DUAL FORMULATION OF NONABELIAN LATTICE MODELS AND RELATED MATHEMATICAL PROBLEMS [1]

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UDC 539

Compact nonabelian lattice models (like $3D$ QCD, $2D$ SU(N) principal chiral models) are formulated in terms of plaquette (or link) variables which correspond to the continuum field-strength representation. Using this representation, we derive an exact dual formulation for the partition function and some observables like a Wilson loop, two-point correlation function, etc. It is applied to the study of the low-temperature region of the models relevant to the construction of the continuum limit. In particular, we compute the leading term of the asymptotic expansion of the dual Boltzmann factor and prove that it converges at low temperatures to a certain Gaussian distribution uniformly in all fluctuations of dual variables. The possible applications of our construction to a calculation of long-distance observables are discussed. Of independent interest might be the derivation of a new asymptotic expansion for matrix elements of the SU(2) rotation matrix in the vicinity of unity which is uniform in representations and magnetic numbers.

Introduction

Dual transformations for lattice spin and gauge models have a long history. In the context relevant to this paper, we would like to mention duals of the abelian $U(1)$ model [1] which have been used to prove the existence of a soft phase at low temperatures with power-like decay of the correlation function in the $2D$ XY model [2] and confinement of static charges at all couplings in the $3D$ gauge model [3]. In these cases, the dual of Abelian models is a local theory for certain discrete variables. No similar representation was known so far for any nonabelian model. The conventional dual transformations [4—6] for nonabelian gauge models also leads to a local dual theory for integers which label irreducible representations of a local or global group, but these transformations are not complete. First of all, the resulting dual variables are not independent but are subject to certain constraints known as triangular conditions, and as such they cannot really be associated with elements of a dual lattice. Secondly, although the local dual formulation is expressed in terms of group invariants (for gauge models) like $6j$-symbols, etc., it involves also the summation over auxiliary representations resulting from the multiplication of nonabelian matrices. Such a formulation is so mathematically involved that one can hardly hope that it can be useful for an analytic study of the model.

On the other hand, there exists a representation of two-dimensional ($2D$) models in terms of link variables [7], and this representation can be formulated directly on the dual lattice. For lattice gauge theories (LGT), there is the so-called plaquette representation [8] which also have a corresponding dual interpretation. It is a first goal of the present paper to use link and plaquette representations for the derivation of exact dual formulations of $2D$ SU(N) principal chiral models and $3D$ LGT. The resulting dual formulations appear to be quite different from the formulations mentioned above. We shall discuss their properties in the corresponding places. Here we want to stress only that, in our opinion, the most essential advantage of our dual formulations is that it is much more suitable for analytic investigations of the model, especially in the low-temperature region. We refer to our papers [9—11] for the detailed explanation of why it is so. In the last of those papers, we have already presented a model dual of the $2D$ SU(2) spin one and the proposed an approximate representation for the dual partition function at low temperatures.

Low-temperature properties of the models under consideration are crucial for the construction of their continuum limit. E.g., in the case of $2D$ nonabelian models, it is widely expected that models possess no phase transition, the correlation function has exponential decay at any coupling, and models are asymptotically free. Despite being more than twenty years old, this expectation has not been proven rigorously. On the contrary, certain percolation theory arguments supported by numerical computations

[1] This article was presented at the II International Symposium "Fundamental Problems in Modern Quantum Theories and Experiments" (September 2—7 2002, Odessa, Ukraine) on the occasion of Professor Walter Thirring's 75th birthday.

suggest that all nonabelian models have soft low-temperature phase with power-like decay of the correlation function [12]. In [11], under certain approximation, within our dual formulation we have shown that a two-point correlation function may indeed decay with power law, thus supporting results of [12]. Another important motivation of the present investigation is to get a deeper insight into the nature of a mass gap in nonabelian spin models and into the string tension of gauge theories. For instance, in many papers devoted to $2D$ nonabelian models, it is written that "there is a nonperturbative mass gap generation at arbitrarily small couplings". It is not really clear, however, what is the precise meaning of this "nonperturbative generation". It cannot be a simple consequence of the link decorrelation which happens in $1D$ models. Then, one could ask if this "nonperturbative generation" follows from the existence of some non-trivial background of defects like vortices of the XY model or is due to the strong but smooth disorder of nonabelian spins, e.g. like center vortices [13].

As is well known, the dual formulations of abelian models have been extremely useful in clarifying all these important physical problems. Moreover, practically all rigorous mathematical results on the behaviour of abelian lattice models at low temperatures have been obtained within dual approaches. It is thus our second goal to develop a technique within dual formulation of nonabelian models which would allow one to investigate them in the limit of bare weak coupling, i.e. in the low-temperature region.

1. Dual of $2D$ SU(N) Spin Models

We begin our consideration with the $2D\,\mathrm{SU}(N)\times\mathrm{SU}(N)$ principal chiral model whose partition function is given by

$$Z = \int \prod_x DU_x \exp\left[\beta \sum_{x,n} \mathrm{Re}\ \mathrm{Tr} U_x U_{x+n}^{\dagger}\right] , \qquad (1)$$

where $U_x \in \mathrm{SU}(N)$, DU_x is the invariant measure, and we impose periodic boundary conditions. The fundamental degrees of freedom U_x belong to sites of a $2D$ lattice. It is quite possible to construct the dual formulation starting from this representation, namely performing the Fourier expansion of the Boltzmann factor on the SU(N) group and integrating over all U_x. This program was recently accomplished in [6] for an arbitrary group in various dimensions. As explained in Introduction, we develop here a different approach to duality transformations based on the so-called link representation for partition and correlation functions. The partition function (1) can be exactly reformulated in terms of link variables $V_l = U_x U_{x+n}^{\dagger}$ as [7]

$$Z = \int \prod_l dV_l \exp\left[\beta \sum_l \mathrm{Tr} V_l\right] \times$$

$$\times \prod_p \left[\sum_r d_r \chi_r \left(\prod_{l\in p} V_l\right)\right] , \qquad (2)$$

where $V_l \in \mathrm{SU}(N)$, dV_l is the invariant measure on the group. $\prod_p$ is a product over all plaquettes of a $2D$ lattice, the sum over r is a sum over all representations of SU(N), $d_r = \chi_r(I)$ is the dimension of the r-th representation. The SU(N) character χ_r depends on a product of link matrices $V_l = V_n(x)$ around a plaquette

$$\prod_{l\in p} V_l = V_n(x) V_m(x+n) V_n^{\dagger}(x+m) V_m^{\dagger}(x) . \qquad (3)$$

For more details on this formulation, we refer the reader to our paper [9], where we have developed a weak coupling expansion for SU(N) spin models using the link representation. From now on, we concentrate on the SU(2) model (extension to any other SU(N) is straightforward; moreover, all general formulae below are applicable to any SU(N)). Now, let x be a site of the dual lattice (center of the original plaquette), and l be a link of the dual lattice (i.e., orthogonal to the original links). We want to reformulate model (2) on the dual lattice only in terms of discrete variables that are, in our case, the SU(2) representations r_p and magnetic quantum numbers $m_i(p)$. On the dual lattice, these variables can all be associated with the sites of this lattice. As follows from (2) and from the definition of the SU(2) character

$$\chi_r(V) = \sum_{n=-r}^{r} V_r^{nn} \qquad (4)$$

on the dual lattice, the partition function may be written as

$$Z = \sum_{r_x=0,\frac{1}{2},1,\ldots}^{\infty} \prod_x \left[(2r_x+1) \sum_{m_i(x)=-r_x}^{r_x}\right] \prod_l \Xi_0(l) . \qquad (5)$$

Due to the trace, there are 4 independent variables m_i at each dual site, thus $i = 1,2,3,4$. The dual weight $\Xi_0(l)$ is given by the following one-link integral:

$$\Xi_0(l) \equiv \Xi_0(r_x, m_1, n_1; r_{x+n}, m_2, n_2; \beta) =$$

$$= \int dV e^{\beta \mathrm{Tr} V} \; V_{r_x}^{m_1 n_1} V_{r_{x+n}}^{\dagger m_2 n_2} \; , \tag{6}$$

where V_r^{mn} is a matrix element of the r-th representation.

A similar form for the two-point correlation function in the representation j reads

$$\Gamma_j(x,y) = \frac{1}{2j+1} \sum_{s_1=-j}^{j} \cdots \sum_{s_R=-j}^{j} \langle \prod_{l \in C_{xy}} \frac{\Xi_j^{s_i s_{i+1}}(l)}{\Xi_0(l)} \rangle \; , \tag{7}$$

where $s_{R+1} = s_1$ and the link integral on $l \in C_{xy}$ is

$$\Xi_j^{s_i s_{i+1}}(l) = \int dV e^{\beta \mathrm{Tr} V} V_{r_x}^{m_1 n_1} V_{r_{x+n}}^{\dagger m_2 n_2} V_j^{s_i s_{i+1}} \; . \tag{8}$$

Here, C_{xy} is some path connecting points x and y and consisting of links dual to links of the original lattice.

Let us give some comments on the formulae obtained. There is the obvious resemblance of this representation to the dual of the XY model

$$Z = \sum_{r_x=-\infty}^{\infty} \prod_l \Xi_0^{XY}(l), \quad \Xi_0^{XY}(l) = I_{r_x - r_{x+n}}(\beta) \; . \tag{9}$$

A similar equation can be written also for the correlation function. It is obvious from here that our dual formulation is much closer to the Abelian analog than one presented in [6]. However, there is also a difference from the Abelian case. While the dual of the XY model is a local theory for integers which label the representation of the $U(1)$ group, this is not exactly the case for a non-abelian model. It is clear from the equations above that the summation over magnetic numbers makes the effective theory for r_x highly non-local, and this non-locality persists at any temperatures. On the other hand, *a priori* it is not obvious that this non-locality has anything to do with expected non-perturbative phenomena like the mass gap generation. Indeed, consider, for example, the partition function of a three-component Gaussian field written in the spherical coordinates. Integration over anglular variables produces a complicated non-local theory for the radial component of the Gaussian field. Such non-locality, however, cannot change the Gaussian nature of the field and has no non-perturbative origin by itself.

It is very easy to make integration in (6) expanding the result in Clebsch—Gordan series. One then finds the following representation for the dual weight of a nonabelian model (an analog of $\Xi_0^{XY}(l)$ given just above)

$$\Xi_0(l) = \frac{1}{2r_2+1} \sum_{J,k} C_J(\beta) C_{r_1 m_1 Jk}^{r_2 n_2} C_{r_1 n_1 Jk}^{r_2 m_2} \; , \tag{10}$$

where we have denoted $r_1 = r_x$, $r_2 = r_{x+n}$ and

$$C_J(\beta) = \frac{2J+1}{\beta} I_{2J+1}(2\beta) \; . \tag{11}$$

For the correlation function in Eq.(8), using the Clebsch—Gordan expansion, one gets

$$\Xi_j^{s_i s_{i+1}}(l) = \sum_{J \alpha_1 \alpha_2} C_{r_1 m_1 j s_1}^{J \alpha_1} C_{r_1 n_1 j s_2}^{J \alpha_2} \times$$

$$\times \Xi_0(J, \alpha_1, \alpha_2; r_2, m_2, n_2; \beta) \; . \tag{12}$$

It is clear from the last equations that, in order to investigate the dual model, it is crucial to understand the properties of the link function $\Xi_0(l)$ which enters both the partition and correlation functions. We therefore finish this section with the brief description of the most important features of $\Xi_0(l)$:

- As follows from the properties of the coefficients of the expansion $C_J(\beta)$, the series in J in (10) gives directly the strong coupling expansion of the model written in closed and compact form. Much less trivial is to get weak coupling expansion for $\Xi_0(l)$ since all J in series (10) become relevant.

- On all configurations $\{r_i, m_i, n_i\}$, $\Xi_0(l)$ is strictly positive, $\Xi_0(l) > 0$. Though we could not prove it rigorously, this claim is supported by the following facts: 1) the first term in the strong coupling expansion is strictly positive, thus $\Xi_0(l)$ is positive at sufficiently small β where the series converges very fast; 2) the leading term of the asymptotic expansion of $\Xi_0(l)$ at large β is strictly positive on all configurations; 3) numerical computations of $\Xi_0(l)$ on a number of configurations and in a wide region of β also support this conclusion. If $\Xi_0(l) > 0$ on all configurations, this gives a chance for numerical Monte-Carlo simulations of the dual model.

- The dominant contribution to $\Xi_0(l)$ at large β comes from the diagonal components of rotation matrices, the non-diagonal contribution is suppressed roughly as $[(m-n)!]^{-1}$. This is, of course, a consequence of the fact that, when $\beta \to \infty$, the link matrix performs only small fluctuations around unity. In turn, this property gives a possibility to compute the low-temperature asymptotic expansion of the $\Xi_0(l)$. In fact, the knowledge of such asymptotics is necessary if one wants to investigate the weak-coupling regime of the theory. We consider this problem below after the presentation of the dual formulation for gauge models.

2. Dual of $3D$ SU(N) LGT

This section deals with nonabelian LGT in three dimensions whose partition function is given by [14]

$$Z = \int DU \ \exp\{\beta S[U_\mu(x)]\} \ , \tag{13}$$

where S is the Wilson action (extension to any other gauge-invariant action is quite simple). The integral is calculated over the Haar measure on the group at every link of the lattice. As in the case of spin models, dual representations of LGT can be obtained in two different ways. The first one starts from the character expansion of the Boltzmann factor in (13). Then, one can explicitly integrate out the gauge degrees of freedom and finally introduce dual variables. This program can be accomplished both for Abelian [1] and for nonabelian [4—6] LGT. In case of nonabelian models, one should also calculate sums over all magnetic numbers (corresponding to calculation of group traces). The resulting dual representation appears to be a local theory of discrete variables which label the irreducible representations of the underlying gauge group and can be written solely in terms of group invariant objects like the $6j$-symbols, etc. Unfortunately, this form of dual theory is rather complicated and hardly can be used for a direct analytical study. Moreover, there is an essential difference between the duals of Abelian and nonabelian LGTs. While, in the first case, the dual variables reside on sites of the dual lattice and are completely independent, it is not the case for nonabelian models. Here, it was not possible to achieve such a level of generalization and, therefore the dual variables are those which reside on links and plaquettes of the original lattice and are still subject to the so-called triangular constraints embedded into $6j$-symbols. The second way to get the dual form is to first rewrite the theory in terms of plaquette variables which can be considered by themselves as certain dual variables. Plaquette variables are subject to the Bianchi constraint which has nonlocal form for nonabelian models. Nevertheless, in this case, one can obtain a dual representation for nonabelian models which is close to the corresponding Abelian analog. In particular, dual variables are those associated with sites of the dual lattice and, at least in certain cases, the difficulties related to triangular constraints can be overcome. The essential advantage of this dual form is that it appears to be more suitable for an analytical study of both the high and low-temperature regions of nonabelian LGT.

The plaquette formulation on the lattice was obtained in [8]. We use here a slightly different form obtained by us in [10], though all general formulae given below are applicable to both formulations. We begin with the following partition function in the maximal axial gauge:

$$Z = \int \prod_p dV_p \exp\left[\beta \sum_p \mathrm{ReTr} V_p\right] \prod_c J(V_c) \ , \tag{14}$$

where

$$J(V_c) = \sum_r d_r \chi_r(V_c) \ , \tag{15}$$

$$V_c = \left(\prod_{p\in A} V_p\right) C \left(\prod_{p\in B} V_p\right) C^\dagger \ , \quad C = \prod_{p\in c} V_p \ . \tag{16}$$

The product over c runs over all cubes of a $3D$ lattice. $J(V_c)$ is the SU(N) delta-function which introduces a constraint on plaquette matrices (the Bianchi identity). C is the connector of nonabelian Bianchi identity. For details of this representation, we refer the reader to our paper [10].

In what follows, we write explicitly all formulae only for the SU(2) gauge group. Generalization to other groups is straightforward. Let x be a site dual to the cube of the original lattice and l be a link dual to the plaquette of the original lattice. Then it follows from (14) and from the expression for the Jacobian that, in the case of the SU(2) gauge group, the partition function on the dual lattice can be written in the following form:

$$Z = \sum_{r_x=0,\frac{1}{2},1,\ldots}^{\infty} \prod_x \left[(2r_x+1) \sum_{m_i(x)=-r_x}^{r_x}\right] \prod_l \Xi_0(l) \ . \tag{17}$$

The summation over r_x corresponds to the summation over all irreducible representations of the SU (2) group. The sums over magnetic numbers $m_i(x)$ correspond to the calculation of SU(2) traces. The index i may run from 6 to $6{+}4L$ depending on the position of the original cube, L is the linear extent of the lattice. The link integral $\Xi_0(l)$ is given by

$$\Xi_0(r_x, m_1, n_1; r_{x+n}, m_2, n_2; r_i, k_i, k_{i+1}, p_i, p_{i+1}; \beta) =$$

$$= \int dV e^{\beta \mathrm{Tr} V} \; V_{r_x}^{m_1 n_1} V_{r_{x+n}}^{\dagger \; m_2 n_2} \times$$

$$\times \prod_{i=1}^{M(x)} \left(V_{r_i}^{k_i k_{i+1}} V_{r_i}^{\dagger \; p_i p_{i+1}} \right) \;, \qquad (18)$$

where V_r^{mn} is a matrix element of the r-th representation. The integer number $M(x) = L - z$ depends on the position of a plaquette on the lattice and indicates how many times a given plaquette (dual link) serves as a connector in the Bianchi identities.

A similar form for the Wilson loop in the representation j reads

$$W_j(C) = \frac{1}{2j+1} \sum_{\{s_i\}=-j}^{j} \langle \prod_{l \in S^d(C)} \frac{\Xi_j^{s_i s_{i+1}}(l)}{\Xi_0(l)} \rangle \;, \qquad (19)$$

where the link integral on $l \in S^d(C)$ is

$$\Xi_j^{s_i s_{i+1}}(l) = \int dV e^{\beta \mathrm{Tr} V} \; V_j^{s_i s_{i+1}} V_{r_x}^{m_1 n_1} V_{r_{x+n}}^{\dagger \; m_2 n_2} \times$$

$$\times \prod_{i=1}^{N(x)} \left(V_{r_i}^{k_i k_{i+1}} V_{r_i}^{\dagger \; p_i p_{i+1}} \right) \;. \qquad (20)$$

In general, $M(x) \neq N(x)$ for some x if Wilson loop also contains connectors. Here, $S^d(C)$ is some surface dual to the surface $S(C)$ which is bounded by the loop C and consists of links dual to plaquettes of the original lattice.

Using the Clebsch—Gordan expansion, one finds

$$\Xi_j^{s_i s_{i+1}}(l) = \sum_{J \alpha_1 \alpha_2} C_{r_x m_1 j s_1}^{J \alpha_1} C_{r_x n_1 j s_2}^{J \alpha_2} \times$$

$$\times \Xi_0(J, \alpha_1, \alpha_2; r_{x+2}, m_2, n_2; r_i, k_i, k_{i+1}, p_i, p_{i+1}; \beta) \;. \quad (21)$$

This representation of the Wilson loop reduces the problem again to the calculation of the basic link integral $\Xi_0(l)$. Similar dual formulae can be obtained for a number of other observables like the plaquette-plaquette correlation function, 't Hooft loop, etc.

3. Low-temperature Asymptotics of Link Functions

As we have mentioned in Introduction, the most important application of the dual formulation could be the investigation of the low-temperature region of nonabelian models. As can be seen from the formulae of last two sections, the basic quantity both in the spin and gauge models is a one-link integral which is essentially the dual Boltzmann factor. The investigation of the low-temperature region is thus reduced to the establishing of the asymptotic expansion for this function at large β. It is necessary to get asymptotics uniformly valid in all fluctuations of dual variables. It turns out that such asymptotics can be constructed, and this is, in our opinion, one of the most important advantages of the dual formulation. Calculation of the asymptotic expansion relies essentially on the fact that, when $\beta \to \infty$, the plaquette matrix performs only small fluctuations around unity both in the finite volume and, most importantly, in the thermodynamic limit. Note that this is not the case for the original link degrees of freedom: in the large volume limit, their fluctuations are not bounded even in the maximal axial gauge.

As a first step, in the investigation of the large-β region, we derive the low-temperature asymptotic expansion of the link function defined by Eq.(6) for spin models and by Eq.(18) for gauge models. We consider first the more general case of gauge models and then give a result for spin models.

For SU(2) matrix elements, we use the Wigner D-function parametrized by Euler angles

$$D_{mn}^r(\alpha, \omega, \phi) = e^{-im\alpha - in\phi} d_{mn}^r(\omega) \;. \qquad (22)$$

Let us recall that the fundamental trace in this parametrization reads

$$\mathrm{Tr} V = 2 \cos \frac{\omega}{2} \cos \frac{1}{2}(\alpha + \gamma) \qquad (23)$$

and the invariant measure on the group takes the form

$$\int dV = \frac{1}{16\pi^2} \int_0^{4\pi} d\gamma \int_0^{2\pi} d\alpha \int_0^{\pi} d\omega \sin\omega \;. \qquad (24)$$

Substituting last expressions into (6), one can exactly integrate over α and γ angles. This gives

$$\Xi_0(l) = \delta_{n_2 - m_2 + \sum_i (p_{i+1} - p_i)}^{m_1 - n_1 + \sum_i (k_i - k_{i+1})} \int_0^{\frac{\pi}{2}} d\omega \sin\omega \cos\omega \times$$

$$\times I_{2s}(2\beta \cos\omega) d_{m_1 n_1}^{r_x}(2\omega) d_{n_2 m_2}^{r_{x+n}}(2\omega) \times$$

$$\times \prod_{i=1}^{M(x)} \left(d^{r_i}_{k_i k_{i+1}}(2\omega) d^{r_i}_{p_{i+1} p_i}(2\omega) \right) \ , \tag{25}$$

where

$$s = m_1 - n_2 + \sum_i (k_i - p_{i+1}) \ . \tag{26}$$

To get the asymptotics when $\beta \to \infty$ uniformly valid in all field configurations, we first use the following asymptotics for the modified Bessel function:

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}} \ \exp(-\frac{1}{2x} n^2) \left(1 + \mathcal{O}(x^{-1})\right) \tag{27}$$

when $x \to \infty$ and such that $n^2/x \sim \mathcal{O}(1)$. It leads to

$$I_{2s}(2\beta \cos\omega) = \frac{e^{2\beta}}{\sqrt{4\pi\beta}} \ \exp(-\frac{1}{\beta} s^2 - \beta \sin^2\omega) \times$$

$$\times \left(1 + \mathcal{O}(\sin^2\omega)\right) \ . \tag{28}$$

Since $\omega \sim \mathcal{O}(\beta^{-1/2})$, the remainder is bounded like $\mathcal{O}(\beta^{-1})$. The second step is to construct asymptotic expansion for matrix elements $d^r_{mn}(\omega)$ uniform in r and in all magnetic numbers when $\omega \to 0$. As we have verified, all the standard asymptotics given in the literature do not satisfy all possible criteria. In particular, in certain important cases, a one-link integral can be computed exactly. All standard asymptotics fail to reproduce such exactly solvable cases. We have derived a new asymptotic expansion for the matrix elements which satisfy all the criteria we are aware of and which we believe is new. Therefore, we give some details of this derivation in Appendix below. The final result is given in Eq.(46). Here we are interested only in the leading term of that expansion. Combining Eqs.(28) and (46), we arrive finally at the following asymptotic representation for the one-link integral:

$$\Xi_0(l) = C(\beta) \delta^{m_1 - n_1 + \sum_i (k_i - k_{i+1})}_{n_2 - m_2 + \sum_i (p_{i+1} - p_i)} \times$$

$$\times \exp\left(-\frac{1}{4\beta}\alpha^2\right) B(l) \left(1 + \mathcal{O}(\beta^{-1})\right) \ , \tag{29}$$

where we have used the notations

$$C(\beta) = \frac{e^{2\beta}}{2\beta\sqrt{\pi\beta}} \ , \tag{30}$$

$$\alpha = m_1 + n_1 - m_2 - n_2 +$$

$$+ \sum_i (k_i + k_{i+1} - p_i - p_{i+1}) \ . \tag{31}$$

Making change of variables $\sin\omega = y/\sqrt{2\beta}$ in the last integral and extending the integration region over y to infinity (what introduces only exponentially small corrections which can be properly bounded), the last integral can be written as

$$B(l) = \int_0^\infty dy \ y \ e^{-\frac{1}{2}y^2} \ J_{m_1 - n_1}\left(\frac{R_x \sin\theta_x}{\sqrt{2\beta}} y\right) J_{n_2 - m_2}\left(\frac{R_{x+n} \sin\theta_{x+n}}{\sqrt{2\beta}} y\right) \times$$

$$\times \prod_{i=1}^{M(x)} \left(J_{k_i - k_{i+1}}\left(\frac{R_i \sin\theta^{(1)}_i}{\sqrt{2\beta}} y\right) J_{p_{i+1} - p_i}\left(\frac{R_i \sin\theta^{(2)}_i}{\sqrt{2\beta}} y\right)\right) \ , \tag{32}$$

where R and $\sin\theta$ are defined in Eq.(47).

In the case of $2D$ models, the one-link integral does not contain any connectors. Also, integrals on time-like links in the gauge model are free of connectors due to the construction of the original plaquette representation. Therefore, in these cases, the integrand includes only two rotation matrices. Then, the last integral can be done exactly. Hence, the leading term of the asymptotic expansion of a one-link integral in $2D$ spin models as well as of time-like integrals in gauge models can be given as

$$\Xi_0(l) = C(\beta) \delta^{m_1 - n_1}_{n_2 - m_2} \times$$

$$\times \exp\left[-\frac{1}{4\beta}(R_x^2 + R_{x+n}^2 - 2R_x R_{x+n}\cos\theta_x \cos\theta_{x+n})\right] \times$$

$$\times I_k\left(\frac{R_x \sin\theta_x R_{x+n}\sin\theta_{x+n}}{2\beta}\right) . \quad (33)$$

4. Discussion

The present article deals with the low temperature asymptotics of $2D$ spin and $3D$ gauge nonabelian models in the dual formulation [9—11]. Since the models under consideration are commonly expected to be asymptotically free and have no phase transition, the low-temperature region plays an extremely significant role. Thus, our results besides a purely academic interest can be engaged as a method in different fields of mathematical physics, solid state physics, and high-energy physics.

First of all, we hope that our formulation will allow one to generalize the analytical study of the $U(1)$ lattice gauge theory of [2] to nonabelian cases.

Among possible physical applications of our approach, we could mention an analytical investigation of two-dimensional quantum Heisenberg ferromagnets. The description of magnetic properties of solid ^{3}He films adsorbed on graphite by means of the $2D$ principal chiral model [15] should be mentioned since the second layer of ^{3}He provides an excellent example of a nearly ideal $2D$ quantum 1/2-spin system on a triangular lattice.

Another field of research, where our asymptotic expansion can be used, the nonabelian gauge models such as QCD. Nowadays, the rigorous investigation of the phase diagram of these models attracts an increasing interest, especially in the low-temperature region. The asymptotic expansion proposed in the present paper can be applied to nonperturbative analysis of physical observables such as a Wilson loop or a plaquette-plaquette correlation function.

Lastly, we would like to mention the possibility of the correspondence between nonabelian spin and gauge models and the simplectic quantum gravity formalism. This would allow our formulation to be applicable to some problems of quantum gravity.

APPENDIX. ASYMPTOTICS OF $d^r_{mn}(\omega)$

Here we compute the asymptotic expansion of the SU (2) matrix elements $d^r_{mn}(\omega)$ in the classical region $R = (2r+1) \gg 1$ uniformly valid in the vicinity of the point $\omega = 0$ for all allowed values of m and n.

To get such an asymptotics, we first present the d-function in terms of the hypergeometric function $F \equiv {}_2F_1$

$$d^r_{mn}(\omega) = \frac{\xi_{mn}}{k!}\left[\frac{(s+k+p)!(s+k)!}{s!(s+p)!}\right]^{\frac{1}{2}}\left(\sin\frac{\omega}{2}\right)^k \times$$

$$\times \left(\cos\frac{\omega}{2}\right)^{-p} F(s+k+1, -s-p; k+1; \sin^2\frac{\omega}{2}) , \quad (34)$$

where $\xi_{mn} = 1$ if $n \geq m$, $\xi_{mn} = -1$ otherwise, and

$$k = |m-n|, \quad p = |m+n|, \; s = r - \frac{1}{2}(k+p) . \quad (35)$$

As is seen from the arguments of the hypergeometric function, the infinite series in F terminates so that the right-hand side of (34) is polynomial in $\sin^2\frac{\omega}{2}$,

$$d^r_{mn}(\omega) = \xi_{mn}\left[A_k(x)A_k(y)\right]^{\frac{1}{2}}\left(\sin\frac{\omega}{2}\right)^k\left(\cos\frac{\omega}{2}\right)^{-p} \times$$

$$\times \sum_{l=0}^{r+\frac{1}{2}(p-k)} (-1)^l \frac{(\sin^2\frac{\omega}{2})^l}{\Gamma(k+1+l)l!}\mathcal{F}_l(x,y) , \quad (36)$$

where we introduced the notations

$$A_k(x) = \frac{\Gamma(x - \frac{1}{2}k + \frac{1}{2})}{\Gamma(x + \frac{1}{2}k + \frac{1}{2})} , \quad (37)$$

$$\mathcal{F}_l(x,y) = \frac{\Gamma(x + \frac{1}{2}k + \frac{1}{2})}{\Gamma(x - \frac{1}{2}k + \frac{1}{2} - l)}\frac{\Gamma(y + \frac{1}{2}k + \frac{1}{2} + l)}{\Gamma(y - \frac{1}{2}k + \frac{1}{2})} , \quad (38)$$

$$x = r + \frac{1}{2}(1+p), \; y = r + \frac{1}{2}(1-p) . \quad (39)$$

The second step consists in expanding the ratio of gamma functions. This can be done with help of the following formula:

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} = x^{a+b}\left(\sum_{s=0}^{N-1}\frac{(-1)^s}{s!x^s}B_s^{(a-b+1)}(a) \times\right.$$

$$\left.\times (b-a)_s + \mathcal{O}(x^{-N})\right) , \quad (40)$$

where $B_s^{(y)}(x)$ is the generalized Bernoulli polynomial. An important point concerns the large expansion parameter we use. We take not simply the classical angular momentum $r + \frac{1}{2}$ but rather the quantities x and y defined above. Such a choice gives a more accurate asymptotics valid in a wider region of parameters. Then, in the case of the quantity $\mathcal{F}_l(x,y)$, the series in (40) terminates because $a - b = k + l$ and representation (40) becomes exact. This leads to

$$\mathcal{F}_l(x,y) = (xy)^{l+k}\sum_{s_1,s_2=0}^{l+k}\frac{1}{x^{s_1}y^{s_2}s_1!s_2!} \times$$

$$\times \frac{[(l+k)!]^2}{(l+k-s_1)!(l+k-s_2)!}B_{s_1}^{(k+l+1)}\left(\frac{1}{2}(k+1)\right) \times$$

$$\times B_{s_2}^{(k+l+1)}\left(\frac{1}{2}(k+1) + l\right) \quad (41)$$

what is essentially the desired expansion at large x and y. It follows from the last representation that

$$\mathcal{F}_l(x,y) = (xy)^{l+k}\left[1 + \frac{1}{2}l(l+k)\left(\frac{1}{y} - \frac{1}{x}\right) - \right.$$

$$-\frac{l^2(l+k)^2}{4xy}+\frac{1}{24}(l+k)(l+k-1)\times$$

$$\times\left.(3l^2-l-k-1)(\frac{1}{x^2}+\frac{1}{y^2})+\mathcal{O}(x^{-3},y^{-3})\right]\ . \qquad (42)$$

For $A_k(x)$, following the same procedure, one finds

$$A_k(x)=x^{-k}\left[1+\frac{1}{24x^2}k(k^2-1)+\mathcal{O}(x^{-4})\right]\ . \qquad (43)$$

Substituting the last expressions into (36), we get after some algebra

$$d^r_{mn}(\omega)=\xi_{mn}(\cos\frac{\omega}{2})^{-p}\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+k)!l!}\left(\frac{t}{2}\right)^{k+2l}\times$$

$$\times\left\{1+\frac{1}{2}l(l+k)\left(\frac{1}{y}-\frac{1}{x}\right)-\frac{l^2(l+k)^2}{4xy}+\right.$$

$$+\frac{k(k^2-1)}{48}\left(\frac{1}{x^2}-\frac{1}{y^2}\right)+$$

$$\left.+\frac{1}{24}(l+k)(l+k-1)(3l^2-l-k-1)\left(\frac{1}{x^2}-\frac{1}{y^2}\right)\right\}. \qquad (44)$$

Here, we have extended the summation over l to infinity since it introduces corrections of the order $\mathcal{O}(\omega^{2r})$ or less. Remembering now the series representation for the Bessel function

$$J_k(t)=\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+k)!l!}\left(\frac{t}{2}\right)^{k+2l}, \qquad (45)$$

we can easily sum up all series in the last formula. Finally, we arrive at the following asymptotic expansion for the d-function

$$d^r_{mn}(\omega)=\xi_{mn}\left\{J_k(t)+\frac{b}{4}\left[J_k(t)+\right.\right.$$

$$+\frac{t}{3\sin^2\theta}(1-2\cos^2\theta)(J_{k-1}(t)-J_{k+1}(t))-$$

$$\left.-\frac{1+\cos^2\theta}{6\sin^2\theta}\left[(k+1)J_{k-2}(t)-(k-1)J_{k+2}(t)\right]\right]+$$

$$\left.+O(\sin^4\frac{\omega}{2})\right\}. \qquad (46)$$

Here, we introduced the following notations:

$$R=2r+1,\ \cos\theta=\frac{p}{R},\ b=\sin^2\frac{\omega}{2},$$

$$t=R\sin\theta\sqrt{b}\ . \qquad (47)$$

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ДУАЛЬНЕ ФОРМУЛЮВАННЯ НЕАБЕЛЕВИХ ГРАТКОВИХ МОДЕЛЕЙ ТА ВІДПОВІДНИХ МАТЕМАТИЧНИХ ПРОБЛЕМ

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Р е з ю м е

Побудовано плакетне та лінкове формулювання компактних неабелевих граткових моделей (таких, як головні кіральні моделі $3D$ КХД $2D$ SU(N)), що відповідають континуальному опису цих моделей в термінах тензора напруженості поля. Вперше одержано точне дуальне формулювання для статистичної суми і для таких фізичних величин, як петля Вільсона, двоточкова кореляційна функція і т.ін. Воно дозволяє вивчати низькотемпературні області неабелевих моделей, важливих для побудови континуальної границі. Знайдено асимптотичний розклад дуальної больцманівської ваги і доведено, що в області низьких температур цей розклад рівномірно збігається за всіма флуктуаціями дуальних змінних до певного гауссівського розподілу. В роботі обговорюється, як одержані результати можуть використовуватися для обчислення різних величин в інфрачервоній області. Особливий інтерес становить побудова нового, рівномірного за представленнями та магнітними числами, асимптотичного розкладу для матричних елементів в матриці обертання SU(2).

ДУАЛЬНОЕ ФОРМУЛИРОВАНИЕ НЕАБЕЛЕВЫХ РЕШЕТОЧНЫХ МОДЕЛЕЙ И СООТВЕСТВУЮЩИХ МАТЕМАТИЧЕСКИХ ПРОБЛЕМ

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Р е з ю м е

Построены плакеточная и линковая формулировки компактных неабелевых решеточных моделей (таких, как главные киральные модели $3D$ КХД $2D$ SU(N)), соответствующие континуальному описанию этих моделей в терминах тензора напряженности поля. Впервые получены точные дуальные формулировки для статистической суммы и для таких физических величин, как петля Вильсона, двухточечная корреляционная функция и т.п. Это позволяет изучать низкотемпературные области неабелевых моделей, важных для построения континуального предела. Найдено асимптотическое разложение дуального больцмановского веса и доказано, что в области низких температур это разложение равномерно сходится по всем флуктуациям дуальных переменных к определенному гауссовскому распределению. В работе обсуждается, как полученные результаты могут использоваться для вычисления различных наблюдаемых в инфракрасной области. Особый интерес может представлять построение нового, равномерного по представлениям и магнитным числам, асимптотического разложения для матричных элементов в матрице вращения SU(2).