SEPARATION OF VARIABLES AND SOME SOLUTION OF A TWO-CENTER PROBLEM WITH A CONFINEMENT-TYPE POTENTIAL [1]

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A group of hidden dynamic symmetry in a model of quantum-mechanical problem of two-centers with Coulomb and oscillator interactions is obtained. The group properties of the system of PDE is studied. The similarity solutions of one-parameter subgroups of the Poincaré and Galiley groups are received. The obtained solutions are used for the calculation of energy terms of the problem.

Introduction

As a rule, when systems possessing a hidden (or higher dynamic) symmetry are considered, two methods are used [1, 2]. The first of them consists in rewriting the Schrödinger equation and putting it in the form where the symmetry, having been hidden before, becomes explicit. The second one implies the construction of integrals of motion which play the role of hidden symmetry group generators. In the paper, based on the example of a physically important model of confinement-type two-centered potential we try to emphasize the deep relationship of the hidden symmetry to the possibility of separation of variables in the Schrödinger equation. The knowledge of such kind of relationships in two recent decades [3] has resulted in the intense application of the method of separation of variables to the equations of mathematical physics and led to a series of important and far from trivial results in this field of mathematics (see, for instance, [4, 5]). The method of separation of variables is much simpler compared to the two above methods. In the framework of this method, the eigenvalues of hidden symmetry group generators acquire the sense of separation constants, and the eigenfunctions, common for the Hamiltonian and the generators, which commutative with the Hamiltonian and with each other, are the solutions of the Schrödinger equation the corresponding coordinates. Below, the group properties of a model quantum problem of the motion of a light particle (a gluon) in the field of two heavy particles (a quark – antiquark pair) are studied. Recently this problem has become the subject of intense studies due to its relation to a wide range of problems of hadron physics: models of baryons with two heavy quarks (QQq baryons) [6] and models of heavy hybrid mesons with open flavor (QQq mesons) [7]. In spite of the lac k of strict theoretical substruction, the potential models give a satisfactory description of mass spectra for heavy mesons and baryons (see e.g. [6—8] and references therein), which, according to modern views, represent bound states of quarks. While modeling the interquark interaction potential, as a rule, confinement-type potentials are used [8, 9]. One of such potentials is the so-called Cornell potential, containing a Coulomb-like term of single-gluon exchange and a term, responsible for string interaction, providing the quark confinement. The confinement part of the potential is most often modeled by a spatial spherically symmetric oscillator potential [6, 7]. Then the motion of a light quark (gluon) in a field of two heavy quarks can be described in the non-relativistic approximation by a stationary Schrödinger equation with a model combined potential, being the sum of the potential of two Coulomb centers and the potential of two harmonic oscillators:

$$V(r_1,r_2) = -\frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \omega^2(r_1^2+r_2^2). \qquad (1)$$

In this formula, r_1 and r_2 are the distances from the particle to the fixed force centers 1 and 2, $Z_{1,2} =$

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$\frac{4}{3}\alpha_s$, α_s is the strong interaction constant, and the phenomenological parameter ω is chosen from the condition of the best agreement of the calculated mass spectra of the quark system with experimental data. In order to avoid ambiguities, we mention that, in our consideration, concerning not only the case of purely Coulomb interaction of the light particle with each of the centers, the notion of the force center is preserved for the $r_{1,2} = 0$ points, where the combined potential (1) has singularities.

In the dimensionless variables, the Schrödinger equation with the model potential (1) is given by

$$H\Psi \equiv \left[-\frac{1}{2}\triangle - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \omega^2(r_1^2 + r_2^2) \right] \Psi(\mathbf{r}; R) =$$

$$= E(R)\Psi(\mathbf{r}; R) \quad (2)$$

where r is the distance from the particle to the midpoint of the intercenter distance R, $E(R)$ and $\Psi(r; R)$ are the particle energy and wave function. Hereinafter, the spectral problem for the Schrödinger equation (2) with the combined potential (1) is conveniently denoted by $eZ_1Z_2\omega$. The sense of such a notation follows from the fact that the traditional quantum-mechanical problem of two purely Coulomb centers [10] has a standard notation eZ_1Z_2. Note that the Schrödinger equation for the eZ_1Z_2 problem can be obtained from Eq. (2) by the limiting transition $\omega \to 0$.

The group properties, eigenvalue, and eigenfunction spectrum for the $eZ_1Z_2\omega$ problem of two Coulomb centers have been studied substantially [10–16]. Namely, the choice of a certain non-canonical basis in a group being a direct product of two groups of motion of three-dimensional spaces $P(3)\otimes P(2)$, or in the wider groups of motions of six-dimensional spaces $P(5{,}1)$ and $P(4{,}2)$ is known to result in the necessity to solve the problem of a linear algebra of two-centerd integrals obtained in [16]. Here we show that, in our case of $eZ_1Z_2\omega$, a generally similar situation takes place. This problem can also be considered as a problem of theory of representations of certain non-compact groups, where the function being a product of a quasiradial and a quasiangular two-centered functions by $\exp(im\alpha + i\tilde{m}\alpha)$ comprises the basis of a degenerate non-canonical representation of the group being a direct product of two three-dimensional space motion groups $P(3) \otimes P(2,1)$, or the wider six-dimensional space motion groups P(5,1),P(4,2), etc.

1. Spheroidal Integral of Motion in the $eZ_1Z_2\omega$ Problem

The variables in Eq. (2) can be separated by introducing a prolate spheroidal (elliptic) coordinate system ξ, η α with the origin at the midpoint of R segment and focus at its endpoints [10]:

$$\left. \begin{array}{ll} \xi = (r_1 + r_2/R)\,, & 1 \le \xi < \infty, \\ \eta = (r_1 - r_2/R)\,, & -1 \le \eta \le 1, \\ \alpha = \arctan\left(\frac{x_2}{x_1}\right), & 0 \le \alpha \le 2\pi \end{array} \right\}. \quad (3)$$

Here, α is the angle of rotation around OX_3 directed from center 1 to center 2. Consider the explicit form of the differential equations resulting from the procedure of separation of the variables in Eq. (2) in the prolate spheroidal coordinates (3). By presenting the wave function $\Psi(\xi, \eta, \alpha, R)$ as a product $F(\xi, R)G(\eta, R)\Phi(\alpha)$ and substituting it into (2), one obtains three ordinary differential equations linked by the separation constants λ and m:

$$\left[\frac{d}{d\xi}(\xi^2 - 1)\frac{d}{d\xi} + a\xi + (p^2 - \gamma\xi^2)(\xi^2 - 1) - \right.$$

$$\left. - \frac{m^2}{\xi 2 - 1} + \lambda \right] F(\xi; R) = 0, \quad (4)$$

$$\left[\frac{d}{d\eta}(1 - \eta^2)\frac{d}{d\eta} + b\eta + (p^2 - \gamma\eta^2)(1 - \eta^2) - \right.$$

$$\left. - \frac{m^2}{1 - \eta^2} - \lambda \right] G(\eta; R) = 0, \quad (5)$$

$$\left[\frac{d^2}{d\alpha^2} + m^2 \right] \Psi(\alpha) = 0. \quad (6)$$

Here we use the notation

$$p = \frac{R}{2}\sqrt{2E'}, \qquad E' = E - \frac{\omega^2 R^2}{2}, \qquad \gamma = \frac{\omega^2 R^4}{4},$$

$$a = (Z_1 + Z_2)R \quad , \quad b = (Z_2 + Z_1)R.$$

In order to have the complete wave function $\Psi(\mathbf{r}, R)$ normalized, the functions $F(\xi, R)$ and $G(\eta, R)$ should obey the boundary conditions

$$|F(1; R)| < \infty, \qquad F(\infty; R) < \infty, \quad (7)$$

$$|G(\pm 1; R)| < \infty. \quad (8)$$

The procedure of obtaining the energy terms $E(R)$ is reduced to the following steps. At first, two boundary problems are considered independently: (4), (7) for the quasiradial equations and (5), (8) for the quasiangular ones, $\lambda^{(\xi)}$ and $\lambda^{(\eta)}$ being considered the eigenvalues and

p being left as a free parameter. Each of eigenfunctions can be conveniently characterized by two quantum numbers n, m and the eigenvalue λ, namely: n_ξ, m, $\lambda^{(\xi)}$ for $F_{n_\xi,m}(\xi,R)$ and $n_\eta, m, \lambda^{(\eta)}$ for $G_{n_\xi,m}(\eta,R)$. The quantum numbers n_ξ, n_η are non-negative integers $0,1,2...$ and coincide with the number of nodes for $F_{n_\xi,m}(\xi,R)$,$G_{n_\xi,m}(\eta,R)$ functions on the radial ($1 \leq \xi < \infty$) and angular ($-1 \leq \eta \leq 1$) intervals, respectively. The general theory of Sturm-Liouville-type one-dimensional boundary problems implies that the quantum numbers n_ξ, n_η, m, remain constant under continuous variation of the intercenter distance R, and the eigenvalues $\lambda^{\xi}_{n_\xi m}(p,a,\gamma)$ or $\lambda^{\eta}_{n_\eta m}(p,b,\gamma)$ are non-degenerate. The pair of one-dimensional boundary problems for $F_{n_\xi,m}(\xi,R)$ and $G_{n_\eta,m}(\eta,R)$ is equivalent to the initial $eZ(1)Z(2)\omega$ problem under condition of equality of the eigenvalues $\lambda^{\xi}_{n_\xi m}(p,a,\gamma) = \lambda^{\eta}_{n_\eta m}(p,b,\gamma)$ and the account of a relationship of p,a,b,γ with the E, Z_1, Z_2, ω, R parameters. The eigenvalues $E_{n_\xi n_\eta m}, \lambda_{n_\xi n_\eta m}$ and eigenfunctions $\Psi_{n_\xi n_\eta m}(r;R)$ of the three-dimensional $eZ_1Z_2\omega$ problem are enumerated by a set of quantum numbers $j = (n_\xi n_\eta m)$ which are conserved under the continuous variation of Z_1, Z_2, ω, R parameters:

$$E_j(R) = E_{n_\xi n_\eta m}(R, Z_1, Z_2, \omega), \tag{9}$$

$$\Psi_j(\mathbf{r};R) = N_j(R)F(\xi;R)G(\eta;R)\frac{e^{im\alpha}}{\sqrt{2\pi}}. \tag{10}$$

The normalization constant $N_j(R)$ is found from the condition

$$\int_\Omega d\Omega \Psi_i^* \Psi_j = \delta_{ij},$$

$$d\Omega = \frac{R^3}{8}(\xi^2-\eta^2)d\xi\, d\eta\, d\alpha = \frac{R^3}{8}d\tau d\alpha, \tag{11}$$

where δ_{ij} is the Kronecker symbol, and $\Omega = \{\xi,\eta,\alpha | 1 \leq \xi < \infty, -1 \leq \eta \leq 1, 0 \leq \alpha < 2\pi\}$.

Hence, the system of functions $\{\Psi_j(r,R)\}$ forms a complete set of orthonormalized wave functions. Now we proceed to establish the relationship between the symmetry properties of the $eZ_1Z_2\omega$ problem and the above separation of variables in the Schrödinger equation (2) in the prolate spheroidal coordinates (3). The very fact of such a separation indicates an additional (with respect to the geometric one) symmetry of Hamiltonian (2) causing the existence of an additional integral of motion, whose operator commutes with $\hat{H}$ and the operator $\hat{L}_3$, the projection of the angular moment on the intercenter axis R. In order to reveal it, we exclude the energy parameter p^2 and the magnetic quantum number m from the above differential equation system (4) – (6). Thus, we derive the equation

$$\widehat{\lambda}\Psi_j(\mathbf{r};R) = \lambda\Psi_j(\mathbf{r};R), \tag{12}$$

where $\hat{\lambda}$ denotes a differential operator

$$\widehat{\lambda} = \frac{1}{\xi^2-\eta^2}\times$$
$$\times\left\{(\xi^2-1)\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} - (1-\eta^2)\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}\right\}+$$
$$+\left[\frac{1}{1-\eta^2} - \frac{1}{\xi^2-1}\right]\frac{\partial^2}{\partial\alpha^2} - RZ_2\frac{\xi\eta+1}{\xi+\eta}+$$
$$+RZ_1\frac{\xi\eta-1}{\xi-\eta} + \frac{\omega^2R^4}{4}(\xi^2-1)(1-\eta^2). \tag{13}$$

The purely geometric symmetry group of the $eZ_(1)Z_(2)\omega$ problem is the O_2 group containing rotations around the intercenter axis R and reflections in the planes containing this axis. In the symmetric case ($Z_1 = Z_2 = Z$), the $eZZ\omega$ system possesses an additional element of geometric symmetry – the reflection in the plane, perpendicular to the $\mathbf{R}$ vector and cutting it at its center. In addition to the geometric symmetry, the $eZ_1Z_2\omega$ problem possesses a higher dynamic symmetry related to the exact separation of variables in the Schrödinger equation (2) in the prolate spheroidal coordinates (3). In the following subsections, we show how, by means of the separation of variables, the dynamic symmetry group of the quantum-mechanical $eZ_1Z_2\omega$ problem can be determined.

2. Asymptotic Expansion for the Wave Functions and Energy Terms

Let us consider a partial case of the considered problem $eZ_(1)Z_(2)\omega$, when parameter ω vanishes. In this case, quasiradial (4) and quasiangular (5) equations are transformed, correspondingly, into equations for the Coulomb spheroidal function (CSF) $\Pi_{mk}(p,a,\xi)$ and $\Xi_{mq}(p,b,\eta)$

$$\left\{\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi} + \left[-p^2(\xi^2-1)+\right.\right.$$
$$\left.\left. + a\xi - \lambda^{(\xi)}_{mk} - \frac{m^2}{\xi^2-1}\right]\right\}\Pi_{mk}(p,a,\xi) = 0, \tag{14}$$

$$\left\{\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta} + \left[-p^2(1-\eta^2)+\right.\right.$$

$$+ b\eta + \lambda_{mk}^{(\eta)} - \frac{m^2}{1-\eta^2}\Bigg]\Bigg\} \Xi_{mk}(p,b,\eta) = 0, \tag{15}$$

with boundary conditions

$$\lim_{\xi\to 1}(\xi^2-1)^{-m/2}\Pi_{mk}(p,a,\xi) = 1,$$

$$\lim_{\xi\to\infty} \Pi_{mk}(p,a,\xi) = 0,$$

$$\lim_{\eta\to\pm 1\mp 0}(1-\eta^2)^{-m/2}\Xi_{mk}(p,b,\eta) = 1. \tag{16}$$

Here, λ is a constant of separation, and we use standard notations as follows [10]:

$$a = (Z_2 + Z_1)R \quad , \quad b = (Z_2 - Z_1)R, \tag{17}$$

$$p^2 = -ER^2/2 \quad , E < 0. \tag{18}$$

Quantum numbers k and q coincide with zeros of CSF according to variables ξ and η, respectively, and the azimuthal quantum number m runs over $0, \pm 1, \pm 2, \ldots$. If we assume that the parameters $\lambda_{mk}^{(\xi)}$ and $\lambda_{mk}^{(\eta)}$ are related by

$$\lambda_{mk}^{(\xi)} = \lambda_{mk}^{(\eta)} = \lambda_{kqm}, \tag{19}$$

so the simultaneous solution of the couple of the boundary-value problems (14) and (15) for radial and angle CSF is equivalent to that of the two-center Coulomb problem related to the motion of an electron in the field of two fixed point-like charges Z_1 and Z_2 separated by the distance R

$$\left[-\frac{1}{2}\triangle_{\mathbf{r}} - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} - E(R)\right]\Psi(\mathbf{r}, R) = 0. \tag{20}$$

Let us restrict ourselves and consider a symmetric case $Z_1 = Z_2$. For this symmetric case, the CSF's are reduced Ξ_{mq} to spheroidal functions $S_{m\ell}(p,\eta)$ related to by $\Xi_{mq}(p,0,\eta) = S_{m\ell}(p,\eta)$.

CSF's are used for the solution of different problems in atomic physics [10, 19, 20]. Some effective numerical algorithms should be developed in order to calculate the CSF's with a good accuracy and for the broad region of variation for spatial variables, free parameters, and quantum numbers.

But general properties of CSF's such as (i) integral equations and integral relations, (ii) double orthogonality, (iii) Green functions, and so on should be investigated to the same extent as the spheroidal functions, which describe a free motion in spheroidal coordinates.

Let us consider the quasiradial equation (14). Two linear independent solutions are denoted as $\Pi_{m\ell}^{(1)}(p,\xi) = \Pi_{m\ell}^{(1)}(p,a,\lambda_{m\ell};\xi)$ and $\Pi_{m\ell}^{(2)}(p,\xi) = \Pi_{m\ell}^{(2)}(p,a,\lambda_{m\ell};\xi)$. The solution $\Pi_{m\ell}^{(1)}(p,\xi)$ is a regular function for $\xi \to 1$ and irregular for $\xi \to \infty$. It is just opposite to the behaviour of the second solution $\Pi_{m\ell}^{(2)}(p,\xi)$ which is irregular for $\xi \to 1$ and regular for $\xi \to \infty$. The points $\xi = 1$ and $\xi = \infty$ are singular points of the coefficients of the differential equation for these functions (14). The asymptotic behaviour of the solutions in the vicinity of singular points reads

$$\Pi_{m\ell}^{(1,2)}(p,\xi) \sim e^{\pm p\xi}(2p\xi)^{-1\pm\alpha} \quad , \quad \alpha = \frac{a}{2p}. \tag{21}$$

Let us introduce new independent variables and a new function in Eq. (14) as follows:

$$x_\pm = p(\xi \pm 1)\,, \; (2p \le x_+ < \infty\,, \; 0 \le x_- < \infty),$$

$$\tilde{\Pi}_{m\ell}^{(\pm)}(x_\pm) = \left(\frac{\xi \pm 1}{\xi \mp 1}\right)^{m/2}\Pi_{m\ell}(p,\xi). \tag{22}$$

Here and below, the upper indices belong to $\tilde{\Pi}_{m\ell}^{(\pm)}(x_\pm)$, and lower one to $\tilde{\Pi}_{m\ell}^{(\pm)}(x_\pm)$, correspondingly. Using (22), Eqs. (14) can be transformed to equations for $\tilde{\Pi}_{m\ell}^{(\pm)}(x_\pm)$:

$$\left[\frac{d}{dx_\pm}\left(x_\pm^2\frac{d}{dx_\pm}\right) - x_\pm^2 + 2\alpha x_\pm - s(s+1)\right]\times$$

$$\times\widetilde{\Pi}_{ml}^{(\pm)}(x_\pm) + \frac{p}{x_\pm \mp 2p}\Bigg[\pm 2(m+1)x_\pm\frac{d}{dx_\pm} +$$

$$+\left(\frac{s(s+1)-\lambda_{ml}}{p} \pm 2\alpha\right)x_\pm \mp 2s(s+1)\Bigg]\widetilde{\Pi}_{ml}^{(\pm)}(x_\pm) \equiv$$

$$\equiv T_s(x_\pm)\widetilde{\Pi}_{ml}^{(\pm)}(x_\pm) + pQ_\pm(x_\pm)\widetilde{\Pi}_{ml}^{(\pm)}(x_\pm) = 0. \tag{23}$$

Operator $T_s(x)$ corresponds to a radial Schrödinger operator in spherical coordinates for the one-center Coulomb problem with charge $2Z$ with orbital momentum s. When $p \to 0$, both equations are transformed in each other $T_l R(x) = 0$ $(0 \le x < \infty)$. These equations have solutions in the form

$$R_\ell^{(1)}(x) \equiv R_{\alpha\ell}^{(1)}(x) = x^\ell e^{-x}\Phi(-\alpha+\ell+1, 2\ell+2, 2x), \tag{24}$$

$$R_\ell^{(2)}(x) \equiv R_{\alpha\ell}^{(2)}(x) = x^\ell e^{-x}\Psi(-\alpha+\ell+1, 2\ell+2, 2x). \tag{25}$$

So, the regular solution $\widetilde{\Pi}_{ml}^{(1,\pm)}(x_\pm)$ and irregular one $\widetilde{\Pi}_{ml}^{(2,\pm)}(x_\pm)$ of each of Eqs. (24) for $x_+ = 2p, x_- = 0$ have to be transformed in the limit in Coulomb radial functions $R_\ell^{(1)}(x_\pm)$ and $R_\ell^{(2)}(x_\pm)$, correspondingly. This permits us to present functions $\Pi_{m\ell}^{(i,\pm)}(x_\pm)$ as follows

$$\widetilde{\Pi}_{m\ell}^{(1,\pm)}(x_\pm) \equiv \widetilde{\Pi}_{m\ell}^{(1,\pm)}(x_\pm)(\alpha, \lambda_{m\ell}, p; x_\pm) =$$

$$= \sum_{s=0}^{\infty} h_s^\pm(p|\alpha, \lambda_{m\ell}) R_s^{(1)}(x_\pm), \tag{26}$$

$$\widetilde{\Pi}_{m\ell}^{(2,\pm)}(x_\pm) \equiv \widetilde{\Pi}_{m\ell}^{(2,\pm)}(x_\pm)(\alpha, \lambda_{m\ell}, p; x_\pm) =$$

$$= \sum_{s=0}^{\infty} \widetilde{h}_s^\pm(p|\alpha, \lambda_{m\ell}) R_s^{(2)}(x_\pm). \tag{27}$$

Let us substitute (26) and (27) into Eqs. (23) and use the recurrent relations for functions $R_s^{(1)}(x)$ and $R_s^{(2)}(x)$. We derived the system of linear algebraic equations for the coefficients $h_s^\pm(p|\alpha, \lambda_{m\ell})$, $\widetilde{h}_s^\pm(p|\alpha, \lambda_{m\ell})$ as follows:

$$\pm p\alpha_s h_{s-1}^{(\pm)} + (\beta_s - \lambda_{m\ell}) h_s^{(\pm)} \mp p\gamma_s h_{s+1}^{(\pm)} = 0,$$

$$s = 0, 1, 2 ..., h_{-1}^{(\pm)} = 0, \tag{28}$$

$$\mp p\widetilde{\alpha}_s \widetilde{h}_{s-1}^{(\pm)} + (\widetilde{\beta}_s - \lambda_{m\ell}) \widetilde{h}_s^{(\pm)} \pm p\widetilde{\gamma}_s \widetilde{h}_{s+1}^{(\pm)} = 0,$$

$$s = 0, 1, 2 ..., \widetilde{h}_{-1}^{(\pm)} = 0, \tag{29}$$

where

$$\alpha_s = \frac{2(s^2 - \alpha^2)(s+m)}{s(2s-1)(2s+1)}, \quad \beta_s = \widetilde{\beta}_s = s(s+1),$$

$$\gamma_s = 2(s+1)(s+1-m); \tag{30}$$

$$\widetilde{\alpha}_s = \frac{4(s-\alpha)(s+m)}{2s-1},$$

$$\widetilde{\gamma}_s = \frac{(s+\alpha+1)(s-m+1)}{2s+3}. \tag{31}$$

Each of the recurrent systems (29)–(30) establishes the coefficients $h_s^{(\pm)}$ ($\widetilde{h}_s^{(\pm)}$) to be exact to an arbitrary constant factor. Eqs. (27), (28) give us no possibility to write the exact expressions for the coefficients $h_s^{(\pm)}$ and ($\widetilde{h}_s^{(\pm)}$). But, according to (27), (28), we can obtain the coefficients in an approximate form. Now we have to know the asymptotic behaviour of solutions of Eqs. (14) – (15) for small values of the distance between centers. So we have to construct the asymptotic expansion of the functions $S_{m\ell}(p, \eta), \widetilde{\Pi}_{m\ell}^{(1,\pm)}(x_\pm)$ and $\widetilde{\Pi}_{m\ell}^{(2,\pm)}(x_\pm)$ in a small parameter for the fixed quantum numbers ℓ and m.

Let us consider angular spheroidal functions $S_{m\ell}(p, \eta)$. We write them using Legendre polynomials as

$$\bar{S}_{m\ell}(p, \eta) = N_{m\ell}^{-1}(p) \sum_{n=\mathrm{Ent}[(m-l)/2]}^{\infty} d_{2n+\delta}^{m\ell} P_{\ell+2n+\delta}^m(\eta);$$

$$\delta = \begin{cases} 0, & \text{if} \quad \ell - m = 2k, \\ 1, & \text{if} \quad \ell - m = 2k+1, k = 0, 1, 2 ... \end{cases} \tag{32}$$

Here and below, $\mathrm{Ent}(\rho)$ is the integer part of a real number ρ. Coefficients $d_{2n+\delta}^{m\ell}$ satisfy the recurrent conditions as follows [10]:

$$p^2 B_{2n+\delta} B_{2n+1+\delta} d_{2n+2+\delta}^{m\ell} +$$

$$+ \Big[\lambda_\delta^{(\eta)} - (\ell + 2n + \delta)(\ell + 2n + 1 + \delta) - p^2 +$$

$$+ p^2 (B_{2n-1+\delta} E_{2n+\delta} + B_{2n+\delta} E_{2n+1+\delta})\Big] \times$$

$$\times d_{2n+\delta}^{m\ell} + p^2 E_{2n+\delta} E_{2n-1+\delta} d_{2n-2+\delta}^{m\ell} = 0, \tag{33}$$

$$B_k = \frac{\ell + k + m + 1}{2\ell + 2k + 3}, \quad E_k = \frac{\ell + k - m}{2\ell + 2k - 1}.$$

Let us present the constant $\lambda_\delta^{(\eta)}$ and coefficients $d_{2n+\delta}^{m\ell}$ via p^2:

$$d_{2n+\delta}^{m\ell} = p^{|2n|} \sum_{j=0}^{\infty} [d_{2n+\delta}^{m\ell}]_{2j} p^{2j}, \quad d_\delta = 1, \tag{34}$$

$$\lambda_\delta^{(\eta)} = \sum_{j=0}^{\infty} [\lambda_\delta]_{2j} p^{2j}. \tag{35}$$

By substituting (34) – (35) into Eqs. (33) beginning with $n = 0$ and nullifying all coefficients with the same degree in p^2, we obtain recurrent relation for coefficients $[d_{2n+\delta}^{m\ell}]_{2j}$ and $[\lambda_\delta]_{2j}$can be expressed in terms of coefficients $[d_{\pm 2+\delta}^{m\ell}]_{2j-4}$ from the equations which correspond to $n = 0$:

$$[\lambda_\delta]_{2j} = -B_\delta B_{1+\delta} \left[d_{2+\delta}\right]_{2j-4} -$$

$$-E_\delta E_{-1+\delta} \left[d_{-2+\delta}\right]_{2j-4}. \tag{36}$$

To continue the procedures mentioned above, we can write the same results for the separation constant for the radial equation in the form of asymptotic series (13) in the parameter p

$$\lambda^\xi = \nu(\nu+1) + 2p^2 \frac{(\nu^2 + \nu - 1 + m^2)}{(2\nu-1)(2\nu+3)} +$$

$$+2p^2 \frac{\alpha^2(\nu^2 + \nu - 3m^2)}{\nu(\nu+1)(2\nu-1)(2\nu+3)} -$$

$$-2p^4\frac{(\nu^2-m^2)(\alpha^2-\nu^2)}{\nu^2(4\nu^2-1)}\left(\frac{(\nu^2-m^2)(\alpha^2-\nu^2)}{\nu(4\nu^4-1)}-\right.$$

$$-\frac{[(\nu+1)^2-m^2][\alpha^2-(\nu+1)^2]}{(\nu+1)(2\nu+1)(2\nu+3)}-$$

$$\left.-\frac{[(\nu-1)^2-m^2][\alpha^2-(\nu-1)^2]}{(2\nu-1)^2(2\nu-3)}\right)-$$

$$-2p^4\{...\}_{\nu\to-\nu-1}+O(p^6). \quad (37)$$

After the replacement $\{...\}_{\nu\to-\nu-1}$ in (36), we put all values multiplied by p^4 into the brackets $\{..\}$. Substituting (34) and (35) in (18) instead of the separation constants $\lambda^{(\xi)}$ and $\lambda^{(\eta)}$, in the case of s-level (36), we obtain the asymptotic formula for energetic terms of the system Z_1eZ_2:

$$E=-\frac{Z^2}{2n^2}\left\{1-\frac{4s}{3n}(ZR)^2+\frac{4s}{3n}(ZR)^3+\frac{4s}{5n}\times\right.$$

$$\times\left[s\left(\frac{5}{3n}+\frac{19}{27}\right)-1\right](ZR)^4+\frac{16s}{9n}\times$$

$$\times\left[s\left(ln\frac{2ZR}{n}+\psi(n+1)+2\gamma-\frac{2}{n}-\frac{139}{60}\right)+\frac{1}{48n^2}+\right.$$

$$\left.\left.+\frac{43}{240}\right](ZR)^5-\frac{16s^2}{9n}(ZR)^6lnR+O(R^6)\right\}. \quad (38)$$

Here, $\psi(x)$ is a logarithmic derivative of the Γ function. $Z=Z_1+Z_2$ is the complete nucleus charge, and $\gamma=0.5775$ is the Euler constant.

Conclusions

Summarizing the results of the work, we focus on its most important points. By means of the separation of variables, an additional spheroidal integral of motion $\hat{\lambda}$ is constructed, whose eigenvalues are the separation constant in the model quantum-mechanical $eZ_1Z_2\omega$ problem. The developed group treatment of the model $eZ_1Z_2\omega$ problem is related to the group treatment of the traditional quantum-mechanical problem of two Coulomb centers $eZ_1Z_2\omega$ [10—16]. But its consequence is a more rich linear algebra of two-center integrals, which contains the corresponding linear algebra of the eZ_1Z_2 problem as a partial case (i.e. at $\omega=0$). A separate publication will be devoted to the construction of such an algebra. Here, we only note that the presence of the both mentioned algebras enables and essentially simplifies the quantum-mechanical calculations of matrix elements and effective potentials in the three-body problem with Coulomb and oscillatory interactions [10]. In particular, the obtained results may appear useful in the calculation of the energy spectra of QQg-mesons. Note also that the model $eZ_1Z_2\omega$ problem can be treated under certain conditions as a step to the solution of a relativized Schrödinger equation [18] with the two-center confinement-type potential (1).

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РОЗДІЛЕННЯ ЗМІННИХ ТА ДЕЯКИЙ РОЗ'ВЯЗОК ДВОЦЕНТРОВОЇ ЗАДАЧІ З ПОТЕНЦІАЛОМ ТИПУ КОНФАЙНМЕНТА

В.Ю. Лазур, І.В. Цогла

Р е з ю м е

Знайдено групу прихованої динамічної симетрії двоцентрової квантово-механічної задачі з кулонівською та осциляторною взаємодіями. Вивчено групові властивості системи диференціальних рівнянь цієї задачі в частинних похідних. Отримано подібні розв'язки однопараметричних підгруп групи Пуанкаре та групи Галілея. Одержані розв'язки використано для обчислення енергетичного спектра цієї задачі.

РАЗДЕЛЕНИЕ ПЕРЕМЕННЫХ И НЕКОТОРОЕ РЕШЕНИЕ ДВУХЦЕНТРОВОЙ ЗАДАЧИ С ПОТЕНЦИАЛОМ ТИПА КОНФАЙНМЕНТА

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Р е з ю м е

Найдено группу скрытой динамической симметрии двухцентровой квантово-механической задачи с кулоновским и осцилляторным взаимодействиями. Изучены групповые свойства системы дифференцальных уравнений этой задачи в частных производных. Получены одинаковые решения для однопараметрических подгрупп группы Пуанкаре и группы Галилея. Полученные решения использованы для вычисления энергетического спектра этой задачи.