\mathsf{x}^i \in Q$, $\dim Q = \dim M - 1$, thus making M into some fibration, $M \approx \mathbb{R} \times Q$, over $\mathbb{R}$. The manifold of jets $J^r(\mathbb{R}, Q)$ provides some local representation of what is known as the manifold $C^r(M, 1)$ of r-contact one-dimensional submanifolds of M. The intrinsically defined global projection of non-zero elements of T^rM onto the manifold $C^r(M, 1)$ in this local and, surely, "non-covariant" representation is given by

$$\wp^r : T^rM \setminus \{0\} \to J^r(\mathbb{R}, Q), \quad (9)$$

and is implicitly defined in the third order by the following formulae, where the local coordinates in $J^r(\mathbb{R}, Q)$ are denoted by $t; \mathsf{x}^i, \mathsf{v}^i, \mathsf{v}'^i, \mathsf{v}''^i, \ldots, \mathsf{v}^i_{(r-1)}$ with $\mathsf{v}^i_{(0)}$ marking v^i sometimes:

$$\dot{t}\,\mathsf{v}^i = u^i,$$

$$(\dot{t})^3 \mathsf{v}'^i = \dot{t}\dot{u}^i - \ddot{t}u^i,$$

$$(\dot{t})^5 \mathsf{v}''^i = (\dot{t})^2 \ddot{u}^i - 3\dot{t}\ddot{t}\dot{u}^i + \left[3(\ddot{t})^2 - \dot{t}t_{(3)}\right] u^i. \quad (10)$$

There does not exist any well-defined projection from the manifold $C^r(M, 1)$ onto the space of independent variable $\mathbb{R}$, so the expression

$$\Lambda = L\left(t; \mathsf{x}^i, \mathsf{v}^i, \mathsf{v}'^i, \mathsf{v}''^i, \ldots, \mathsf{v}^i_{(k-1)}\right) dt \quad (11)$$

will vary in the dependence on the way of local representation $M \approx \mathbb{R} \times Q$. We say that two different expressions of type (11) define the same variational problem in parametric form if their difference expands into nothing but only the pull-backs to $C^k(M, 1)$ of the following contact forms which live on the manifold $C^1(M, 1)$,

$$\theta^i = d\mathsf{x}^i - \mathsf{v}^i dt. \quad (12)$$

These differential forms obviously vanish along the jet of every curve $\mathbb{R} \to Q$.

Let the components of the variational equation

$$\mathsf{E}_i = 0 \quad (13)$$

of Lagrangian (11) be treated as the components of the following vector one-form:

$$\mathbf{e} = \{\mathsf{E}_i dt\}. \quad (14)$$

We intend to give a "homogeneous" description to (14) and (11) in terms of some objects that would live on T^sM and T^kM, respectively. But we cannot apply directly the pull-back operation to Lagrangian (11) because the pull-back of one-form is a one-form again, and what we need on T^kM is a Lagrange *function*, not a differential form. However, it is possible to pull (11) all the way back along the composition of projections (6) and (9),

$$p^k = \wp^k \circ p_0^k, \quad (15)$$

ultimately to the manifold $J^k(\mathbb{R}, M)$. In such a way, we obtain the differential form $\left(L \circ p^k\right)dt$. But what we do desire is a form that should involve $d\zeta$ solely (i.e., a semi-basic with respect to the projection $J^k(\mathbb{R}, M) \to \mathbb{R}$). Fortunately, the two differential forms, dt and $\dot{t}d\zeta$, differ not more than only by the contact form

$$\vartheta = dt - \dot{t}d\zeta. \quad (16)$$

Now, we recall that equivalent Lagrangians that have the structure of (11) differ by multiplies of the contact forms (12). It remains to notice that, by the course of (7) and (10), the pull-backs of the contact forms (12) expand only into the contact forms (16) and

$$\vartheta^i = dx^i - u^i d\zeta \quad (17)$$

alone,

$$p^{1*}\theta^i = dx^i - \left(\mathsf{v}^i \circ p^1\right)dt = \vartheta^i - \left(\mathsf{v}^i \circ p^1\right)\vartheta.$$

Thus, every variational problem posed on $J^k(\mathbb{R},Q)$ and represented by (11) transforms into an equivalent variational problem

$$\lambda = \big(L\circ p^k\big)\dot{t}d\zeta \tag{18}$$

posed on $J^k(\mathbb{R},M)$. But the Lagrange function of this new variational problem,

$$\mathcal{L}_0 = \big(L\circ p^k\big)\dot{t}, \tag{19}$$

does not depend upon the parameter ζ and substantially may be thought of as a function defined on T^kM.

We prefer to cast the variational equation (of some order $s \le 2k$) generated by Lagrangian (18) into the framework of the theory of vector-valued exterior differential systems by introducing the following vector differential one-form defined on the manifold $J^s(\mathbb{R},M)$:

$$\varepsilon = \mathcal{E}_\alpha\left(x^\alpha,\dot{x}^\alpha,\ldots,x^\alpha_{(s)}\right)d\zeta. \tag{20}$$

The expressions $\mathcal{E}_\alpha\left(x^\alpha,\dot{x}^\alpha,\ldots,x^\alpha_{(s)}\right)$ in (20) may also be treated as ones defined on T^sM similarly to $\mathcal{L}_0$. Altogether the constructions built above allow the formulation of the following statement:

Proposition 1. *If the differential form (14) corresponds to the variational equation of Lagrangian (11), then the expressions*

$$\mathcal{E}_\alpha = \{u^i\mathsf{E}_i, \dot{t}\,\mathsf{E}_i\} \tag{21}$$

correspond to the Lagrange function (19).

In this case, the (sth-order) equation (5) describes, "in homogeneous form", the same non-parametrized world lines of a particle governed by the variational problem (19), as does Eq. (13) with the Lagrangian given by (11), and also $\mathcal{L}_0$ obviously satisfies the Zermelo conditions. As to more sophisticated details, paper [21] may be consulted.

2. Criterion of Variationality

Our main intention is to find a Poincaré-invariant ordinary (co-vector) differential equation of the third order in three-dimensional space-time. With this goal in mind we organize the expressions E_i in (14) into a single differential object, the exterior one-form

$$\mathsf{e}_0 = \mathsf{E}_i d\mathsf{x}^i \tag{22}$$

defined on the manifold $J^s(\mathbb{R},Q)$, so that the vector differential form (14) should now be treated as the coordinate representation of an intrinsic differential geometric object

$$\mathsf{e} = \mathsf{e}_i d\mathsf{x}^i = \mathsf{E}_i dt\otimes d\mathsf{x}^i = dt\otimes\mathsf{e}_0. \tag{23}$$

The differential form e constructed in this way is an element of the graded module of differential forms on $J^s(\mathbb{R},Q)$ semi-basic with respect to $\mathbb{R}$ with values in the bundle of the graded algebras $\wedge T^*Q$ of scalar forms on TQ. Of course, due to the dimension of $\mathbb{R}$, actually only functions (i.e., semi-basic zero-forms) and semi-basic one-forms (i.e., in dt solely) exist. We also wish to mention that every (scalar) differential form on Q is naturally treated as a differential form on T^rQ, i.e., as an element of the graded algebra of cross-sections of $\wedge T^*\big(T^rQ\big)$.

For arbitrary $s\in\mathbb{N}$, let $\Omega_s(Q)$ denote the algebra of (scalar) differential forms on T^sQ with coefficients depending on $t\in\mathbb{R}$ and arbitrary s. It is possible to develop some calculus in $\Omega_s(Q)$ by introducing the operator of vertical (with respect to $\mathbb{R}$) differential d_v and the operator of total (or formal "time") derivative D_t by means of the prescriptions:

$$d_vf = \frac{\partial f}{\partial\mathsf{x}^i}d\mathsf{x}^i + \frac{\partial f}{\partial\mathsf{v}^i_{(r)}}d\mathsf{v}^i_{(r)},\quad {d_v}^2 = 0,$$

so that $d_v\mathsf{x}^i$ and $d_v\mathsf{v}^i_{(r)}$ coincide with $d\mathsf{x}^i$ and $d\mathsf{v}^i_{(r)}$, respectively, and

$$D_tf = \frac{\partial f}{\partial t} + \mathsf{v}^i\frac{\partial f}{\partial\mathsf{x}^i} + \mathsf{v}^i_{(r+1)}\frac{\partial f}{\partial\mathsf{v}^i_{(r)}},\quad D_td_v = d_vD_t.$$

There exists a notion of *derivation* in graded algebras endowed with generalized commutation rule, as $\Omega_s(Q)$ is. An operator D is called a derivation of degree q if, for any differential form ϖ of degree p and any other differential form w, it is true that $D(\varpi\wedge w) = D(\varpi)\wedge w + (-1)^{pq}\varpi\wedge D(w)$. To complete the above definitions, it is necessary to demand that d_v be a derivation of degree 1 whereas D_t be a derivation of degree 0. But still this is not the whole story. We need one more derivation of degree 0, denoted here as ι and defined by its action on functions and one-forms which altogether locally generate the algebra $\Omega_s(Q)$,

$$\iota f = 0, \iota d\mathsf{x}^i = 0,\ \iota d\mathsf{v}^i = d\mathsf{x}^i, \iota d\mathsf{v}^i_{(r)} = (r+1)\,d\mathsf{v}^i_{(r-1)}.$$

Let the operator the deg means evaluating the degree of a differential form. The *Lagrange differential* δ is first introduced by its action upon the elements of $\Omega_s(Q)$,

$$\delta = \left(\deg + \frac{(-1)^r}{r!}{D_t}^r\iota^r\right)d_v,$$

and next trivially extended to the whole graded module of semi-basic differential forms on $J^s(\mathbb{R},Q)$ with values in $\wedge T^*\big(T^rQ\big)$ by means of

$$\delta(\omega_i dt\otimes d\mathsf{x}^i) = dt\otimes\delta(\omega_i d\mathsf{x}^i),$$

$$\delta(\omega_i^r dt\otimes d\mathsf{v}^i_{(r)}) = dt\otimes\delta(\omega_i^r d\mathsf{v}^i_{(r)}).$$

This δ turns out to possess the property $\delta^2=0$. We have that, for the differential geometric objects (23) and (11), the following relation holds:

$$\mathsf{e} = \delta\Lambda = dt\otimes\delta L. \tag{24}$$

Now the criterion for an arbitrary set of expressions $\{\mathsf{E}_i\}$ in (14) to be the variational equations for some Lagrangian reads

$$\delta\mathsf{e} = dt\otimes\delta\mathsf{e}_0 = 0, \tag{25}$$

with e constructed from $\{\mathsf{E}_i\}$ by means of (22) and (23).

Of course, one may apply the above constructions literally to analogous objects living on the manifold $J^s(\mathbb{R},M)$ in (6) and obtain the operator, the Lagrange differential, δ^Y acting upon semi-basic, with respect to $\mathbb{R}$, differential forms on $J^s(\mathbb{R},M)$ with values in the bundle $\wedge T^*\big(T^sM\big)$. In the algebra $\Omega_s(M)$, the operator δ^Y preserves the sub-algebra of forms that do not depend on the parameter $\zeta\in\mathbb{R}$. The restriction of δ^Y to the algebra of differential forms truly defined on T^sM sole will be denoted by δ^T. It was introduced in [22]. If the Lagrange function $\mathcal{L}_0$ in (8) does not depend on the parameter $\zeta\in\mathbb{R}$, as is the case of (18), (19), instead of to apply δ^Y to the forms λ from (8) and

$$\varepsilon = \varepsilon_\alpha dx^\alpha = \mathcal{E}_\alpha d\zeta\otimes dx^\alpha \tag{26}$$

from (20), we may apply the restricted operator δ^T to the Lagrange function $\mathcal{L}_0$ and to the differential form

$$\epsilon_0 = \mathcal{E}_\alpha dx^\alpha. \tag{27}$$

In case of (19), the criteria $\delta^Y\varepsilon=0$,

$$\delta^T\varepsilon_0 = 0, \tag{28}$$

and (25) are all equivalent, and the variational equations produced by the expressions $\varepsilon=\delta^Y\lambda$ from (26), (18), $\varepsilon_0=\delta^T\mathcal{L}_0$ from (27), (19), and e from (24) all are equivalent to (5). Expressions (14) and (11) are not "generally covariant" whereas (27) is. But the criterion (28) needs to be solved along with Zermelo conditions, whereas (25) is self-contained.

The presentation of a system of variational expressions $\{\mathsf{E}_i\}$ under the guise of a semi-basic (i.e., in dt solely) differential form that takes values in the bundle of one-forms over the configuration manifold Q is quite natural:

- the Lagrange density (called *Lagrangian* in this work) is a one-form in dt only;
- the destination of the Euler—Lagrange expressions in fact consists in evaluating them on the infinitesimal variations, i.e., the vector fields tangent to the configuration manifold Q along the critical curve; consequently, the set of E_i constitutes a linear form on the cross-sections of TQ with the coefficients depending on higher derivatives.

More details can be found in [23] and [24].

3. Lepagean Equivalent

The system of partial differential equations, imposed on E_i, that arises from (25) takes a more tangible shape in the specific case of third-order Euler—Poisson (i.e., ordinary Euler—Lagrange) expressions. The reader may consult [25] and references therein. Let a skew-symmetric matrix $\mathbf{A}$, symmetric matrix $\mathbf{B}$, and column $\mathbf{c}$ all depend on t, x^i, and v^i and satisfy the following system of partial differential equations:

$$\begin{aligned}
&\partial_{\mathsf{v}^{[i}}\mathsf{A}_{jl]} = 0,\\
&2\,\mathsf{B}_{[ij]} - 3\,\mathbf{D_1}\mathsf{A}_{ij} = 0,\\
&2\,\partial_{\mathsf{v}^{[i}}\mathsf{B}_{j]l} - 4\,\partial_{\mathsf{x}^{[i}}\mathsf{A}_{j]l} + \partial_{\mathsf{x}^l}\mathsf{A}_{ij} + 2\,\mathbf{D_1}\partial_{\mathsf{v}^l}\mathsf{A}_{ij} = 0,\\
&\partial_{\mathsf{v}^{(i}}\mathsf{c}_{j)} - \mathbf{D_1}\mathsf{B}_{(ij)} = 0,\\
&2\,\partial_{\mathsf{v}^l}\partial_{\mathsf{v}^{[i}}\mathsf{c}_{j]} - 4\,\partial_{\mathsf{x}^{[i}}\mathsf{B}_{j]l} + \mathbf{D_1}^2\partial_{\mathsf{v}^l}\mathsf{A}_{ij} +\\
&\quad + 6\,\mathbf{D_1}\partial_{\mathsf{x}^{[i}}\mathsf{A}_{jl]} = 0,
\end{aligned}$$

$$4\,\partial_{\mathsf{x}^{[i}}\mathsf{c}_{j]} - 2\,\mathbf{D_1}\partial_{\mathsf{v}^{[i}}\mathsf{c}_{j]} - \mathbf{D_1}^3\mathsf{A}_{ij} = 0\,, \tag{29}$$

where the differential operator $\mathbf{D_1}$ is the lowest order generator of the Cartan distribution,

$$\mathbf{D_1} = \partial_t + \mathbf{v}\,.\,\boldsymbol{\partial}_{\mathsf{x}}\,.$$

It is obvious and commonly well known that the Euler—Lagrange expressions are of affine type in the highest derivatives. The most general form of the Euler—Poisson equation of the third order reads:

$$\mathbf{A}\,.\,\mathbf{v}'' + (\mathbf{v}'\!.\,\partial_{\mathsf{v}})\,\mathbf{A}\,.\,\mathbf{v}' + \mathbf{B}\,.\,\mathbf{v}' + \mathbf{c} \,=\, \mathbf{0}\,. \tag{30}$$

Due to the affine structure of the left-hand side of Eq. (30), we may, alongside with the differential form (23), introduce the next one whose coefficients do not depend on third-order derivatives:

$$\epsilon = \mathsf{A}_{ij} d\mathsf{v}'^{j} \otimes d\mathsf{x}^{i} + \mathsf{k}_i dt \otimes d\mathsf{x}^{i}\,,$$

$$\mathsf{k} = (W\!.\,\boldsymbol{\partial}_v)\,A\,.\,W + \mathbf{B}\,.\,W + \mathbf{c}\,. \tag{31}$$

From the point of view of searching only holonomic local curves in $J^3(\mathbb{R}, Q)$, those exterior differential systems which differ not more than merely by multipliers of the contact forms (12) and

$$\theta'^{i} = d\mathsf{v}^{i} - \mathsf{v}'^{i} dt, \quad \theta''^{i} = d\mathsf{v}'^{i} - \mathsf{v}''^{i} dt,$$

are considered equivalent. The differential forms (31) and (23) are equivalent:

$$\epsilon - \mathsf{e} = \mathsf{A}_{ij} \theta''^{j} \otimes d\mathsf{x}^{i}.$$

The differential form (31) may be accepted as an alternative representation of the *Lepagean equivalent* [1] of (23).

4. Invariant Euler—Poisson Equation

We are preferably interested in those variational equations that expose some symmetry. Let $X(\boldsymbol{\epsilon})$ denote the component-wise action of an infinitesimal generator X on a vector differential form $\boldsymbol{\epsilon}$. The fact that the exterior differential system generated by the form $\boldsymbol{\epsilon}$ possesses the symmetry of X means that there exist some matrices Φ, Ξ, and Π which depend on $\mathbf{v}$ and $\mathbf{v}'$, and are such that

$$X(\boldsymbol{\epsilon}) = \boldsymbol{\Phi}\,.\,\boldsymbol{\epsilon} + \boldsymbol{\Xi}\,.\,(\mathsf{x} - \boldsymbol{V} dt) + \boldsymbol{\Pi}\,.\,(d\boldsymbol{V} - \boldsymbol{W} dt). \tag{32}$$

Equation (32) expresses the condition that two vector exterior differential systems, the one generated by the vector differential form $\boldsymbol{\epsilon}$ and the other generated by the shifted form $X(\boldsymbol{\epsilon})$, are algebraically equivalent. For systems generated by one-forms (as in our case), this is completely the same thing as to demand that the set of local solutions be preserved under the one-parametric Lie subgroup generated by X. We see two advantages of this method:

– the symmetry conception is formulated in reasonably most general form;

– the problem of invariance of a differential equation is reformulated in algebraic terms by means of undetermined coefficients Φ, Ξ, and Π;

– the order of the underlying non-linear manifold is reduced (to $J^2(\mathbb{R}, Q)$ instead of $J^3(\mathbb{R}, Q)$).

Further details may be found in [26].

In the case of the Poincaré group, we assert that $\mathbf{A}$ and $\mathbf{k}$ in (31) do not depend on t and $\mathbf{x}$. For the sake of reference, it is worthwhile to put down the general expression of the generator of the Lorentz group parametrized by a skew-symmetric matrix Ω and some vector $\boldsymbol{\pi}$:

$$X = \,-\,(\boldsymbol{\pi}\cdot\mathbf{x})\,\partial_t + g_{00}\,t\,\boldsymbol{\pi}\,.\,\boldsymbol{\partial}_{\mathsf{x}} + \boldsymbol{\Omega}\cdot(\mathbf{x}\wedge\boldsymbol{\partial}_{\mathsf{x}} +$$

$$+ g_{00}\,\boldsymbol{\pi}\,.\,\partial_{\mathsf{v}} + (\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}\,.\,\partial_{\mathsf{v}} + \boldsymbol{\Omega}\cdot(\mathbf{v}\wedge\partial_{\mathsf{v}}) +$$

$$+2\,(\boldsymbol{\pi}\cdot\mathbf{v})\,\mathbf{v}'\!.\,\partial_{\mathsf{v}'} + (\boldsymbol{\pi}\cdot\mathbf{v}')\,\mathbf{v}\,.\,\partial_{\mathsf{v}'} + \boldsymbol{\Omega}\cdot(\mathbf{v}'\wedge\partial_{\mathsf{v}'})\,.$$

Here, the centered dot symbol denotes the inner product of vectors or tensors and the lowered dot symbol denotes the contraction of a row-vector and the subsequent column-vector.

The system of equations (29), (32) may possess many solutions or no solutions at all, depending on the dimension of the configuration manifold. For example, for the dimension one, the skew-symmetric matrix $\mathbf{A}$ does not exist. For the dimension three, there is no solution to the P.D.E. system (29), (32) (see [27]). Fortunately, for the dimension two, the solution exists *and is unique* up to a single scalar parameter μ (see also [28]):

Proposition 2. *The invariant Euler—Poisson equation of a relativistic two-dimensional motion is:*

$$-\frac{*\mathbf{v}''}{(1+\mathbf{v}\cdot\mathbf{v})^{3/2}} + 3\,\frac{*\mathbf{v}'}{(1+\mathbf{v}\cdot\mathbf{v})^{5/2}}\,(\mathbf{v}\cdot\mathbf{v}') -$$

$$-\frac{\mu}{(1+\mathbf{v}\cdot\mathbf{v})^{3/2}}\,\big((1+\mathbf{v}\cdot\mathbf{v})\,\mathbf{v}' - (\mathbf{v}'\cdot\mathbf{v})\,\mathbf{v}\big) = \mathbf{0}\,. \tag{33}$$

The dual vector above is defined in commonly used notations, $(*\mathbf{w})_i = \epsilon_{ji}\mathrm{w}^j$. We know two different Lagrange functions for the left-hand side of (33):

$$L_1 = -\frac{\mathsf{v}'^2\mathsf{v}^1}{\sqrt{1+\mathsf{v}_i\mathsf{v}^i}(1+\mathsf{v}_2\mathsf{v}^2)} + \mu\sqrt{1+\mathsf{v}_i\mathsf{v}^i}, \qquad (34)$$

$$L_2 = \frac{\mathsf{v}'^1\mathsf{v}^2}{\sqrt{1+\mathsf{v}_i\mathsf{v}^i}(1+\mathsf{v}_1\mathsf{v}^1)} + \mu\sqrt{1+\mathsf{v}_i\mathsf{v}^i}. \qquad (35)$$

With the help of the prescriptions of Proposition 1, we immediately obtain the "homogeneous" counterpart of (33):

$$-\frac{\ddot{\boldsymbol{u}}\times\boldsymbol{u}}{\|\boldsymbol{u}\|^3} + 3\,\frac{\dot{\boldsymbol{u}}\times\boldsymbol{u}}{\|\boldsymbol{u}\|^5}\,(\dot{\boldsymbol{u}}\cdot\boldsymbol{u}) -$$

$$-\frac{\mu}{\|\boldsymbol{u}\|^3}\left((\boldsymbol{u}\cdot\boldsymbol{u})\,\dot{\boldsymbol{u}} - (\dot{\boldsymbol{u}}\cdot\boldsymbol{u})\,\boldsymbol{u}\right) = \mathbf{0} \qquad (36)$$

with the corresponding family of Lagrange functions,

$$\mathcal{L}_1 = \frac{u^1\left(\dot u_2 u_3 - \dot u_3 u_2\right)}{\|\boldsymbol{u}\|\left(u_2u^2+u_3u^3\right)} + \mu\|\boldsymbol{u}\|,$$

$$\mathcal{L}_2 = \frac{u^2\left(\dot u_3 u_1 - \dot u_1 u_3\right)}{\|\boldsymbol{u}\|\left(u_1u^1+u_3u^3\right)} + \mu\|\boldsymbol{u}\|,$$

$$\mathcal{L}_3 = \frac{u^3\left(\dot u_1 u_2 - \dot u_2 u_1\right)}{\|\boldsymbol{u}\|\left(u_1u^1+u_2u^2\right)} + \mu\|\boldsymbol{u}\|\,.$$

To produce a variational equation of the third order, the Lagrange function should be of affine type in second derivatives. It makes no sense to even try finding such a Poincaré-invariant Lagrange function in space-time with dimension greater than two [27]. But the generalized momentum

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{u}} - \frac{d}{d\zeta}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{u}}} = \frac{\dot{\boldsymbol{u}}\times\boldsymbol{u}}{\|\boldsymbol{u}\|^3} + \mu\,\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$

does not depend on the particular choice of one of the above family of Lagrange functions. This expression for the generalized momentum was (in different notations) in fact obtained in [11] by means of introducing an abundance of Lagrange multipliers into the formulation of the corresponding variational problem.

4.1. Free Relativistic Top in Two Dimensions

Equation (36) carries a certain amount of physical sense. We leave it to the reader to ensure (see also [29]) that, in terms of spin vector (4), the Mathisson—Papapetrou equations (1), (2) under the Mathisson-Pirani auxiliary condition (3) are equivalent to the system of equations

$$\varepsilon_{\alpha\beta\gamma\delta}\ddot u^\beta u^\gamma\sigma^\delta - 3\,\frac{\dot{\boldsymbol{u}}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|^2}\,\varepsilon_{\alpha\beta\gamma\delta}\dot u^\beta u^\gamma\sigma^\delta +$$

$$+\frac{m_0}{\sqrt{|g|}}\left[(\dot{\boldsymbol{u}}\cdot\boldsymbol{u})\,u_\alpha - \|\boldsymbol{u}\|^2\dot u_\alpha\right] = \mathcal{F}_\alpha,$$

$$\|\boldsymbol{u}\|^2\dot\sigma_\alpha + (\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}})\,u_\alpha = 0,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{u} = \mathbf{0}\,. \qquad (37)$$

It should be clear that the four-vector $\boldsymbol{\sigma}$ is constant in all its components if the force $\mathcal{F}_\alpha$ vanishes. Equation (37) admits a planar motion, when $u_3 = \dot u_3 = \ddot u_3 = 0$. In this case it reads

$$\eta_3\sigma_3\left(\frac{\ddot{\boldsymbol{u}}\times\boldsymbol{u}}{\|\boldsymbol{u}\|^3} - 3\,\frac{\dot{\boldsymbol{u}}\times\boldsymbol{u}}{\|\boldsymbol{u}\|^5}\,(\dot{\boldsymbol{u}}\cdot\boldsymbol{u})\right) +$$

$$\frac{m_0}{\|\boldsymbol{u}\|^3}\,\left[(\boldsymbol{u}\cdot\boldsymbol{u})\,\dot{\boldsymbol{u}} - (\dot{\boldsymbol{u}}\cdot\boldsymbol{u})\,\boldsymbol{u}\right] = \mathbf{0}\,,$$

where we have set $g_{\alpha\beta} = \mathrm{diag}\left(1,\eta_1,\eta_2,\eta_3\right)$. Comparing with (36) imposes $\mu = \dfrac{m_0}{\eta_3\sigma_3}$.

5. Poisson Structure

It is instructive that each of the two Lagrange functions (34), (35) is of minimal order and produces the same Poisson structure. In constructing the Hamilton function, we chose to start from (35) and then follow the prescriptions of [30]. First, it is necessary to build up the energy function

$$H_2 = \mathsf{p}_1\mathsf{v}^1 + \mathsf{p}_2\mathsf{v}^2 + \mathsf{p}'\mathsf{v}'^1 - L_2 \qquad (38)$$

and reduce the number of independent variables to the set $\{\mathsf{x}^i,\mathsf{p}_i,\mathsf{q},\mathsf{p}'\}$ by means of the following equations:

$$\frac{\partial H_2}{\partial\mathsf{v}^2} = 0 \quad \frac{\partial H_2}{\partial\mathsf{v}'^1} = 0 \qquad \mathsf{q} = \mathsf{v}^1.$$

In these new independent variables, the Hamilton function (38) reads

$$H = \mathsf{p}_1\mathsf{q} + \mathsf{p}_2\mathsf{p}'\frac{\left(1+\eta_1\mathsf{q}^2\right)^{3/2}}{\sqrt{1-\eta_2\mathsf{p}'^2\left(1+\eta_1\mathsf{q}^2\right)^2}} +$$

$$-\mu\sqrt{\frac{1+\eta_1\mathsf{q}^2}{1-\eta_2\mathsf{p}'^2\left(1+\eta_1\mathsf{q}^2\right)^2}}\,.$$

The Poisson structure is implemented by the Poisson bracket

$$\left\{F,G\right\}=\frac{\partial F}{\partial \mathsf{x}^i}\frac{\partial G}{\partial \mathsf{p}_i}-\frac{\partial F}{\partial \mathsf{p}_i}\frac{\partial G}{\partial \mathsf{x}^i}+\frac{\partial F}{\partial \mathsf{q}}\frac{\partial G}{\partial \mathsf{p}'}-\frac{\partial F}{\partial \mathsf{p}'}\frac{\partial G}{\partial \mathsf{q}},$$

and the generalized Hamilton equations read:

$$\frac{d\mathsf{x}^i}{dt}=\left\{\mathsf{x}^i,H\right\}\qquad \frac{d\mathsf{p}_i}{dt}=\left\{\mathsf{p}_i,H\right\},$$
$$\frac{d\mathsf{q}}{dt}=\left\{\mathsf{q},H\right\}\qquad \frac{d\mathsf{p}'}{dt}=\left\{\mathsf{p}',H\right\}.$$

6. Concluding Remarks

Problems with higher order derivatives entering the Lagrange function have been the subject of continuous interest among physicists, but some renewed interest arose due to the attempts to introduce terms responsible for rigidity into the action functional of the relativistic string. From this point of view, the model considered in this paper might be thought of as a point-like limit of the relativistic string, as suggested in [14]. On the other hand, the inner degrees of freedom of, say, a spinning particle demand the introduction of additional variables along the orbits of a coadjoint representation of the Poincaré group. But the question of the space-time origin of these additional variables remains open. Roughly speaking, we may try to construct spin variables from the higher derivatives of ordinary coordinate variables, as suggested in [15]. All such models demand quantization, as an ultimate target. In particular, an alternative way to the quantization of a free relativistic top opens up [11]. Higher derivative terms produce some amendments to the higher momenta in the generalized Legendre transformation and, after quantization, may be viewed as quantum corrections to the states of a point particle without spin. Generalization to the four-dimensional space-time further would prescribe some helicity to a quantum particle [15]. Contrary to the models, proposed by others, as in [11] or in [15], our approach is free from the abundance of a preliminary constraint, imposed *ad hoc*. And it produces the only possible variational model with the third-order term in three-dimensional space-time. Moreover, our model is in perfect agreement with the Mathisson—Papapetrou equation of motion of a classical relativistic spherical top. With these arguments in mind, we may justifiably expect interesting deviations, arising from different ways of quantization of the Poisson structure introduced above.

Finally, let us mention that the symmetry group of the dynamical system (36) without mass term with a multiplier μ is the conformal group whereas the presence of mass μ breaks the symmetry down to the mere pseudo-Euclidean group. Without the μ-term, equation (36) describes geodesic circles, that is, plane curves with constant curvature, which is the mathematical equivalent to the notion of relativistic uniformly accelerated motion. The subject of uniformly accelerated frames of reference gained recently a renewed attention due to the theory of maximal acceleration suggested by Caianiello [17].

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ДИФЕРЕНЦІАЛЬНО-ГЕОМЕТРИЧНІ МЕХАНІЗМИ В ДИНАМІЦІ ОСТРОГРАДСЬКОГО ДЛЯ РЕЛЯТИВІСТСЬКОЇ СФЕРИЧНОЇ ДЗИГИ

Р.Я. Мацюк

Р е з ю м е

Застосування варіаційного числення вищого порядку до деяких класичних моделей руху релятивістської частинки, яке було започатковано в 1937 році, є актуальною проблемою до цього часу. Диференціальна геометрія механіки Остроградського була предметом жвавого інтересу багатьох сучасних математиків протягом останніх трьох десятиріч. В даній роботі ми показуємо, як працюють деякі внутрішньо притаманні підходи із всього арсеналу засобів формальної теорії варіаційних рівнянь в застосуванні до одного конкретного прикладу, що стосується рівняння руху третього порядку вільної релятивістської дзиги в тривимірному просторі-часі. Основною метою є побудова комбінованого підходу, що одночасно використовує симетрійні принципи та розгляд оберненої варіаційної задачі в термінах векторнозначних диференціальних форм. Знайдено деякий простий алгоритм, який пов'язує автономну варіаційну задачу з варіаційною задачею в параметричній формі. Цей приклад чітко демонструє, що не існує глобального і внутрішньо узгодженого визначеного лагранжіана для Пуанкаре-інваріантного і добре визначеного варіаційного рівняння, яке досліджується в даному випадку. Також розглянуто гамільтоновий аналог в термінах дужок Пуассона. Здається, що ця модель узагальнено реалізує канонічний опис руху квазікласичної частинки зі спіном, що підтверджує рівняння Матіссона—Папапетроу в плоскому просторі-часі.

ДИФФЕРЕНЦИАЛЬНО-ГЕОМЕТРИЧЕСКИЕ МЕХАНИЗМЫ В ДИНАМИКЕ ОСТРОГРАДСКОГО ДЛЯ РЕЛЯТИВИСТСКОЙ СФЕРИЧЕСКОЙ ЮЛЫ

Р.Я. Мацюк

Р е з ю м е

Применение вариационного исчисления высшего порядка к некоторым классическим моделям движения релятивистской частицы, начатое в 1937 году, является актуальной проблемой и в наше время. Дифференциальная геометрия механики Остроградского была предметом живого интереса многих современных математиков на протяжении последних трех десятилетий. В данной работе мы показываем, как работают некоторые внутренне присущие подходы из всего арсенала способов формальной теории вариационных уравнений в применении к одному конкретному примеру, что касается уравнения движения третьего порядка свободной релятивистской юлы в трехмерном пространстве-времени. Основной целью является построение комбинированного подхода, что одновременно использует симметрийные принципы и рассмотрение обратной вариационной задачи в терминах векторнозначных дифференциальных форм. Найден некий простой алгоритм, связывающий автономную вариационную задачу с вариационной задачей в параметрической форме. Этот пример четко демонстрирует, что не существует глобального и внутренне согласованного определенного лагранжиана для Пуанкаре-инвариантного и хорошо определенного вариационного уравнения, рассматриваемого в данном случае. Также рассмотрен гамильтоновый аналог в терминах скобок Пуассона. Кажется, что эта модель обобщенно реализует каноническое описание движения квазиклассической частицы со спином, что подтверждает уравнение Матиссона—Папапетроу в плоском пространстве-времени.