D-BRANES ON CALABI—YAU MANIFOLDS[1]

I.M.BURBAN

Bogolyubov Institute for Theoretical Physics, Nat. Acad. Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03143, Ukraine; e-mail: mmtpitp@bitp. kiev.ua)

UDC 537.8:530.145

The description of BPS D-branes of type II superstring theories compactified on Calabi—Yau manifolds is discussed. For a subclass of D-branes defined by the derived category of coherent sheaves on a Calabi—Yau manifold at an arbitrary point of the moduli space, the property of a D-brane Π-stability is clarified.

Introduction

It turned out that, besides of the one-dimensional objects, strings, string theories describe extended higher-dimensional obiects, branes [1, 2]. Branes, in particular D-branes carrying Ramond—Ramong (RR) charges, of the superstring theory represent the theoretical and phenomenological interest. They are the probes for the study of the true nature of the stringy background and its quantum geometry. BPS D-branes are the most important objects for the better understanding of nonperturbative effects in the string theory and supersymmetric gauge field theories. They provide a nontrivial check for the conjectured web of dualities in the string theory.

D-branes have been investigated from the geometric and the field theoretical points of view. In the latter case, they appear as "defects" of various dimensions to which closed strings can be coupled. The backgrounds of the supersymmetric compactifications of superstring theories are described mainly by two-dimensional superconformal field theories.

The most general point of view of a vacuum structure is *a priori* not geometric but "highly stringy", the ambient space being defined by the abstract conformal field theory. In this regime, the equations of motion of field theories (the Yang Mills eq., the Einstein eq., and others) will not be valid and should be drastically modified. In some cases (for example, the Gepner models), there are points in the moduli space where these modified equations are connected to the ordinary equations by variations of parameters. They are called "large volume limit" points of the Kähler moduli space.

The particular cases of the vacua described by the $N = (2,2)$ superconformal field theories have geometric realizations by Calabi—Yau manifolds M. The $N = 2$ supersymmetric algebra admits a topological twisting, and the physical interest to these models implies that they lead to string theories with spacetime supersymmetries. Therefore, in a compactification of the superstring theory, we restrict ourselves by configurations of the form

$$M \times R^D, \quad D = 10 - 2d, \tag{1}$$

where M is a compact Calabi—Yau manifold of the complex dimension d, R^D is the D-dimensional Minkowski space. The Calabi—Yau threefolds provide natural arena for studying the nonperturbative string geometry.

These Ricci flat Kähler manifolds are T^2 for $d = 1$ (the resulted $D = 8$ theory has 32 supercharges), T^4 (32 supercharges) and $K3$ (16 supercharges) for $d = 2$, and Calabi—Yau threefolds M (8 supercharges) for $d = 3$. The moduli space of the Ricci flat Kähler metrics on a Calabi—Yau space M has the dimension $h^{11}(M) + 2h^{12}(M)$, h^{11} defines the dimension of module space of Kähler forms and $2h^{21}$ is the dimension of the space of inequivalent complex structures on M.

The starting point of view for the description within a world-sheet conformal field theory of D-branes wrapped around supersymmetric cycles in the Calabi—Yau space is superconformal invariant boundary conditions on world-sheet superconformal fields. The D-branes in the superstring theory are introduced by imposing the appropriate boundary conditions on the closed string coordinates in the world-sheet superconformal field theory.

The phenomenological interest is served to study the space-filling BPS D-branes in the type II superstring theories. These configurations are described by the boundary conditions preserving of the $N = 1$ world-sheet symmetry and the half of the space-time symmetry. The general classification of these conditions for the

[1]This article was presented at the II International Symposium "Fundamental Problems in Modern Quantum Theories and Experiments" (September 2—7 2002, Odessa, Ukraine) on the occasion of Professor Walter Thirring's 75th birthday.

$N = (2, 2)$ superconformal field theory has been done in [5]: the A-type boundary conditions are

$$J^+ = -\tilde{J}^-, G^+ = \pm\tilde{G}^-, e^{i\phi^+} = e^{-i\tilde{\phi}^-}, \quad (2)$$

and the B-type boundary conditions are

$$J_+ = \tilde{J}^-, G^+ = \pm\tilde{G}^-, e^{i\phi^+} = (\pm)^d e^{i\theta} = e^{-i\tilde{\phi}^-}, \quad (3)$$

where $G^{\pm}(\tilde{G}^{\pm})$ are the superpartners of the energy-momentum tensor $T(\bar{T})$ of the conformal weight $\frac{3}{2}$, $J^{\pm}(\tilde{J}^{\pm})$ are $U(1)$ currents of the conformal weight 1, $\phi^{\pm}(\tilde{\phi}^{\pm})$ are scalars associated with the bosonization of $U(1)$ currents in the left- (right-) moving sectors. The parameter θ is called the "phase" of supersymmetric cycles.

Such a compactification leads to the gauge field theories with space-time supersymmetry. The geometric realization of these boundary conditions is specified by the supersymmetric cycles in Calabi—Yau manifolds, i.e. by submanifolds X on which the fields must take the boundary values at the boundary of a Riemann surface (the complex linear bundles E over it together with their $U(1)$ connections), and supersymmetry and conformal symmetry are conserved in the boundary field theory.

In the case of the compactification type II superstring theories on the Calabi—Yau threefold, the $N = (2, 2)$ superconformal field theory with $\hat{c} = 3$ admits the tpological twisting redefinition of supercharges.

Each of the two supercharges G_+ and G_- can be choosen as the BRST charge Q, (i.e. $Q^2 = 0$), and we obtain two distinct A-type and B-type twisted topological superstring theories.

Topological BPS D-branes are the boundary conditions of the open string world-sheet of the twisted topological field theory preserving the half world-sheet superconformal and $N = 1$ spacetime supersymetries. These BPS topological D-branes of the superstring IIA and IIB theories are called, respectively, A-type and B-type topological D-branes. We consider the situation where some of spatial directions of a D-brane are wrapped around some cycles in the Calabi—Yau manifold.

The main problem in the world-sheet description of the D-branes is to find their connection with geometry beyond of the large volume points of the moduli space. One of the main conjecture in this direction claims that the holomorphic properties of the B-type D-branes do not depend on the Kähler moduli space (i.e. the holomorphic properties D-branes in the Kähler moduli space are the same as at the large volume points). In general, the properties of the D-branes are changed if Kähler moduli are varied.

The natural language for the study of classical BPS D-branes at an arbitrary point of the compactified moduli space of the type II superstring theories on the Calabi—Yau manifold is a homological algebra and the theory of derived categories. In this approach, topological B-type D-branes are descibed by the category of the boundary conditions of topological open strings which, in turn, form a category equivalent to the derived category of the coherent sheaves $D(\mathrm{coh}M)$ on the Calabi—Yau manifold [8]. As we mentioned above, the topological D-branes decribed by the derived category of coherent sheaves do not depend on the Kähler moduli space. To proceed from topological branes to physical ones, one has to add the Kähler dependent information, in particular, a notion of stability.

1. Calabi—Yau Spaces

The investigations of possible options of background configurations of the superstring theory have shown that they are constrained. The only known way to satisfy these constrains is to take Calabi—Yau manifolds which are compact, complex Kähler manifolds with Ricci flat Kähler metric. We recall the primary definitions and mathematical statements connected with these manifolds which are of their application to the D-brane theory.

- A linear connection on a differentiable manifold M generated by a bundle $O(M)$ of orthogonal frames is called the metric connection.
- Every Riemannian manifold admits only one metric connection with zero torsion (the Riemannian connection).
- A complex structure on a real manifold M is a tensor field J_μ^ν such that $J_\mu^\nu J_\nu^\sigma = -\delta_\sigma^\mu$ and a certain integrability condition.
- A complex manifold is a real manifold suplied with complex structure.
- Let $g_{i\bar{k}}$ be a Hermitian metic. If the form $J = g_{i\bar{k}} dz^i \wedge d\bar{z}^k$ is closed, then J and $g_{i\bar{k}}$ are called, respectively, the Kähler form and the Kähler metric.
- A Hermitian metric is a Riemaniann metric $g_{\mu\nu}$ such that $J_\rho^\mu J_\sigma^\nu g_{\mu\nu} = g_{\rho\sigma}$. A Hermitian metric is a Kähler one if and only if J_ν^μ is covariantly constant with respect to the connection defined by $g_{\mu\nu}$.

- If we denote the Hodge numbers $h^{p,q} = \dim H^{p,q}_{\bar\partial}(M)$, then, for every Kähler threefold, we have the Hodge diamond:

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0}. & & & \end{array} \qquad (4)$$

- A Ricci form is defined by $R = iR_{i\bar k}dz^i \wedge d\bar z^{\bar k}$, where $R_{i\bar k} = \partial^2(\ln\det g)/\partial z^i \partial \bar z^{\bar k}$ is the Ricci tensor of the Kähler metric. The Ricci form is closed, $dR = 0$. It defines a class of the cohomology droup $H^2(M,R)$. This class does not depend on a Kähler metric and is the first Chern class c_1 of the manifold M.

- If $c_1 \neq 0$, then the form R is not exact and there is no Ricci flat metric on the manifold M.

- The Hodge diamond for Calabi—Yau threefold take the following form:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & & & h^{1,1} & & & \\ & & 0 & & 0 & & \\ & & & 1. & & & \end{array} \qquad (5)$$

We see that the only nonzero Hodge numbers in this case are $h^{1,1} = h^{2,2}$ and $h^{2,1} = h^{1,2}$. The fact is that $h^{3,0} = 1$ reflects the property of a canonical linear bundle (canonical sheaf) on M to be trivial and therefore it admits a global holomorphis section Ω. The three-form Ω allows us to establish the canonical isomorphism between $H^{2,1}(M)$ and $H^1(M,T)$, where T is a holomorphic tangent bundle. We have $\omega : H^1(M,T) \to H^{2,1}(M)$ or, in the local coordinats, $B^j_{\bar i} dX^{\bar i} \frac{\partial}{\partial X^j} \to \Omega_{ijl} B^j_{\bar i} dX^i dX^l dX^{\bar i}$. These groups are connected with the special class of fields (marginal operators) which generate deformations of the theory preserving the superconformal structure in a superconformal field theory.

A calibration ϕ on a Riemannian manifold (M,g) is a p-form on M which is closed, $d\phi = 0$, and provides a lower bound for the volume form: for any p-dimensional oriented linear subspace V of the tangent space at any point of the manifold, one has $\phi|_V \leq (\text{volume})_V$ (here $\phi|_V = \alpha \cdot (\text{vol})_V$ for some $\alpha \in R$ and $\phi|_V \leq (\text{vol})_V$ if $\alpha \leq 1$).

Let Σ be an oriented submanifold of the manifold M of the dimension p. Then each tangent space $T_x\Sigma$ for $x \in \Sigma$ is an oriented tangent p-plane. We say that Σ is a calibrated submanifold if $\phi|_{T_x\Sigma = (\text{vol})|T_x\Sigma}$ for all x. All calibrated submanifolds are minimal submanifolds. This is true for compact calibrated manifolds but noncompact calibrated submanifolds are locally volume minimized.

A calibration ϕ on (M,g) can only have nontrivial calibrated submanifolds if there exists an oriented tangent p - plane V on M with $\phi_{|V} = \text{vol}_V$. For instance, $\phi = 0$ is the calibration on M, but it has no calibrated submanifolds.

Every Calabi—Yau manifold M ($\dim M = m$) with the complex volume m-form Ω and Kähler form J admits only two types of calibrations:

- $\phi = J$, the calibrated submanifolds correspond to the cyles defined in (2).

- $\phi = \text{Re}\Omega$, the calibrated submanifolds correspond to the cycles in (3).

Let us consider the complex projective space CP^n formed by taking $n+1$ the complex coordinates and identify

$$(z_1, \ldots, z_{n+1}) \approx (\lambda z_1, \ldots, \lambda z_{n+1}) \qquad (6)$$

for any complex number λ. This identification is important because it makes CP^n to be a compact manifold. CP^n is a Kähler manifold but not a Calabi—Yau one. Many Calabi—Yau manifolds can be obtained as its submanifolds. In particular, let P be a homogeneous polynomial

$$P(z_1, \ldots, z_{n+1}) = \lambda^k P(\lambda z_1, \ldots, \lambda z_{n+1}). \qquad (7)$$

The submanifold M of the manifold CP^n defined by equation

$$P(z_1, \ldots, z_{n+1}) = 0 \qquad (8)$$

is a Kähler manifold in general. But if we choose $k = n+1$, then this manifold has $c_1 = 0$ and therefore will be a Calabi—Yau manifold. For $n = 3$ the Hodge numbers of a quintic hypersurface in CP^4 are $(h^{11}, h^{21}) = (1, 101)$.

2. The Donaldson—Uhlenbeck—Yau Theorem

A holomorphic bundle E of rank r on M is given by a collection of trivial bundles $U_i \times C^r$ over the open cover $\{U_i\}_{i\in I}$ of M patched together by the holomorphic transition functions $\phi_{ij} : U_i \cap U_j \rightarrow GL(r, C)$. A holomorphic section of E (locally) are the the r-tuple of holomorphic functions which can be patched under ϕ. A Cauchy—Riemann $\bar{\partial}$-operator is patched together of the local operators $\bar{\partial} \circ \phi_{ij} = \phi_{ij} \circ \bar{\partial}$ to give a linear map

$$\bar{\partial}_E : \Omega^0(E) \rightarrow \Omega^{0,1}(E), \tag{9}$$

whose kernel is coincides with the space of the holomorphic sections of E. Here, $\Omega^{p,q}(E)$ denotes C^∞ forms of the type (p, q) with values in E. Any connection

$$d_A : \Omega^0(E) \rightarrow \Omega^1(E) = \Omega^{1,0}(E) \oplus \Omega^{0,1}(E) \tag{10}$$

splits into $\partial_A \oplus \bar{\partial}_A$ according to the above decomposition. d_A is compatible with the holomorphic structures of E if $\bar{\partial}_A = \bar{\partial}_E$. If we define Hermitian metrics h on E, then there exists a unique connection d_A compatible with the metric and holomorphic structure (i.e. $d_A(h) = 0, \bar{\partial}_A = \bar{\partial}_E.), (d_A s_i = \partial_A s = \Sigma \partial h_{ij}(h^{-1})_{jk} s_k)$. The curvature of this connection is

$$F_A = d_A^2 = F_A^{0,2} \oplus F_A^{1,1} \oplus F_A^{2,0} =$$

$$= \partial^2 \oplus (\partial\bar{\partial}_A + \partial_A \bar{\partial}_A) \oplus \bar{\partial}_A^2. \tag{11}$$

We have $\bar{\partial}_A^2 = 0$ so that $F_A^{0,2} = 0$ and $F_A^{2,0} = 0$ by conjugation.

On the contrary, a connection on any complex bundle satisfying $F_A^{0,2} = 0$ defines a holomorphic structure on that bundle, $\bar{\partial}_A^2 = 0$. The $F_A^{0,2} = 0$ is the integrability condition for that equation. If we attempt to solve the remaining Hermitian—Yang—Mills equation

$$F_A^{1,1} = 0 \tag{12}$$

on a Calabi—Yau manifold M with $c_2(E) = c_2(TM)$, then solutions will exist for E holomorphic if and only if

$$\mu(E') < \mu(E) \tag{13}$$

for all $0 \longrightarrow E' \longrightarrow E$, where

$$\mu(E) = \frac{1}{rkE} \int_{\mathcal{M}} c_1(E) \wedge J^{d-1}, \tag{14}$$

i.e. when the holomorphic bundle E is μ-stable (the Donaldson — Uhlenbek—Yau theorem).

At large volume points of a D-brane moduli space, the type-B BPS branes are described by Hermitian holomorphic vector bundles whose stability defines one of the D-branes.

3. Central $Z(E)$ Charges and D-brane RR Charges $Q(E)$

There exists the relation between RR charges of D-branes and the topology of a Chan—Paton vector bundle. The RR charge of a wrapped D-brane is defined by the topology of the embedded cycle $f : S \rightarrow M$ and topology of the Chan—Paton vector bundle E :

$$Q = ch(f_! E)\sqrt{\hat{A}(TM)}, \tag{15}$$

where TM is the tangent bundle to M, $f_!$ is the Gysin map. But D-brane charges can be understood most naturally as classes in topological K-theory. The crucial observation for this was the following: addional brane — antibrane pairs with the same gauge bundle do not change the total charge. In what follows, we shall consider the type II superstring theory on a complex variety and wrapped branes on a complex subvariety. The BPS central charge $Z(E)$ of a D-brane is determined by its RR charges Q_i by the relation

$$Z(E) = \Sigma Q_i(E)\Pi^i = \int\limits_S \exp\{-B - iJ\} ch(E)\sqrt{td(T_x)} +$$

$$= \text{quantum} \quad \text{corrections}. \tag{16}$$

4. The Classical Geometry and the Boundary Conformal Field Theory of D-branes on Calabi—Yau Manifolds

At a large volume limit points and large-complex structure limit, BPS D-branes are wrapped on holomorphic cycles or special Lagrangian submanifolds of the Calabi—Yau manifold M. These are submanifolds X on which the open strings can end. In the presence of D-branes, the boundary conditions for open strings of sigma models are modified in such a way that Dirichlet boundary conditions are allowed in addition to Neumann boundary conditions. The role of the equations of motion in this case is played by the conformal invariance. The requirement that the boundary conditions preserve the superconformal invariance impose the constraints on a submanifold X. Ooguri at al. [5] specified the boundary conditions on the world sheet for $N = (2, 2)$ supersymmetry generators and explained their geometric significance. The further efforts have been concentrated on extending the boundary conformal field description of D-branes to the case of Calabi—Yau manifolds. In the supersymmetric Calabi—Yau sigma model, there are two kinds of supersymmetric cycles:

- the A-type supersymmetric cycles (X, E). The submanifold X is a special Lagrangian submanifold of M with a flat $U(1)$ connection such that

$$\begin{aligned} i^*\omega &= 0, (\text{“Lagrangian“}), \\ i^*[\text{Im}e^{-i\theta}\Omega] &= 0, (\text{“special“}), \\ F_A &= 0, \end{aligned} \tag{17}$$

where F_A is the curvature of A, and $i: X \to M$.

- the B-type supersymmetric cycles (X, E). The submanifold X is a complex submanifold of M with a flat $U(1)$ connection such that

$$F_A^{0,2} = 0, \quad \text{Im}(e^{-i\theta}(\omega + F_A)^n) = 0, \tag{18}$$

where $n = \text{dim}M$.

For a more general conformal field theory, we don't have the Lagrangian description of the theory, so the classification and interpretation of boundary conditions is not so straightforward. The branes define Dirichlet and Neumann boundary conditions in nonlinear sigma models with a Calabi—Yau treefold target space:

- the A-type D-branes are 3-branes wrapped on what are called special Lagrangian submanifolds;
- the B-type D-branes are $2p$-branes wrapped on holomorphic cycles carrying holomorphic vector bundles.

At the first sight, this notation may contradict the discussion of the previous section. Since $2p$ cycles and masses of B branes are controlled by Kähler moduli (and thus are calculable in the A-twisted topological closed string theory), 3-cycles and masses of A branes are controlled by the complex structure moduli. Nevertheless, it is correct because, in going from the open to closed string channel, the boundary conditions of the $U(1)$ current change a sign interchanging A and B twistings. This switch has an important consequence (if we combine it with the known properties of CY sigma models): the A-twisted models receive world-sheet instanton corrections, and the B-twisted models receive no quantum corrections. This means (physically) that the $N = 2$ superpotential in the compactified IIB theory (it depends only on complex structure moduli) is classicaly exact. In the large volume limit, the central charges and the masses of A-type branes are already exact. The masses of the BPS B-type D-branes of type II superstring theories receive world-sheet instanton corrections.

In spite of a detailed undrestanding of the D-brane dynamics in a flat space, their behaviour in the abstract conformal field theory is less understood. The string vacua, where the D-branes spectra are essentialy interesting, form $N = (2, 2)$ superconformal field theory. The moduli space of the $N = (2, 2)$ SCFT are affected by the string quantum corrections and has a rich phase structure. In the geometric phase of the string vacua, one can use the classical geometry description. In the non-geometric phase, the semiclassical description is broken. In the geometric phase, a new degree of freedom can be described by semiclassically as a gauge field living on the various submanifolds of the space-time. Therefore, in the geometric phase, the D-branes can be described by K-theory classes (rather than by singular cohomology classes).

However, such an explicit and intuitive description is lacking in the deep stringy regimes. In these regions of the moduli space, one has to rely on the abstract SCFT techniques in order to classify the D-brane charges and study their dynamics. In the closed string case, the string propagating on the Calabi—Yau manifolds can be described by variety techniques depending on which region of the complex structures and Kähler moduli of the Calabi—Yau manifolds one concentrates.

A complete microscopic description of D-branes wrapped on supersymmetric cycles is available in the case where these cycles are submanifolds in a flat space like tori. This description is extended to spaces where the technique of conformal field theories constructed from fields can be easily applied as in the case of orbifolds. However only recently, the case of D-branes living in nontrivial curved spaces and wrapped on the supersymmetric cycles in these spaces has been investigated from the microscopic point of view.

The spectrum of a brane can depend on the particular vacuum (point in the moduli space). For a given $N = (2, 2)$ supersymmetry, this dependence of the spectrum of moduli is highly constrained: it is well known that the BPS spectrum can be change only on lines of *marginal stability* , defined by the condition

$$\text{Im}\frac{Z(Q_1)}{Z(Q_2)} = 0. \tag{19}$$

The problem of finding the spectrum of wrapped branes on Calabi—Yau manifolds and deciding whether it changes on string scales is not trivial.

The mirror symmetry will exchange the spectrum and world-sheet theories of A-type branes on a Calabi—Yau M isomorphic to that of the B-type branes on its mirror W. The branes define Dirichlet and Neumann boundary conditions in nonlinear sigma models with the Calabi—Yau treefold target space.

5. The Category and Topological D-branes of the Type IIB Superstring Theory

The topological string theory is obtained by twisting the $N = (2,2)$ superconformal field theory. We will consider the type IIB model. Recall that the $N = (2,2)$ superconformal algebra is generated by the holomorphic $T(z), G^{\pm}(z), J(z)$ and antiholomorphic $\tilde{T}(\bar{z}), \tilde{G}^{\pm}(\bar{z}), \tilde{J}(\bar{z})$ currents corresponding to the generators of the superconformal algebra from relations (2),(3). The twisted topological sigma model for the type IIB superstring theory with boundaries $C_k \in M$ and with the Chan—Paton factors living in the finite-dimensional complex vector bundle E on the Calabi—Yau manifold M was studied in [6].

$$\begin{array}{rcl} T(z) \longrightarrow T(z)_{\rm top} & = & T(z) \pm \frac{1}{2}\partial J(z) \\ \tilde{T}(\bar{z}) \longrightarrow \tilde{T}(\bar{z})_{\rm top} & = & \tilde{T}(\bar{z}) \pm \frac{1}{2}\tilde{J}(\bar{z}) \end{array} \qquad (20)$$

and, as a result, we will have shift of the conformal weight of all operators in the theory, $h \to h_{\rm top} = h - \frac{1}{2}q$, where q is the $U(1)$ charge.

Deformation (20) generates two types of topological string models: type A when signs are opposite in (20), and type B when, in (20), signs are the same. One of the supercharges, for example, $G^{+}_{-\frac{1}{2}}$ of the $N = (2,2)$ superalgebra can be reinterpreted as the BRST operator $Q = G^{+}_{-\frac{1}{2}}$. Hence, $Q^2 = 0$ and the Hilbert space of the twisted theory can be defined as the Q-cohomology of the Hilbert space states of the original conformal field theory. The $U(1)$ charge becomes a ghost charge and current $J(z)$ becomes the ghost number operator.

Note that type A and type B twists (20) are comparable only with A and, respectively, B boundary conditions (2),(3). The open-closed type B theory was investigated in [6]. The open string in this case can be consistently coupled to a holomorphic bundle E on M [6].

The Fermi fields $\eta^{\bar{i}}, \theta^{\bar{i}}$ are the sections of $\Phi^*(T^{0,1}(M))$, ρ^i is the section of $T^*(\Sigma) \otimes \Phi^*(T^{1,0})$, and $\theta_j = g_{j,\bar{i}}\theta^{\bar{i}}$. The Lagrangian of the B model is

$$L = t\int\limits_{\Sigma} d^2z\big(g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + i\eta^{\bar{i}}(D_z\rho^i_{\bar{z}} + D_{\bar{z}}\rho^i_{\bar{z}})g_{i\bar{i}} +$$

$$+ i\theta_i(D_{\bar{z}}\rho^i_z - D_z\rho^i_{\bar{z}}) + R_{i\bar{i}j\bar{j}}\rho^i_z\rho^j_{\bar{z}}\eta^i\theta_k g^{kj}\big). \qquad (21)$$

This Lagrangian is invariant under the fermionic supersymmetry. The supersymmetries of the model are generated by the infinitesimal BRST transformations

$$\begin{array}{rcl} \delta\phi^i & = & 0, \\ \delta\phi^{\bar{i}} & = & i\epsilon\eta^{\bar{i}}, \\ \delta\eta^{\bar{i}} & = & \delta\theta_i \quad = \quad 0, \\ \delta\rho^i & = & -\epsilon d\phi^i. \end{array} \qquad (22)$$

This model can be coupled to a background gauge field A on each boundary C_k:

$$L_{C_k} = \int\limits_{C_k} \Phi^*(A) - \eta^{\bar{i}}F_{\bar{i}j}\rho^j. \qquad (23)$$

The Lagrangian $L + L_{C_k}$ preserves the BRST symmetry if and only if

$$F_{\bar{i}\bar{j}}(A) = 0, \qquad (24)$$

that is, the operator $\partial_A = \bar{\partial} + A^{0,1}$ defines an integrable holomorphic structure on the gauge bundle E and $\bar{\partial}_E$ denotes the covariant Dolbeault operator on E. The BRST operator Q is defined by $\delta\Lambda = -i\alpha\{Q,\Lambda\}$ for any operator Λ. The Hilbert space of states in this topological field theory is given by the cohomology of the BRST operator Q. The operators corresponding to the open string states ϕ belong to the Dolbeault cohomology $H^{0,p}_{\bar{\partial}}(M, EndE)$ [6].

The $U(1)$ supercharge will define a grading of the superalgebra, in which Q will have degree 1. For open string states, the operators correspond to the bundle-valued Dolbeault cohomology

$$\phi \in H^{p,0}(M, {\rm End}E), \qquad (25)$$

where p is the ghost number of the operator ϕ. The topological operators will have integral $U(1)$ charges.

To each D-brane the phase shift between the left-moving and right-moving spectral flow operators at the boundary ending on that D-brane corresponds. This is the grade of a D-brane [8].

The grading of the B-type branes is defined by

$$\frac{1}{\pi}\,{\rm Im}\log\int_X e^{B+iJ}{\rm ch}E\,\sqrt{Td\,(T_X)} + \ldots.$$

This grading depends on $B + iJ$, and therefore it will not appear as physical data in the topological B-model. The Witten's B-model needs to be modified in order to include grading. For this, the collection of D-branes is decomposed as

$$E = \bigoplus_{-\infty}^{\infty} E^n,$$

where only the finite number of bundles E^n is nonzero. The notation of the grading in the Witten's B-model leads to the substitution of the relation (25) by

$$\phi \in H^{0,p}(M,(E^m)^\vee \otimes E^n), \tag{26}$$

and now the ghost number of the operator ϕ is $p-m+n$. Using the properies of the sheaf cohomology [11], relation (26) can be rewritten as

$$\phi \in \mathrm{Ext}^p(\mathcal{E}^m,\mathcal{E}^n), \tag{27}$$

where $\mathcal{E}^n$ is the locally free sheaf corresponding to the vector bundle E of rank n. The product of operators is defined by the Yoneda pairing:

$$\mathrm{Ext}^p(\mathcal{E}^m,\mathcal{E}^n)\otimes \mathrm{Ext}^q(\mathcal{E}^n,\mathcal{E}^p)\to \mathrm{Ext}^{p+q}(\mathcal{E}^m,\mathcal{E}^p). \tag{28}$$

It is easy to see that, for the finite collection of nontrivial sheaves $\mathcal{E}^n$, and the sets of operators ϕ of the form (27) define the category of D-branes:

- The objects are locally free sheaves $\mathcal{E}^n$ (D-branes).
- The morphisms are operators ϕ defined in (27) (the open strings stratched between D-branes).

The verification of the axiom of the category is trivial. For this, it is necessary to take a deformed topological conformal field theory by adding a topological world-sheet action given by the term

$$\delta_\phi = t\int_{G_k}\{G,\phi\} \tag{29}$$

to it. It is amounted to the adding of the boundary term equation $\delta Q = t\phi = d$ to the BRST current Q_0. The operator ϕ is supported on the boundary C_k. The deformed topological field theory will conserve the properties of the twisted $N=2$ superconformal field theory if the ghost number h of the operator ϕ will be equal to one (in the nontwisted theory, ϕ is a marginal operator). The nontrivial and most important step is the choice of the deformation operators ϕ living in $\mathrm{Ext}^0(\mathcal{E}^n,\mathcal{E}^{n+1}) = \mathrm{Hom}(\mathcal{E}^n,\mathcal{E}^{n+1})$. Now the content of the deformed topological field theory is defined by the collection of locally free sheaves $\mathcal{E}^n$ and by the holomorphic mapping d_n (which corresponds to the marginal operators of the deformed topological field theory)

$$d_n : \mathcal{E}^n \to \mathcal{E}^{n+1}. \tag{30}$$

The BRST operator of the theory becomes

$$Q = Q_0 + \delta Q = Q_0 + d(\sigma=0) + d(\sigma=\pi). \tag{31}$$

The nilpotency of this operator yields

$$d_{n+1}d_n = 0, \tag{32}$$

that is, the deformed topological field theory is defined by the bounded complexes

$$\mathcal{E}^\bullet : \ldots \xrightarrow{d_{-2}} \mathcal{E}^{-1} \xrightarrow{d_{-1}} \mathcal{E}^0 \xrightarrow{d_0} \mathcal{E}^1 \xrightarrow{d_1} \ldots . \tag{33}$$

For the new deformed topological field theory, one needs to find the cohomology of the new BRST operator Q. Let us find the operator algebra of the new deformed topological field theory. One needs to include both boundary operators: a boundary operator defined for a given boundary condition and a boundary condition-changed operator. For each n we consider the direct sum of two sheaves $\mathcal{E}^n \oplus \mathcal{F}^n$ and homomorphisms

$$\ldots \longrightarrow \mathcal{E}^n \oplus \mathcal{F}^n \xrightarrow{\begin{pmatrix} d_n^E & 0 \\ 0 & d_n^F \end{pmatrix}} \mathcal{E}^{n+1} \oplus \mathcal{F}^{n+1} \longrightarrow \ldots \tag{34}$$

This complex gives a notation of two branes:

- $\mathcal{E}^\bullet$ — the boundary condition for the start of the string,
- $\mathcal{F}^\bullet$ — the boundary condition for the end of the string.

In order to work with such strings, we have to develop a technique collapsing a double complex into a single complex. Using relation (34), we can form the double complex of the sheaves as

$$\begin{array}{ccccc} & \downarrow d_1^E & & \downarrow d_1^E & \\ \xrightarrow{d_{-1}^F} & \mathrm{Hom}(\mathcal{E}^1,\mathcal{F}^0) & \xrightarrow{d_0^F} & \mathrm{Hom}(\mathcal{E}^1,\mathcal{F}^1) & \xrightarrow{d_1^F} \\ & \downarrow d_0^E & & \downarrow d_0^E & \\ \xrightarrow{d_{-1}^F} & \mathrm{Hom}(\mathcal{E}^0,\mathcal{F}^0) & \xrightarrow{d_0^F} & \mathrm{Hom}(\mathcal{E}^0,\mathcal{F}^1) & \xrightarrow{d_1^F} \\ & \downarrow d_{-1}^E & & \downarrow d_{-1}^E & . \end{array} \tag{35}$$

From (35) we can form the single complex

$$\ldots \longrightarrow \mathrm{Hom}^0(\mathcal{E}^\bullet,\mathcal{F}^\bullet) \xrightarrow{\vec{d}_0} \mathrm{Hom}^1(\mathcal{E}^\bullet,\mathcal{F}^\bullet) \xrightarrow{\vec{d}_1} \ldots, \tag{36}$$

where $\mathrm{Hom}^q(\mathcal{E}^\bullet,\mathcal{F}^\bullet) = \oplus_n \mathrm{Hom}(\mathcal{E},\mathcal{F}^{n+q})$ and $\vec{d} = d^E + d^F$. The cohomology of complex (36) is $H^{n-m}\mathrm{Hom}^\bullet(\mathcal{E}^\bullet,\mathcal{F}^\bullet)$ (the calculation can be fulfiled using the spectral sequence with the initial term $(E_0)_m^n = \mathrm{Hom}(\mathcal{E}^m,\mathcal{F}^n)$. The another complex

$$\begin{array}{ccccc} & \uparrow Q_0 & & \uparrow Q_0 & \\ \xrightarrow{\bar{d}} & \Omega^1(\mathrm{Hom}^0(\mathcal{E}^\bullet,\mathcal{F}^\bullet)) & \xrightarrow{\bar{d}} & \Omega^1(\mathrm{Hom}^1(\mathcal{E}^\bullet,\mathcal{F}^\bullet)) & \xrightarrow{\bar{d}} \\ & \uparrow Q_0 & & \uparrow Q_0 & \\ \xrightarrow{\bar{d}} & \Omega^0(\mathrm{Hom}^0(\mathcal{E}^\bullet,\mathcal{F}^\bullet)) & \xrightarrow{\bar{d}} & \Omega^0(\mathrm{Hom}^1(\mathcal{E}^\bullet,\mathcal{F}^\bullet)) & \xrightarrow{\bar{d}} \\ & \uparrow Q_0 & & \uparrow Q_0 & \end{array} \tag{37}$$

defines the spectral sequence with the initial term $E_2^{p,q} = H^p(X, H^q(\mathrm{Hom}^q(\mathcal{E}^\bullet, \mathcal{F}^\bullet)))$ which converges to the cohomology group $\mathrm{H}_Q^{p+1} = \mathrm{Hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$.

It turns out futhermore that this group coincides with the cohomology group of the chain complex

$$\mathrm{hom}^\bullet : \ldots \to \mathrm{hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet_{\mathrm{inj}}) \to \\ \to \mathrm{hom}^{P+1}(\mathcal{E}^\bullet, \mathcal{F}^\bullet_{\mathrm{inj}}) \to \ldots. \quad (38)$$

Here,

$$\mathrm{hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet_{\mathrm{inj}}) = \oplus_n \mathrm{Hom}(\mathcal{E}^n, \mathcal{F}^{n+P}_{\mathrm{inj}}). \quad (39)$$

Every locally free sheaf $\mathcal{F}^n$ has the injective resolution

$$0 \to \mathcal{F}^n \to I^0(\mathcal{F}^n) \to I^1(\mathcal{F}^n) \to \ldots, \quad (40)$$

where $I^s(\mathcal{F}^n)$ are the injective objects of quasicoherent sheaves. We can construct the complex $\mathcal{F}^\bullet_{\mathrm{inj}}$ of the injective objects of a locally free sheaf $\mathcal{F}^n$

$$\mathcal{F}^\bullet_{\mathrm{inj}} : \ldots \to \mathcal{F}^0_{\mathrm{inj}} \to \ldots \to \mathcal{F}^s_{\mathrm{inj}} \to \ldots, \quad (41)$$

where $\mathcal{F}^n_{\mathrm{inj}} = \oplus_s I^s(\mathcal{F}^{n-s})$.

The operators of the deformed topological field theory corresponding to open string states belong to $\mathrm{Hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$, where P is their ghost number and the operator product is defined by the Yoneda product

$$\mathrm{Hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \times \mathrm{Hom}^Q(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}^{P+Q}(\mathcal{E}^\bullet, \mathcal{F}^\bullet). \quad (42)$$

5.1. The Category of Topological D-branes

The category of all possible topological D-branes $T(M)$ on a Calabi—Yau manifold M up to the physical equivalence can be defined as the category of all possible topological field theories of the type considered above with the target space M. In another way, $T(M)$ is the category with objects of all possible bounded complexes of locally free sheaves on M and the morphisms of the open string operators given by $\mathrm{Hom}^P(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. These definitions take place up to physical equivalence. For a final definition of the category $T(M)$, it is necessary to take into account the equivalence relations. Two D-branes described by complexes of locally free sheaves are physically equivalent if and only if

$$\begin{array}{rcl} \mathrm{Hom}^P(\mathcal{E}_1^\bullet, \mathcal{F}^\bullet) &=& \mathrm{Hom}^P(\mathcal{E}_2^\bullet, \mathcal{F}^\bullet), \\ \mathrm{Hom}^P(\mathcal{F}^\bullet, \mathcal{E}_1^\bullet) &=& \mathrm{Hom}^P(\mathcal{F}^\bullet, \mathcal{E}_2^\bullet). \end{array} \quad (43)$$

for all P and $\mathcal{F}$.

Now we can give the precise definition of the category of topological D-branes $T(M)$ as the category of all complexes of locally free sheaves on M with the above-mentioned morphisms of the modulo above-mentioned relations.

Let us consider the functor $F : K_{LF}(M) \to \mathrm{T}(M)$ from the category of complexes of locally free sheaves to the category of D-branes such that:

- chain complexes in $\mathrm{K}_{LF}(M)$ map to the corresponding D-branes in $T(M)$;
- chain maps of $K_{LF}(M)$ transform to those of $\mathrm{Hom}^0(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ in $\mathrm{T}(M)$.

A quasi-isomorphism $f : \mathcal{E}_1^\bullet \to \mathcal{F}_2^\bullet$ of the category $K_{LF}(M)$ induces a map

$$f^* : \overset{P}{\mathrm{Hom}}(\mathcal{E}_1^\bullet, \mathcal{F}^\bullet) \to \overset{P}{\mathrm{Hom}}(\mathcal{E}_1^\bullet, \mathcal{F}^\bullet), \quad (44)$$

or, graphically for $P = 1$,

$$\begin{array}{ccccccccc} \longrightarrow & \mathcal{E}_1^0 & \longrightarrow & \mathcal{E}_1^1 & \longrightarrow & \mathcal{E}_1^2 & \longrightarrow & \\ & \downarrow f & & \downarrow f & & \downarrow f & & \\ \longrightarrow & \mathcal{E}_2^0 & \longrightarrow & \mathcal{E}_2^2 & \longrightarrow & \mathcal{E}_2^2 & \longrightarrow & \\ \searrow & & \searrow & & \searrow & & \searrow & \\ \longrightarrow & \mathcal{F}^0 & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \mathcal{F}^2 & \longrightarrow, & \end{array} \quad (45)$$

which is an isomorphism in $T(M)$. Indeed, f^* in (44) is the canonical isomorphism.

The category $T(M)$ satisfies almost all the conditions to say that it is the derived category of the category $\mathrm{K}_{LF}(M)$. The last is not Abelian. Therefore, we change the category of the complexes of locally free sheaves $\mathrm{K}_{LF}(M)$ by the category of complexes of coherent sheaves $\mathrm{Kom}(M)$ and denote the corresponding derived bounded category by $D^b(M)$.

The derived category $D(\mathcal{A})$ of an Abelian category $\mathcal{A}$ is defined by the following universal properties:

1. there exists a functor Q from category $\mathrm{Kom}(\mathcal{A})$ to $D(\mathcal{A})$ which transform quasi-isomorphisms to isomorphisms;
2. for any other category $D'(\mathcal{A})$ and a functor F satisfying similar to the previous condition, there exists a functor G such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Kom}(\mathcal{A}) & \xrightarrow{Q} & D(\mathcal{A}) \\ & \overset{F}{\searrow} & \downarrow G \\ & & D', \end{array} \quad (46)$$

i.e. $F = GQ$.

In such a way, we can obtain the bounded derivative category $D^b(M)$ of a category of coherent sheaves $\mathrm{Kom}(M)$ for which $K_{LF}(M)$ is a subcategory.

If we replace the category of the complexes of coherent sheaves Kom(M) in (46) for this case by the category of the complexes of locally free sheaves $K_{LF}(M)$ and the category D' by the category of the topological D-branes $T(M)$, take the functor F constructed above for $T(M)$ which has the same objects as $T(M)$ but the set of the morhisms is restricted to $\mathrm{Hom}^0(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$. In this case, there is the functor $G : D^b(M) \to T_0(M)$ which establishes the equivalence of the categories. It is easy to see that the functor G is full, faitful, and dense i.e., it determines the equivalence of the categories. The relation $\mathrm{Hom}^p(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}^0(\mathcal{E}, \mathcal{F}[p])$ ensures the existence of branes in the category $T(M)$ connected by the strings with nonzero ghost numbers.

5.2. The Category of Physical D-branes

In the previous sections, we have elaborated the construction of the category of boundary conditions in the topological string theory (a category of topological D-branes). The category of BPS boundary conditions in the conformal field theory (a category of physical D-branes) will be a subcategory of the category of topological D-branes. Every physical D-brane has a topological analog but not vice versa. The physical D-branes are, first of all, determined by the stability property [9, 10, 15]. These are objects of $D^b(\mathrm{coh}M)$ which are Π stable against all possible decays. The basic characteristcs of the BPS D-brane E are its RR charge $Q(E)$ and its BPS central charge $Z(E)$ depending on $Q(E)$ and a point of the Kähler moduli space u.

Recall the stability property of solutions of the Yang—Mills equations. The solutions of the Yang—Mills equations preserving the $N = 1$ supersymmetry correspond to holomorphic vector bundles satisfying the condition of μ-stability: the holomorphic vector bundle E is μ-stable if, for all subbundle E', we have (14). All quantities which depend on the Kähler class J are modified in the string theory by world-sheet instanton corrections. In (14), the dependence on the Kähler class is replaced by the dependence on the period Π on the Calabi—Yau manifold M. For BPS D-branes of the superstring theory, the slope of (14) is replaced by the "grade"

$$\phi(E, u) = \frac{1}{\pi} \mathrm{Im} \log Z(E),$$

$$\phi(E, u) - \phi(\bar{E}, u) = 1 (\mathrm{mod} 2), \qquad (47)$$

where u is a point of the moduli space and $Z(E)$ is the central charge of the object E. The central charges of the brane E and its antibrane $\bar{E}$ are connected by the relation $Z(E) = -Z(\bar{E})$. It would be natural to replace the condition μ-stability by the comdition of Π-stability, the string version of the μ-stability condition. But the direct generalization of condition (13) such as the BPS D-brane will be a Π-stable object of the category of topological branes at a point u if $\phi(E') < \phi(E)$ for all subobjects E' of the object E is incorrect. The derived category is not Abelian and we have no natural definition of subobjects, a kernel, and a cokernel.

For the conformal field theoretical generalization of stability arguments, we need the physical interpretation of the grading p of the morphisms $\mathrm{Ext}^p(E, F)$. It is, in fact, the world-sheet $U(1)$ charge of a bosonic open string connecting the branes E and F. Its most direct physical meaning is developed by the relation

$$m^2 = \frac{1}{2}(p - 1), \qquad (48)$$

where m (in string units) is the mass of the boson in string theory. In category terms, two branes (or brane-antibrane pair) E and F may make up a bound state formation G if all gradnigs p of all the morphisms $\mathrm{Ext}^p(E, F)$ will be less than one. For any exact sequence

$$0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0, \qquad (49)$$

the object G goes unstable if the grading p of $\mathrm{Ext}^p(E, F)$ between any of its subobjects E and quotients F becomes greater than one. This is because the negative $U(1)$ charges of chiral operators in the unitary superconformal field theory are not allowed. The contradiction can only be resolved by the decay of G.

The flow of gradings is induced by the variation of the Kähler moduli space (B-type D-branes) and complex structures (A-type D-branes). In the large volume limit, each brane corresponds to some stable coherent sheaf, and for each pair of branes E and F, we have graded morphisms $\mathrm{Ext}^n(E, F) = \mathrm{Hom}^1(E, F[n])$. The morphism of degree n at starting point K of the Kähler moduli space will undergo "flow grading" determined by the degree of the phase

$$n \to n' = n + \phi_L(F) - \phi_K(E) - \phi_L(E) + \phi_K(F). \qquad (50)$$

An equivalent rule can be expressed in a similar way by grading varying with ϕ: $\mathrm{Hom}(E[\phi_K(E)], F[n + \phi_K(F)]) \to \mathrm{Hom}(E[\phi_L(E)], F[n + \phi_L(F)])$.

As we said above, holomorphic B-type D-branes are described everywhere in the moduli space by the formalism of derived categories. The fundamental reason why one is forced to use derivative category in the study of D-branes is discussed in [16, 17]. Those autors showed that Fourier—Mukai transforms corresponding to monodromies are associated with general loops in the Kähler moduli space and act naturally on the derived category (not a category coherent sheaves).

By analogy with the exact sequence (49) of an Abelian category, we introduce "distinguished triangles" in a derived category:

$$\begin{array}{ccc} & C_f & \\ \overset{[1]}{\swarrow} & & \nwarrow \\ A & \xrightarrow{f} & B, \end{array} \qquad (51)$$

where "[1]" denotes the morphism from C_f to $A[1]$. Every morphism $f : A \to B$ in the derived category is completed to a distinguished triangle

$$\ldots \longrightarrow C_f[-1] \xrightarrow{\psi} A \xrightarrow{f} B \xrightarrow{\phi} C_f[1] \longrightarrow \ldots, \qquad (52)$$

where $C_f = \mathrm{Cone}\{A \xrightarrow{f} B\}$ is the complex with the terms $A[1] \oplus B$ and the morphisms $\begin{pmatrix} d^A & 0 \\ -f & d^B \end{pmatrix}$.

$$\ldots \longrightarrow A^{n+1} \oplus B^n \xrightarrow{\begin{pmatrix} d_n^A & 0 \\ -f & d_{n-1}^B \end{pmatrix}} A^n \oplus B^{n-1} \longrightarrow \ldots (53)$$

The morphism "[1]" can be shuffled around any edge, and triangle (51) can be rewritten as

$$\begin{array}{ccc} & C & \\ \swarrow & & \nwarrow \\ A[1] & \xrightarrow{[1]} & B, \end{array} \qquad (54)$$

or as

$$\begin{array}{ccc} & C[-1] & \\ \swarrow & & \nwarrow [1] \\ A & \xrightarrow{f} & B. \end{array} \qquad (55)$$

Certain triplets of D-branes $\{A, B, C\}$ are distinguished because of the tachyon condensation between a pair of them can produce the third one as a bound state. The rules for Π-stability of D-branes are explained in [10]. The distinguished triangles (51), (54),(55) tell us that

- the object $C = \mathrm{Cone}(f : A \to B)$ is potentially a bound state of objects $A[1]$ and B,
- the object B is potentially a bound state of objects A and C,
- the object A is a bound state of objects B and $C[-1]$.

In distinguished triangles, the gradings involving open strings need to keep track. The data for Π-stability are the grading $\phi \in R$ associated to the objects in $D^b(\mathrm{coh}M)$.

The D-brane C in (51) is stable with respect to A and B if and only if $\phi(A) - \phi(B) < 1$. The "stability" of a given vertex of some triangle is relative to this triangle only, a given D-brane may decay by other channels.

If the difference in ϕ's exceeds one on any of the edges of triangles (51),(54),(55), then the D-brane in the opposite vertex will decay.

As we mentioned above, the stringy version of the stability condition which reduces to the conditions in large volume and orbifold limits is called Π-stability. In short, this problem can be stated as follows. Let $\mathrm{Stab}_{\,p}$ be the set of Π-stable objects at p. All triangles must satisfy the constraints given above. Consider the path from p_1 to p_2.

Suppose M contains a rational curve S which can contracted down to a point by varying the complexified Kähler form. Let us consider the basic set of objects $\{A, B, C\}$ in $D(M)$:

- A — the skyscraper sheaf $\mathcal{O}_x$, where $x \in S$, $\phi(\mathcal{O}_x = = 0)$;
- B — the skyscraper sheaf $\mathcal{O}_y$, where $y \notin S$, $\phi(\mathcal{O}_y) = 0$;
- C — the structure sheaf $\mathcal{O}_S$ of the flopping curve S ($D2$-brane wrapped on S), $\phi(\mathcal{O}_S) = -\frac{\theta}{\pi}$,

where $e^{i\theta} \sim t = \int_S D + iJ$. The grading ϕ depends continuously on the complexified Kähler form $B + iJ$. We consider the following cases:

$$\begin{array}{ccc} & A & \\ u \overset{[1]}{\swarrow} & & \nwarrow w \\ \mathcal{O}_S & \xrightarrow{v} & \mathcal{O}_X. \end{array} \qquad (56)$$

$A = \mathrm{Cone}(\mathcal{O}_C \to \mathcal{O}_X) = \mathcal{O}_C(-1)[1]$ $u = 1 - \frac{\theta}{\pi}, v = \frac{\theta}{\pi}, w = 0$. If $\theta < 0$ (we pass from phase M to phase M'), $\mathcal{O}_X$ brane will decay into A and $\mathcal{O}_S$ branes.

$$\begin{array}{ccc} & B & \\ u \overset{[1]}{\swarrow} & & \nwarrow w \\ \mathcal{O}_C[-1] & \xrightarrow{v} & A. \end{array} \qquad (57)$$

$\mathrm{Hom}(\mathcal{O}_S[-1] \to A) = C^2$,

$B = \mathrm{Cone}(\mathcal{O}_C[-1] \to A) = \mathcal{O}_C(-1)[1]$, $u = \frac{\theta}{\pi}$, $v = 1 + \frac{\theta}{\pi}$, $w = 0$.

If $\theta < 0$ (we pass from phase M to phase M'), the B-brane jumps into existence.

$$\begin{array}{ccccc} & & C & & \\ & u \overset{[1]}{\swarrow} & & \nwarrow w & \\ \mathcal{O}_C[-1] & & \xrightarrow{v} & & B. \end{array} \qquad (58)$$

$\mathrm{Hom}(\mathcal{O}_S[-1] \to B) = C = \mathrm{Cone}(\mathcal{O}_C[-1] \to B) = \mathcal{O}_C(-1)[1]$, $u = -\frac{\theta}{\pi}, v = 1 + \frac{\theta}{\pi}, w = 0$. If $\theta < 0$ (we pass from phase M to phase M'), the C-brane jumps into existence.

Conclusions

The existence of D-branes in string theories gives convincing arguments to conclude that five consistent and perturbatively non-equvivalent supersymmetric theories in ten dimensions belong to the unique eleven-dimensional M-theory. The D-brane dynamics used to be an object of numerous investigations in last years.

In this report, we consider the fundamental picture of BPS B-type D-branes on Calabi—Yau manifolds. The topological B-type D-branes are not, in general, just holomorphic vector bundles or coherent sheaves on analytic submanifolds of these Calabi—Yau manifolds. They can be identified with arbitrary objects of the derived category of coherent sheaves. The triangulated structure of this category allows one to formulate the conditions of Π-stability of D-branes, which generalizes the conditions of μ-stability for vector bundles. The Π-stability enables us to extract the physical D-branes from the topological D-branes.

The direct physical applicationes of such investigations extends the nonperturbative methods of field theories, promotes a better understanding of the dualities in $N = 1$ and $N = 2$ supersymmetric compactifications, enables computations of the black hole entropy, and gives a qualitative approach to the study of the supersymmetry breaking.

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D-БРАНИ НА МНОГОВИДАХ КАЛАБІ—ЯУ

І.М.Бурбан

Р е з ю м е

Обговорюється проблема опису БПЗ D-бран суперструнних теорій типу II на многовидах Калабі—Яу. Для підкласу D-бран, визначених похідною категорією когерентних пучків на многовидах Калабі—Яу в довільній точці їх простору модулів, вивчається їх властивість Π-стабільності.

D-БРАНЫ НА МНОГООБРАЗИЯХ КАЛАБИ—ЯУ

И.М.Бурбан

Р е з ю м е

Обсуждается проблема описания БПЗ D-бран суперструнных теорий типа II на многообразиях Калаби—Яу. Для подкласса D-бран, определенных производной категорией когерентных пучков на многообразиях Калаби—Яу в произвольной точке их пространства модулей, изучается их свойство Π-стабильности.