SYMMETRY PROPERTIES OF QUATERNIONIC AND BIQUATERNIONIC ANALOGS OF JULIA SETS [1]

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The symmetries intrinsic to quaternions and biquaternions give rise to a class of identical Julia sets, which does not exist in the complex number case. In the case of quadratic mapping $X_{k+1} = \varepsilon X_k^2 + C$ (X, C are biquaternions of special case), these symmetries mean that the shape of a fractal Julia set is completely defined by just two numbers, C_0 and $|\mathbf{C}|$. The involutions defined for the set of biquaternions allow us to investigate discrete symmetries of the geometric images of sets. The introducing of three special types of biquaternions provides a convenience in studying the symmetry properties of these algorithms under discrete transformations as well.

Introduction

Beautiful and unexpected fractal properties of Julia—Fatou and Mandelbrot sets spawned by a simple iteration rule

$$z_{k+1} = z_k^2 + C, \tag{1}$$

with z and a control parameter C being complex numbers, have been intensively studied by many authors [1—3]. Iterations of an arbitrary starting point z_0 in accordance with (1) ultimately result in the confinement of all $z_k, k \geq N_{\max} \gg 1$, in a finite region, a basin of attraction, of some z_i. These z_i are called attractors; they are completely defined by the control parameter C.

A key concept of fractal set research is the special case of $z = \infty$ attractor. In fact, each C of (1) classifies all points of the complex plane as belonging to either runaway subset or prisoners one. If z_0 belongs to a runaway subset, then (1) leads to $z = \infty$. In the other case, ultimate cycles of z_k will reside in a basin of some finite attractor.

A fascinating boundary of the basin of attraction of $z = \infty$ is called Julia set. All information about shapes and topologies of these sets is encoded in C. These sets are mostly fractals.

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$$C_{k+1} = C_k^2 + C_k \tag{2}$$

(i.e. of quadratic mapping (1) with the starting point $z_0 = 0$) form the famous Mandelbrot set. It classifies all Julia sets as either connected or unconnected ones. The latter are known as Cantor dust or Fatou sets.

There were many attempts to apply fractal sets to various fields of physics (see [4, 5]). It was clear from the very beginning that many attractive features of these sets were due to remarkable properties of the algebra of complex numbers. This suggests to generalize the quadratic mapping (1) to different algebras and to study their behavior.

In [6], such a generalization was undertaken for double numbers, which differ from complex ones just in definition of the imaginary unit ε, i.e. $\varepsilon^2 = 1$. Replacement of complex numbers by double numbers actually means a transition from an Euclidean plane to a pseudoeuclidean one, the latter being of special interest for physics, in particular for relativistic kinematics. Another example of validity of double numbers is given by two-dimensional relativistic models, which are prevalent in modern field theories. They also may be successfully described in terms of double numbers [7].

Generally speaking, all hypercomplex algebras, i.e. the algebra of double and dual numbers, of quaternions, biquaternions, and octanions, have proved to be very convenient in numerous physical applications. It may be explained by their close relations with geometries of Euclidean and pseudoeuclidean spaces and with spaces of constant curvature [8].

In this paper, we focus on the quaternion algebra. Really, the 'quaternion language' seems to be especially natural for the physics of our four-dimensional space-time. It has been already successfully exploited in a describing of rigid body motion [9], in searching for instanton solutions of Yang—Mills equations [10], in the

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problem of supersymmetric oscillator [11], and many others.

1. Quaternions and Biquaternions

Let us first briefly recall the fundamentals of the quaternion algebra. The quaternion X is a set of 4 real numbers x_0, x_1, x_2, x_3 with 3 imaginary units e_1, e_2, e_3, and unit e_0. The definition and commutation properties are as follows:

$$X = e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3, \tag{3}$$

where

$$e_1^2 =: e_2^2 = e_3^2 = -e_0 = -1;$$

$$e_1e_2 = -e_2e_1 = e_3;$$

$$e_3e_1 = -e_1e_3 = e_2;$$

$$e_2e_3 = -e_3e_2 = e_1; \tag{4}$$

Hence quaternions are non-commutative with respect to multiplication, so that

$$X_1 \cdot X_2 \neq X_2 \cdot X_1. \tag{5}$$

In some applications, a representation of quaternion (3) as a pair of complex numbers,

$$X = (x_0 + e_1x_1) + (x_2 + e_1x_3)e_2, \tag{6}$$

may turn to be useful. The traditional notation of arbitrary quaternion (3) as a combination of scalar x_0 and three-dimensional vector $\mathbf{x}$ parts,

$$X = x_0 + \mathbf{x}, \tag{7}$$

may be very convenient in many problems. In terms of (7), the operation of quaternion conjugation is simply

$$\bar{X} = x_0 - \mathbf{x}.$$

The product of two arbitrary elements of the quaternion algebra (i.e. of two four-dimensional vectors) in the traditional notations reads:

$$XX' = x_0x_0' - (\mathbf{x} \cdot \mathbf{x}') + x_0\mathbf{x}' + x_0'\mathbf{x} + [\mathbf{x}\mathbf{x}']. \tag{8}$$

Consequently, the square of any quaternion X looks like

$$X^2 = XX = x_0^2 - |\mathbf{x}|^2 + 2x_0\mathbf{x}. \tag{9}$$

It should be noted that the operation of division by a nonzero element ($q \neq 0$) is well defined for quaternions.

The set of quaternions forms a group with respect to the operations of addition and multiplication. However, in the following, we consider in parallel the biquaternion case as well.

Besides a) the complex numbers $z = x + \varepsilon y$ with $\varepsilon^2 = -1$, one may discuss b) the double numbers (sometimes they are referred to as hyperbolic complex numbers) with $\varepsilon^2 = 1$ and c) the dual numbers with $\varepsilon^2 = 0$ [12]. Now, an arbitrary biquaternion may be defined as a linear combination of the four basis elements (4) with coefficients being either complex or double or dual numbers. Moreover, to obtain the object with only four components, we restrict our consideration to the special case of biquaternions; they may be derived from general biquaternions by application of the following condition:

$$\bar{X}^* = -X, \tag{10}$$

where the star-operation denotes the corresponding conjugation in a) – c) coefficient systems, and the bar-operation denotes quaternion conjugation. The introduced biquaternions have the following structure:

$$X = \varepsilon x_0 - \mathbf{x}, \tag{11}$$

where

$$\varepsilon^2 = \begin{cases} -1 & \text{a)} \\ 1 & \text{b)} \\ 0 & \text{c)} \end{cases} . \tag{12}$$

The biquaternions of case a) in (11) are four dimensional vectors of the Minkowski space. There are at least three reasons to introduce biquaternions of the above three types:

1) these biquaternions have interesting physical applications;
2) the algebraic and transformation properties of the biquaternions defined by (10),(11) are closed with respect to quadratic iterations analogical to (1) and allow to discuss the algorithms analogical to the Julia—Fatou ones;
3) the introducing of these three types of biquaternions provides a convenience in studying the symmetry properties of these algorithms under discrete transformations.

2. Quaternionic Analogs of Fractal Sets

The above-mentioned algebraic properties of quaternions permit us to introduce a quaternionic analog of the original complex number Julia—Fatou algorithm (1). Namely,

$$X_{k+1} = X_k^2 + C, \tag{13}$$

with X and the control parameter C being quaternions. This iteration rule maps the four-dimensional Euclidean space on itself. Direct computer experiments justify that (13) classifies again all quaternions with respect to $X = \infty$ as belonging to either prisoner or runaway subsets. The border of these subsets is the quaternionic analog of a Julia set; it is fully defined by coefficients of a mapping rule, e.g. C in case of (13). However, in contrast to complex and double numbers, these fractal sets are four-dimensional.

The first treatment of this problem was given in [13], where some aspects of quadratic mappings

$$X_{k+1} = AX_k^2 + B \tag{14}$$

with A, B and X being quaternions were discussed. It was noted that the general case of the problem is very difficult both for discussion and for representation of results.

To simplify the problem, the consideration of [13] was restricted to a special case of coefficients A and B belonging to the complex number subalgebra of quaternions. It was argued that this choice allowed one to reduce the dimensions of the resulting Julia sets. However, no good arguments were proposed to support this statement.

A particular attention of [13] had been paid to the analysis of topological properties of quaternionic Julia sets. It was noted that, due to noncommutativity of quaternions, there actually exist three different non-equivalent generalizations of (1). Besides (14), one is also to discuss

$$\begin{aligned} X_{k+1} &= X_k A X_k + B, \\ X_{k+1} &= X_k^2 A + B. \end{aligned} \tag{15}$$

The main goal of our research is to provide an instrument for establishing the common features and differences of fractal sets realized by algorithms (13)–(15) from the algebraic, theoretical group point of view.

Let us consider the transformations connected with multiplications and additions of quaternions as transformations of the group of motion of a four-dimensional Euclidean space [8, 15]. It is easy to check that (13)–(15) are invariant under the following transformations:

$$\begin{aligned} X_k^{'} &= Q X_k \bar{Q}, \quad A^{'} = QA\bar{Q}, \\ B^{'} &= QB\bar{Q}, C^{'} = QC\bar{Q}, \end{aligned} \tag{16}$$

where a quaternion Q satisfies the condition

$$Q\bar{Q} = 1, \tag{17}$$

i.e. $\bar{Q} = Q^{-1}$.

Note that, due to commutativity of complex and double numbers, the analogous transformations $q = e^{i\varphi}$ mean just a trivial rotation by an angle φ of the whole plane.

Transformations (16) are inner automorphisms of the division ring of quaternions. They are isomorphic to the transformations of the group of three-dimensional rotations $\mathrm{SO}(3.R)$. Note that x_0 and $|\mathbf{x}|$ stay invariant under such transformations.

Thus, we actually deal with a class of equivalent iteration rules [7]

$$X'_{k+1} = A'(X'_k)^2 + B', \tag{18}$$

which with (14) satisfy relations (16)—(17). Applying these transformations to mapping (13) we obtain the equivalent rules

$$X'_{k+1} = (X'_k)^2 + C'. \tag{19}$$

To illustrate a manifestation of this symmetry, let us show that there is a freedom in orientation of the vector $\mathbf{C}'$ (remember that $C'_0 = C_0, |\mathbf{C}'| = |\mathbf{C}|$). These transformations are analogous to the plane transformations of the Lorentz group found in [16]. For the vector parts of quaternions C and C', the plane transformed quaternion Q and its conjugate $\bar{Q}$ are

$$Q = \frac{\mathbf{C} + \mathbf{C}'}{\sqrt{-(\mathbf{C} + \mathbf{C}')^2}} \, \frac{\bar{\mathbf{C}}}{\sqrt{-\mathbf{C}^2}}, \tag{20}$$

$$\begin{aligned} \bar{Q} &= \frac{\mathbf{C}}{\sqrt{-\mathbf{C}^2}} \, \frac{\bar{\mathbf{C}} + \bar{\mathbf{C}}'}{\sqrt{-(\mathbf{C} + \mathbf{C}')^2}} = \\ &= \frac{\mathbf{C} + \mathbf{C}'}{\sqrt{-(\mathbf{C} + \mathbf{C}')^2}} \, \frac{\bar{\mathbf{C}}'}{\sqrt{-\mathbf{C}'^2}}. \end{aligned} \tag{21}$$

It may be convenient to rotate a vector $\mathbf{C}$ so that to orient a vector $\mathbf{C}'$ along, say, a vector $\mathbf{i}$:

$$C' = C_0 + |\mathbf{C}| i. \tag{22}$$

If all starting points X_0 of (14) lie in this complex plane, the following points X_k will also lie in this plane. Thus, one obtains a complex number Julia set which is a subset of the full quaternionic one.

The most important example of a benefit from (16) in studies of quaternionic Julia set symmetries is given by the transformations that leave the control parameter C intact, i.e.

$$C' = QC\bar{Q} = C, \quad Q\bar{Q} = 1. \tag{23}$$

It is straightforward to check that the proper quaternion is (see [8]):

$$Q = \frac{1 + \mathbf{n}\,\mathrm{tg}\frac{\varphi}{2}}{\sqrt{1 + \mathrm{tg}^2\frac{\varphi}{2}}} = \cos\frac{\varphi}{2} + \mathbf{n}\sin\frac{\varphi}{2}, \tag{24}$$

where $\mathbf{n} = \mathbf{C}/|\mathbf{C}|$.

These transformations form the $O(2)$ group. They are similar to the gauge symmetries of field theories. One can consider the resulting Julia set as a projective space and bundle manifold. The invariance of fractal sets under transformation (24) means that projections of these sets on a three-dimensional subspace are axisymmetric.

These symmetries does not change the zero quaternion component. They do exist for each C_0, but the corresponding two-dimensional Julia subsets are different even for the same $|\mathbf{C}|$. It fact, quaternionic symmetries mean that one needs just two numbers, C_0 and $|\mathbf{C}|$, to describe all possible shapes of quaternionic Julia sets.

It may be interesting to mention that the special choice of parameters A and B in [13] as complex numbers was not obligatory. Symmetries (16) imply that it would suffice just to orient the vector parts of both quaternions in the same direction. The resulting Julia sets would be obtained by rotation of an arbitrary plane Julia subset around the axis oriented along the direction of their vector parts. The transverse cross-sections of these Julia sets consist of concentric circles.

3. Biquaternion Analogs of Fractal Sets and Their Symmetries

The above-mentioned algebraic properties of biquaternions allow us to introduce a biquaternionic analog of the original complex number Julia—Fatou algorithm (1). Namely,

$$X_{k+1} \longrightarrow \varepsilon X_k^2 + C, \tag{25}$$

with X and the control parameter C being biquaternions. This iteration rule maps the four-dimensional Euclidean space ($\varepsilon^2 = 1$), Minkowski space ($\varepsilon^2 = -1$), and four-dimensional Galilei—Newton space ($\varepsilon^2 = 0$) on itself. In contrast to complex and double numbers, these fractal sets are four-dimensional. The symmetry properties of the quaternion set defined by algorithm (14) analogous to (1) under continuous transformations of the $\mathrm{SO}(3.R)$ group have been set up above. Evidently, the symmetry properties of the biquaternion set defined by algorithm (25) under continuous transformations of the $\mathrm{SO}(3.R)$ group are the same.

Now we investigate the symmetry properties of sets generated by algorithm (25) under discrete transformations. The operation of discrete symmetries is connected in our approach with involutions defined for biquaternions of the considered types. Together with algorithm (25), we introduce the algorithms

$$-X_k \longrightarrow -\varepsilon X_k^2 - C, \tag{26}$$

$$\bar{X}_k \longrightarrow \varepsilon \bar{X}_k^2 + \bar{C}, \tag{27}$$

$$X_k^* \longrightarrow -\varepsilon {X^*}_k^2 + C^*, \tag{28}$$

$$-\bar{X}_k \longrightarrow -\varepsilon \bar{X}_k^2 - \bar{C}, \tag{29}$$

$$\bar{X}_k^* \longrightarrow -\varepsilon \bar{X}_k^{*2} + \bar{C}^*. \tag{30}$$

Algorithm (28), due to condition (10), coincides with algorithm (29), and algorithm (30) coincides with (26). Algorithms (25), (26), and (27), (29) generate equivalent but different sets which do not cross in the general case. Unification of the sets generating by (25), (26) and (27), (30) leads to the set which is invariant under a reflection defined by a quaternion conjugation (three-dimensional space reflection) and a full reflection.

Conclusions

Quaternion and biquaternion algebras provides non-trivial generalizations of usual Julia sets. Although these sets are much more complicated, the intrinsic quaternionic symmetries allow one to simplify the problem. It turns out that, in case of quadratic mapping (13), (25), all essentially different quaternionic analogs of Julia sets may be enumerated by just two numbers,

C_0 and $|\mathbf{C}|$. Due to the $O(2)$ symmetry, the three-dimensional part of quaternionic Julia sets may be restored by a rotation of some arbitrary two-dimensional (e.g., a complex number) Julia subset around the axis $\mathbf{n} = \mathbf{C}/|\mathbf{C}|$ which lies in this plane. Three types of biquaternions defined by conditions (10) provide a convenience in studying the symmetry properties of these algorithms under discrete transformations.

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СИМЕТРІЙНІ ВЛАСТИВОСТІ КВАТЕРНІОННИХ І БІКВАТЕРНІОННИХ АНАЛОГІВ МНОЖИН ДЖУЛІЯ

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Р е з ю м е

Симетрії, внутрішні по відношенню до кватерніонів і бікватерніонів, приводять до класу тотожніх множин Джулія, які не існують у випадку комплексних чисел. У випадку квадратичного відображення $X_{k+1} = \varepsilon X_k^2 + C$ (X, C – бікватерніони спеціального виду) ці симетрії вказують на те, що форма фрактальних множин Джулія цілком визначається двома числами C_0 і $|\mathbf{C}|$. Інволюції, визначені для множини бікватерніонів, дозволяють дослідити дискретні симетрії геометричних образів множин. Введення трьох спеціальних типів бікватерніонів також забезпечує деяку зручність у вивченні симетрійних властивостей алгоритмів при дії дискретних перетворень.

СИММЕТРИЙНЫЕ СВОЙСТВА КВАТЕРНИОННЫХ И БИКВАТЕРНИОННЫХ АНАЛОГОВ МНОЖЕСТВ ДЖУЛИЯ

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Р е з ю м е

Внутренние по отношению к кватернионам и бикватернионам симметрии приводят к классу тождественных множеств Джулия, несуществующих в случае комплексных чисел. В случае квадратичного отображения $X_{k+1} = \varepsilon X_k^2 + C$ (X, C – бикватернионы специального вида) эти симметрии указывают на то, что форма фрактальных множеств Джулия полностью определяется двумя числами C_0 и $|\mathbf{C}|$. Инволюции, определенные для множества бикватернионов, позволяют исследовать дискретные симметрии геометрических образов множеств. Введение трех специальных типов бикватернионов также обеспечивает некоторое удобство в изучении симметрийных свойств алгоритмов при действии дискретных преобразований.