A NOTE ON THE TRIVIALITY OF κ-DEFORMATIONS OF GRAVITY[1]

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Some simple observations concerning certain κ-deformations in the framework of $2D$ dilaton gravity formulated as Poisson-sigma models are discussed, and the question of (non-)triviality is addressed.

Introduction

Certain astrophysical observations have motivated the study of κ-deformed Poincaré-algebras, e.g. in the context of "Doubly Special Relativity" (henceforth DSR; for a recent review and more references cf. e.g. [1, 2, 3]). It might be helpful to consider the simpler case of $D = 2$ first to settle open conceptual questions (especially concerning gravity). The first work in this direction appeared only recently [4]. In the present note when talking about κ-deformations[2] we will exclusively refer to the deformation introduced by Magueijo and Smolin [3], which differs from the original proposal [5].[3]

Moreover, it seems to be an interesting task by itself for purely mathematical reasons. Indeed, Izawa has shown a few years ago, that the most general consistent deformation (in the sense of Barnich and Henneaux [7]) of a Poisson-σ model (PSM) is again a PSM with the same number of target space coordinates [8]. Since dilaton gravity without matter in $2D$ is merely a very special PSM [9] this result applies to it as well. Thus, generic consistent deformations of dilaton gravity in $2D$ are mathematically feasible.

The description of dilaton gravity in terms of a PSM has turned out to be very fruitful – e.g. all classical solutions have been obtained locally and globally within this approach [10] and even in the presence of matter an exact quantization of geometry has been achieved [11], with interesting phenomenological applications for scattering processes [12]. For a recent review on dilaton gravity in $2D$ ref. [13] can be consulted.

In the present note we intend to discuss a very simple question in the framework of twodimensional dilaton gravity: are κ-deformations trivial or not (and in what sense are they (non)trivial)?

1. Triviality of Deformations

Let the momenta p_a and the generator of boosts J satisfy the undeformed Poincaré algebra in $2D$:

$$[p_0, p_1] = 0, \quad [J, p_0] = p_1, \quad [J, p_1] = p_0 \,. \tag{1}$$

An elementary calculation shows that the generators

$$P_a = \frac{p_a}{1 + p_0/\kappa}\,, \quad J\,, \tag{2}$$

with $a = 0, 1$ satisfy the κ-deformed relations:

$$\begin{aligned} &[P_a, P_b] = 0\,, && [J, P_0] = P_1 - \kappa^{-1} P_1 P_0\,, \\ &[J, J] = 0\,, && [J, P_1] = P_0 - \kappa^{-1} P_1^2\,. \end{aligned} \tag{3}$$

Obviously for $\kappa \to \infty$ the undeformed algebra is recovered. It is also instructive to invert (2):

$$p_a = \frac{P_a}{1 - P_0/\kappa}\,. \tag{4}$$

Clearly the deformation is algebraically trivial.[4] The whole effect is a change of variables (2). Therefore, it

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[2]Of course, κ-deformations do not automatically imply DSR, nor vice versa.

[3]By κ-deformation of Poincaré algebra only the deformation of the algebraic sector is considered in the present work, characterized by some fundamental mass parameter κ. In [5] there was proposed the framework with quantum κ deformation, introducing as well the deformed nonsymmetric coproducts in the coalgebra sector, which leads to noncommutative space-time [6].

[4]This has been pointed out in ref. [2]. This statement also reminds one of the rigidity results on deformations of Lie algebras [14].

seems natural to do the *same* change of variables in the PSM, the action of which reads [9]

$$L_{\text{PSM}} = \int_{M_2} \left[dX^I \wedge A_I + \frac{1}{2}\mathcal{P}^{IJ} A_J \wedge A_I \right] . \tag{5}$$

For dilaton gravity (assuming Lorentzian signature for definiteness) the three gauge field 1-forms are $A_I = (\omega, e^-, e^+)$, where $e^\pm$ is the dual basis of 1-forms in light-cone gauge for the anholonomic frame and ω is related to the spin-connection 1-form via $\omega^a{}_b = \varepsilon^a{}_b\omega$. The three target space coordinates $X^I = (X, X^+, X^-)$ contain the so-called "dilaton" X and two auxiliary fields $X^\pm = (X^0 \pm X^1)$ (again in light-cone representation). The Poisson-tensor $\mathcal{P}^{IJ}$ depends solely on the target space coordinates, it is antisymmetric and fulfills the Jacobi-identity $\mathcal{P}^{IN}\partial_N\mathcal{P}^{JK} + \text{cycl.}(IJK) = 0$. Due to the odd number of target space coordinates this tensor cannot have full rank and hence for non-trivial models exactly one Casimir function exists, related to the "energy" of the spacetime [15].

The simplest dilaton gravity model is obtained with the "free" Poisson tensor

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & -X^1 \\ 0 & 0 & -X^0 \\ X^1 & X^0 & 0 \end{pmatrix} , \tag{6}$$

yielding vanishing curvature and torsion. In primed coordinates

$$X^{a'} = \frac{X^a}{1 + X^0/\kappa} , \qquad X' = X \tag{7}$$

the Poisson tensor transforms as

$$\mathcal{P}^{I'J'} = \frac{\partial X^{I'}}{\partial X^I}\frac{\partial X^{J'}}{\partial X^J}\mathcal{P}^{IJ} . \tag{8}$$

Consequently, in the primed coordinates we obtain again the free PSM with an extra term,

$$\kappa^{-1} X^{1'} X^{a'} \omega \wedge e^{a'} , \tag{9}$$

which is the result of Mignemi [4].

It seems natural to deform other dilaton models (spherically reduced gravity, the CGHS model [16], the JT model [17], etc.) in the same way [18] and to study the corresponding global structure. Thus, the starting point is the (undeformed) dilaton gravity action

$$L = \int_{M_2} \left[X_a De^a + X d\omega + \epsilon \left(X^+X^-U(X) + V(X) \right) \right] , \tag{10}$$

with $\epsilon = e^+ \wedge e^-$ being the volume 2-form and $X_a De^a = X^+(d - \omega) \wedge e^- + X^-(d + \omega) \wedge e^+$ contains the torsion 2-form in light-cone representation. It is actually not necessary to restrict oneself to potentials of the type $X^+X^-U(X) + V(X)$ – a particular class of relevant counter examples can be found in ref. [19] – but for sake of simplicity this special form will be assumed in the present work because it covers all special models referred to above. The following equations will be useful

$$\frac{\partial X^{0'}}{\partial X^0} = \left(1 - \frac{X^{0'}}{\kappa}\right)^2 , \quad \frac{\partial X^{1'}}{\partial X^1} = 1 - \frac{X^{0'}}{\kappa} ,$$
$$\frac{\partial X^{1'}}{\partial X^0} = -\frac{X^{1'}}{\kappa}\left(1 - \frac{X^{0'}}{\kappa}\right) . \tag{11}$$

By employing (8), (11) together with the relation

$$X^+X^- = \frac{X^{+'}X^{-'}}{\left(1 - \frac{X^{0'}}{\kappa}\right)^2} , \tag{12}$$

the transformation law for the potentials is established:

$$U \to U' = U\left(1 - \frac{X^{0'}}{\kappa}\right) , \quad V \to V' = V\left(1 - \frac{X^{0'}}{\kappa}\right)^3 \tag{13}$$

Of course, the change of variables considered above is not an equivalence transformation: (i) it is singular; (ii) it changes the definition of the metric.

Invariants of the deformed model can be traced back easily to the undeformed case. For example, the "line element"

$$(ds)^2 = e^{0'} \otimes e^{0'}\left(1 - \frac{\eta_0}{\kappa}\right)^4 - \left(e^{1'} \otimes e^{0'} + e^{0'} \otimes e^{1'}\right)\frac{\eta_1}{\kappa} \times$$
$$\times \left(1 - \frac{\eta_0}{\kappa}\right)^3 - e^{1'} \otimes e^{1'}\left(1 - \frac{\eta_0}{\kappa}\right)^2\left(1 - \frac{\eta_1^2}{\kappa^2}\right) \tag{14}$$

with $\eta_a := X^{a'}$ results from a target space diffeomorphism (8) of the PSM, applied to the standard $(ds)^2$ and therefore must be invariant under the deformed transformations. Clearly, a description of the deformed geometry by means of (14) does not make much sense because the deformation would be without effect on classical solutions since all singularities in $e^{0'}, e^{1'}$ are compensated by corresponding zeros. This follows already from the previous general considerations, but it can be checked explicitly by plugging the explicit solutions (15) below into (14). The result is the original line element $(ds)^2 = e^0 \otimes e^0 - e^1 \otimes e^1$.

On the other hand, taking a non-invariant object as "metric" seems to be questionable from a physical point of view. For instance, the curvature scalar related to the metric presented in [4, 18] depends on the choice of gauge, which from a relativistic point of view is an inconvenient feature.

Thus there seems to be no way to evade both, the Scylla of triviality and the Charybdis of non-invariance.

2. Deformed Solutions

For a more explicit analysis we have to express the deformed "zweibein" in terms of the undeformed variables,

$$e^{0'} = \left(1 + \frac{X^0}{\kappa}\right)^2 e^0 + \frac{X^1}{\kappa}\left(1 + \frac{X^0}{\kappa}\right) e^1 ,$$
$$e^{1'} = \left(1 + \frac{X^0}{\kappa}\right) e^1 . \tag{15}$$

If we require non-triviality of the deformation a non-invariant "metric" has to be employed, as discussed in the previous section. Since there does not seem to exist a better alternative (there is no canonical choice), let us assume that the "physical metric" is constructed in the usual way from the deformed "zweibein" (15), namely $g'_{\mu\nu} = e^{a'}_\mu \otimes e^{b'}_\nu \eta_{ab}$, where η_{ab} is the flat metric.[5]

As a demonstration one can choose spherically reduced gravity (from arbitrary dimension D) as undeformed starting point. Equations of motion imply

$$X^a = \epsilon^{\mu\nu} e^a_\mu \partial_\nu X . \tag{16}$$

The discussion will be restricted to the behavior of the deformed solution in the asymptotic region of the undeformed one. This is the simplest test for the validity of the deformed model. In this region we can apply Schwarzschild gauge $(ds)^2 = \xi(dt)^2 - \xi^{-1}(dr)^2$ with $\xi(r)$ and $X(r)$ defined by [20]

$$\xi\,(r) = 1 + 2\mathcal{C}_0\,|1-a|^{\frac{a}{a-1}}\, r^{\frac{a}{a-1}} \left(\frac{B}{a}\right)^{\frac{2-a}{2(a-1)}} , \tag{17}$$

and

$$r = \sqrt{\frac{a}{B}}\frac{1}{|1-a|} X^{1-a} , \tag{18}$$

respectively. The constants in these equations have the following meaning: $a = (D-3)/(D-2)$, B is a normalization constant (which can be set to 1) and $\mathcal{C}_0$ is the value of the Casimir function (for positive/negative values a naked singularity/black hole is described, respectively; if it vanishes the Minkowskian ground state is reached). The (singular) limit $D \to \infty$, if treated with care, yields the CGHS model.

Asymptotically ($r \to \infty$) one readily obtains

$$X \sim r^{1/(1-a)}, \qquad X^1 = 0, \qquad X^0 \sim r^{a/(1-a)} . \tag{19}$$

One observes that for $\frac{1}{2} \leq a < 1$ (i.e. for spherical reduction with $4 \leq D < \infty$) the function

$$\chi(r) = 1 + \frac{X^0}{\kappa} \tag{20}$$

which characterizes the "strength" of the deformation is large, implying that asymptotically the model is being deformed noticeably.[6] This indicates that we are dealing in fact with the old Fock version [21] of the deformations which modifies physics at large distances. Indeed, the way in which the generators X^I have been interpreted corresponds to a deformation in coordinate space rather than in momentum space: for instance, the dilaton X has been regarded as some (power of) a "radius" and thus as a coordinate space entity. However, these interpretational issues are not pivotal for the present discussion.

Despite of this sizable asymptotic deformation the scalar curvature for the primed zweibein reads

$$R(r) \sim r^{-2/(1-a)} . \tag{21}$$

Thus, the asymptotically flat region remains asymptotically flat after the deformation, which is an attractive feature and indicates that it may be sensible to talk of an "asymptotic region" even in the deformed case.

It should be emphasized strongly, though, that one should not over-interpret results extracted from a "metric" which does not possess the (deformed) Lorentz invariance.

[5] The words "metric" and "zweibein" have been put under quotation marks to indicate that this nomenclature is too suggestive. Neither $e^{a'}$ nor $g'_{\mu\nu}$ transform as the notation seems to promise (unless κ is taken to ∞).

[6] It is somewhat amusing that $D = 4$ is the limiting case.

Conclusion

We have shown that κ-deformed dilaton gravity in $2D$ is either classically equivalent to a corresponding undeformed model (and thus the deformation would be trivial) or one has to deal with a "non-invariant metric".

However, this does not imply that all deformations are without effect. In particular, the question of the "correct" metric turns out to be a non-trivial one, even after imposing the invariance condition. This will be the subject of work in progress dealing with these issues in a much more comprehensive manner [22, 23].

We stress that only the classical part of the full κ-deformed Poincaré bialgebra is being used in our construction. Our procedure may be as well called "deformed gravity with an invariant energy scale". An interesting development may consist in a combination of the Poisson-σ models with the quantum algebra approach to DSR described in the recent papers [24].

It also would be worthwhile to investigate to which extent our conclusions depend on a particular choice of the basis in the κ-deformed Poincaré algebra.

During the final preparations of this proceedings contribution an e-print appeared which has partial overlap with our discussion [25]. The authors of that paper conclude that DSR is operationally indistinguishable from special relativity.

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ЗАУВАЖЕННЯ ПРО ТРИВІАЛЬНІСТЬ κ-ДЕФОРМАЦІЙ ГРАВІТАЦІЇ

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Р е з ю м е

В рамках $2D$ дилатонної гравітації, сформульованої в термінах пуассонівської сигма-моделі, обговорюються прості спостереження стосовно певних κ-деформацій, увага зосереджується навколо (не-)тривіальності цих деформацій.

ЗАМЕЧАНИЕ О ТРИВИАЛЬНОСТИ κ-ДЕФОРМАЦИЙ ГРАВИТАЦИИ

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Р е з ю м е

В рамках $2D$ дилатонной гравитации, сформулированной в терминах пуассоновской сигма-модели, обсуждаются простые наблюдения относительно определенных κ-деформаций, вопрос сосредоточен вокруг (не-)тривиальности этих деформаций.