ISOMORPHISM CONNECTING THE FREE FIELD DIRAC EQUATION AND MAXWELL'S EQUATIONS

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It is known that if one considers the state spinor $\psi(x)$ as a field operator satisfying the Dirac equation in interaction with a Maxwell field, then it satisfies [1]

$$(i\gamma\frac{\partial}{\partial x}-m)\langle 0|\psi(x)|p\rangle=-\bar{\Sigma}_1^{(0)}(m^2)\Big\langle 0|\psi^{(0)}(x)|p\Big\rangle,$$

where the right-hand side comes from higher order interactions in the current operator $j^{\mu}(x)=\langle\bar{\psi}(x)|\gamma^{\mu}\psi(x)\rangle$. In the above equation, the theoretical mass of an electron should be infinite to compensate with the mass renormalization part $\bar{\Sigma}_1^{(0)}(m^2)$. In the present paper, in discussing the connection between the free field Maxwell's equations and the free field Dirac equation, we assume the bare mass of a lepton to be finite. The isomorphism connecting an extended Dirac equation [2, 3] with the free Maxwell's equation is also discussed.

Introduction

The connection between the massless Dirac equation and the free field Maxwell's equations has already been shown [4, 5]. Since the bare mass of the electron in the free field Dirac equation cannot be put to zero [6, 7], an equivalent connection will be given here, without setting $m=0$ in the free field Dirac equation.

For this goal, we first consider the connection to the Weyl equation (1922)

$$H_{\rm W}\psi=E_{\rm W}\psi \qquad (1)$$

which is known to describe the massless neutrino, and where

$$H_{\rm W}=-\vec{\sigma}\cdot\vec{p} \qquad (2)$$

where $\vec{\sigma}$ are the Pauli matrices, giving

$$E_{\rm W}=\pm|\vec{p}| \qquad (3)$$

for neutrino and antineutrino, respectively. With the helicity defined as

$$s=\frac{1}{p}\vec{\sigma}\cdot\vec{p}, \qquad (4)$$

we find that the eigenvalues of helicity are opposite to those of energy. Thus the neutrino ($E_{\rm W}=+p$) has negative helicity ($s=-1$); and the antineutrino ($E_{\rm W}=-p$) has positive helicity ($s=+1$). It may be noted that the Weyl Hamiltonian $H_{\rm W}$ commutes with the total angular momentum

$$\vec{M}=\vec{r}\wedge\vec{p}+\tfrac{1}{2}\vec{\sigma}. \qquad (5)$$

The eigenfunctions of the Weyl Hamiltonian $H_{\rm W}$ in terms of the electromagnetic fields $\vec{\mathcal{E}},\vec{\mathcal{B}}$ satisfying the free field Maxwell's equations are

$$\mathrm{rot}\vec{\mathcal{E}}+\frac{\partial\vec{\mathcal{B}}}{\partial t}=0,\ \ \mathrm{rot}\vec{\mathcal{B}}-\frac{\partial\vec{\mathcal{E}}}{\partial t}=0,\ \ \mathrm{div}\vec{\mathcal{B}}=\mathrm{div}\vec{\mathcal{E}}=0 \qquad (6)$$

are Ψ_1,Ψ_2.

For a time independent field, the Weyl equation (1) is expressed as

$$(-\vec{p}\cdot\vec{\sigma}\mp p_0)\psi=0 \qquad (7)$$

for neutrino and antineutrino (negative and positive helicity), respectively. Consequently, for the time independent Maxwell field

$$\vec{\mathcal{E}},\vec{\mathcal{B}}=(\vec{\mathcal{E}}_0,\vec{\mathcal{B}}_0)\,e^{-i\vec{p}\cdot\vec{x}\pm ip_0t}, \qquad (8)$$

the Maxwell's equations (6) can be expressed as

$$\pm p_0\vec{\mathcal{B}}-(\vec{p}\wedge\vec{\mathcal{E}})=0,\ \ \pm p_0\vec{\mathcal{E}}+(\vec{p}\wedge\vec{\mathcal{B}})=0,$$
$$\vec{\mathcal{B}}\cdot\vec{p}=\vec{\mathcal{E}}\cdot\vec{p}=0. \qquad (9)$$

Expressing Eq.(7) in the components of ψ

$$(p_0+p_3)\psi_1+(p_1-ip_2)\psi_2=0,$$
$$(p_1+ip_2)\psi_1+(p_0-p_3)\psi_2=0,\ \ \text{for}\ \ E_{\rm W}=+p_0,$$
$$(-p_0+p_3)\psi_1+(p_1-ip_2)\psi_2=0,$$
$$(p_1+ip_2)\psi_1-(p_0+p_3)\psi_2=0,\ \ \text{for}\ \ E_{\rm W}=-p_0, \qquad (10)$$

we then find that there is only one solution for ψ_1,ψ_2, depending linearly on the components of the fields $\vec{\mathcal{E}},\vec{\mathcal{B}}$, that satisfies the Maxwell's equation (6). For each of the neutrino and antineutrino cases (or negative and positive helicity, respectively),

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{E_M^2 - 2P_3}} \begin{pmatrix} \mathcal{E}_- - i\mathcal{B}_- \\ -\mathcal{E}_3 + i\mathcal{B}_3 \end{pmatrix} \qquad (11)$$

corresponding to the negative helicity case, and

$$\begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \frac{1}{\sqrt{E_M^2 - 2P_3}} \begin{pmatrix} \mathcal{E}_3 + i\mathcal{B}_3 \\ \mathcal{E}_+ + i\mathcal{B}_+ \end{pmatrix} \qquad (12)$$

for positive helicity. Here,

$$\mathcal{E}_\pm = \mathcal{E}_1 \pm i\mathcal{E}_2 \ , \ \ \mathcal{B}_\pm = \mathcal{B}_1 \pm i\mathcal{B}_2$$

$$E_M^2 = \vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2 \ , \ \ \vec{P} = \mathcal{E} \wedge \mathcal{B}. \qquad (13)$$

1. Helicity Eigenfunctions—Dirac Equation

For the other leptons (electron, μ-meson, and τ-meson), the Dirac equation

$$H_\mathrm{D}\psi = E_\mathrm{D}\psi \qquad (14)$$

shows a symmetry behaviour between left and right handed leptons. The four-dimensional Dirac Hamiltonian H_D is

$$H_\mathrm{D} = \vec{\alpha}\cdot\vec{p} + m\beta, \qquad (15)$$

the known $\vec{\alpha}$ and β matrices are given by

$$\vec{\alpha} = \begin{pmatrix} \cdot & \vec{\sigma} \\ \vec{\sigma} & \cdot \end{pmatrix} \ , \ \ \beta = \begin{pmatrix} \sigma_0 & \cdot \\ \cdot & -\sigma_0 \end{pmatrix}, \qquad (16)$$

where σ_0 stands for the two-dimensional unit matrix. Eq.(14) gives then for the eigenvalues

$$E_\mathrm{D} = \pm\sqrt{m^2 + p^2} \qquad (17)$$

for leptons and antileptons, respectively (taken twice).

Similar to Eq.(5), the Dirac Hamiltonian H_D also commutes with the total angular momentum

$$\vec{M} = \vec{r}\wedge\vec{p} + \tfrac{1}{2}\vec{\Sigma}, \qquad (18)$$

where now $\vec{\Sigma}$ is given as

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & \cdot \\ \cdot & \vec{\sigma} \end{pmatrix}. \qquad (19)$$

The Dirac Hamiltonian also commutes with the helicity

$$s = \frac{1}{|\vec{p}|}(\vec{\Sigma}\cdot\vec{p}) \qquad (20)$$

which is an extension of definition (4).

We can then proceed to find the electromagnetic solutions for the extended helicity operator $\vec{p}\cdot\vec{\Sigma}$ as a solution of

$$(\vec{p}\cdot\vec{\Sigma})\psi_\mathrm{h} = \pm p_0\psi_\mathrm{h} \qquad (21)$$

corresponding to positive and negative extended helicity. Owing to the form of $\vec{\Sigma}$ as given by Eq. (14), any solution of the above equation can be expressed as a linear combination of the following two solutions:

$$\psi_\mathrm{h} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \end{pmatrix}, \ \begin{pmatrix} \cdot \\ \cdot \\ \psi_1 \\ \psi_2 \end{pmatrix} \qquad (22)$$

for negative extended helicity, and similarly

$$\psi_\mathrm{h} = \begin{pmatrix} \psi_1' \\ \psi_2' \\ \cdot \\ \cdot \end{pmatrix}, \ \begin{pmatrix} \cdot \\ \cdot \\ \psi_1' \\ \psi_2' \end{pmatrix} \qquad (23)$$

for positive extended helicity. Here, $\psi_1, \psi_2; \psi_1', \psi_2'$ are given by Eqs. (11) and (12). The combinations of the above solutions which satisfy, at the same time, the Dirac equation (14) can then be easily obtained as

$$\psi_+ = \begin{pmatrix} \psi_1\cos\theta_m \\ \psi_2\cos\theta_m \\ -\psi_1\sin\theta_m \\ -\psi_2\sin\theta_m \end{pmatrix}, \ \begin{pmatrix} \psi_1\sin\theta_m \\ \psi_2\sin\theta_m \\ \psi_1\cos\theta_m \\ \psi_2\cos\theta_m \end{pmatrix}, \qquad (24)$$

being the eigenfunctions of H_D corresponding to the eigenvalues $E_\mathrm{D} = \pm\sqrt{p^2 + m^2}$, respectively. Similarly, for positive helicity eigenfunctions of H_D,

$$\psi_- = \begin{pmatrix} \psi_1'\cos\theta_m \\ \psi_2'\cos\theta_m \\ \psi_1'\sin\theta_m \\ \psi_2'\sin\theta_m \end{pmatrix}, \ \begin{pmatrix} -\psi_1'\sin\theta_m \\ -\psi_2'\sin\theta_m \\ \psi_1'\cos\theta_m \\ \psi_2'\cos\theta_m \end{pmatrix} \qquad (25)$$

corresponding to the eigenvalues of H_D, $E_\mathrm{D} = \pm\sqrt{p^2 + m^2}$, respectively. Here, θ_m is given by

$$\cos\theta_m = \sqrt{\tfrac{E+m}{2E}}, \ \ \sin\theta_m = \sqrt{\tfrac{E-m}{2E}}. \qquad (26)$$

The solutions of the free field Dirac equation, given by Eqs. (24), (25) are then the required solutions in terms of the electromagnetic fields $\vec{\mathcal{E}}, \vec{\mathcal{B}}$ that satisfy the free field Maxwell's equations. Thus this completes the isomorphism between the free field Dirac equation and the free field Maxwell's equations.

2. Higher Dimensions — the Extended Hamiltonian

Similar to the relativistic covariant form of the free field Dirac equation

$$(i\gamma\frac{\partial}{\partial x} - m)\psi(x) = 0, \tag{27}$$

an extended equation can be expressed as

$$i\Gamma^\mu \frac{\partial\psi(x)}{\partial x^\mu} = 0, \tag{28}$$

where $\psi(x)$ is an eight column spinor and Γ^μ ($\mu = 1$ to 7) are 8×8 matrices. The seven-dimensional vector x^μ has three usual space components $x^k = -x_k$ ($k = 1,2,3$) and the observable time component t; another two unobserved space components $x^5 = -x_5$, $x^7 = -x_7$; and one unobserved time component $x^6 = x_6$. For a free field, we can set

$$\psi(x) = u(p)e^{-i(p,x)} \quad ((p,x) = p_\mu x^\mu). \tag{29}$$

Thus,

$$\frac{\partial\psi(x)}{\partial x^\mu} = -ip_\mu\psi(x). \tag{30}$$

We take Γ^7 in Eq. (28) to be the unit 8×8 matrix and the corresponding $p_7 = -p^7 = -m$. Hence, Eq. (28) can be set in the equivalent form as

$$i\Gamma^\lambda \frac{\partial\psi(x)}{\partial x^\lambda} - m\psi(x) = 0, \tag{31}$$

where λ now runs from 1 to 6.

Before considering the possible forms of the extended Dirac matrices Γ^λ, it may be worth referring to the Onsager exact solution of the order-disorder transition in the two-dimensional Ising model [9, 10], using the basis of quaternion algebra. The partition function was solved by direct product decomposition of the 2^n-dimensional quaternion matrices into the two-dimensional representation of the quaternion group $i\sigma_k$ ($k = 1,2,3$) and σ_0. We notice that the Dirac 4×4 γ-matrices can be expressed as a direct product of σ-matrices:

$$\gamma^k = i\sigma_2\times\sigma_k,\ \ \gamma^0 = \beta = \sigma_3\times\sigma_0. \tag{32}$$

Following a similar procedure, one can proceed further in forming higher dimension Dirac matrices, as a product of several σ-matrices, following the prescription in [6]

$$\Gamma^k = i\sigma_2\times\sigma_2\times\sigma_k = \begin{pmatrix} \cdot & -i\gamma^k \\ i\gamma^k & \cdot \end{pmatrix},$$

$$\Gamma^4 = \sigma_2\times\sigma_3\times\sigma_0 = \begin{pmatrix} \cdot & -i\beta \\ i\beta & \cdot \end{pmatrix},$$

$$\Gamma^5 = i\sigma_2\times\sigma_1\times\sigma_0 = \begin{pmatrix} \cdot & \gamma^5 \\ -\gamma^5 & \cdot \end{pmatrix},$$

$$\Gamma^6 = \sigma_2\times\sigma_0\times\sigma_0 = \begin{pmatrix} \cdot & -iI \\ iI & \cdot \end{pmatrix}. \tag{33}$$

The idea of postulating more time components was already proposed [10] on the assumption that spacetime comprises a union of subspaces of four dimensions. In the present investigation, the fact that the extended equation describes two particle states, instead of one, was only possible by the addition of another unobservable time component.

The relativistic covariance of the extended Dirac equation in the form of Eq. (31) can be confirmed by assuming an extended Lorentz transformation in the space 4+2, similar to that suggested [11] in the space 3+3.

In (33), we notice that Γ^k , Γ^5 are anti-Hermitian, while Γ^4, Γ^6 are Hermitian. Also Γ^4 commutes with Γ^6 and anticommutes with Γ^k, Γ^5. Defining the spinor adjoint by

$$\bar\psi(x) = \psi^+(x)\Gamma^4, \tag{34}$$

then the adjoint of Eq.(28) becomes

$$-i\frac{\partial\bar\psi(x)}{\partial x^\mu}\Gamma^\mu = 0. \tag{35}$$

A seven-dimensional current density operator vector can then be expressed as

$$j^\mu(x) = \bar\psi(x)\Gamma^\mu\psi(x) \tag{36}$$

satisfying the divergence-free equation

$$\frac{\partial j^\mu(x)}{\partial x^\mu} = 0. \tag{37}$$

We express Eq.(14) in the form

$$i\Gamma^4\frac{\partial\psi(x)}{\partial t} + i\Gamma^\lambda\frac{\partial\psi(x)}{\partial x^\lambda} = 0$$

or

$$i\frac{\partial\psi(x)}{\partial t} + i\Gamma^4\Gamma^\lambda\frac{\partial\psi(x)}{\partial x^\lambda} = 0 \quad (\lambda = 1,2,3,5,6,7) \tag{38}$$

which can be set in the Schrödinger form

$$i\frac{\partial\psi(x)}{\partial t} = H\psi(x), \tag{39}$$

where

$$H = -\Gamma^4\Gamma^\lambda p_\lambda. \tag{40}$$

On using the forms of Γ^λ given by Eq. (33) (Γ^7 is the unit matrix), we find

$$\Gamma^4\Gamma^k = \begin{pmatrix} \alpha^k & \cdot \\ \cdot & \alpha^k \end{pmatrix}, \quad \Gamma^4\Gamma^5 = \begin{pmatrix} i\gamma'^5 & \cdot \\ \cdot & i\gamma'^5 \end{pmatrix},$$

$$\Gamma^4\Gamma^6 = \begin{pmatrix} \beta & \cdot \\ \cdot & \beta \end{pmatrix}, \tag{41}$$

where $\vec{\alpha}$ is given by Eq. (16), $\gamma'^5 = \beta\gamma^5 = \begin{pmatrix} \cdot & \sigma_0 \\ -\sigma_0 & \cdot \end{pmatrix}$. Since $p_k = -p^k$, $p_5 = -p^5$, $p_7 = -p^7 = -m$, we find that the expression of H given by Eq. (40) becomes

$$H = \\ = \begin{pmatrix} (-\alpha^k p_k - i\gamma'^5 p_5 - \beta p_6) & -i\beta m \\ i\beta m & (-\alpha^k p_k - i\gamma'^5 p_5 - \beta p_6) \end{pmatrix}. \tag{42}$$

The eigenvalues can be obtained from the equation

$$Hu = Eu, \tag{43}$$

where E represents the observed eigenvalues. The above equation can be expressed in the eight components of $u(p)$ as

$$\begin{aligned} -p_6u_1 + (p_3 + ip_5)u_3 + (p_1 - ip_2)u_4 - imu_5 = Eu_1, \\ -p_6u_2 + (p_1 + ip_2)u_3 - (p_3 - ip_5)u_4 - imu_6 = Eu_2, \\ (p_3 - ip_5)u_1 + (p_1 - ip_2)u_2 + p_6u_3 + imu_7 = Eu_3, \\ (p_1 + ip_2)u_1 - (p_3 + ip_5)u_2 + p_6u_4 + imu_8 = Eu_4, \end{aligned} \tag{44}$$

$$\begin{aligned} imu_1 - p_6u_5 + (p_3 + ip_5)u_7 + (p_1 - ip_2)u_8 = Eu_5, \\ imu_2 - p_6u_6 + (p_1 + ip_2)u_7 - (p_3 - ip_5)u_8 = Eu_6, \\ -imu_3 + (p_3 - ip_5)u_5 + (p_1 - ip_2)u_6 + p_6u_7 = Eu_7, \\ -imu_4 + (p_1 + ip_2)u_5 - (p_3 + ip_5)u_6 + p_6u_8 = Eu_8. \end{aligned} \tag{45}$$

Substituting from the first four equations into the last four, we get four equations in u_1, u_2, u_3, u_4, from which we obtain

$$u_3 = \frac{2p_6\{(p_3 - ip_5)u_1 + (p_1 - ip_2)u_2\}}{m^2 + p^2 + p_5^2 - (p_6 - E)^2},$$

$$u_4 = \frac{2p_6\{(p_1 + ip_2)u_1 - (p_3 + ip_5)u_2\}}{m^2 + p^2 + p_5^2 - (p_6 - E)^2}, \tag{46}$$

$$[\{m^2 + p^2 + p_5^2 - (p_6 - E)^2\}\{m^2 + p^2 + p_5^2 - \\ -(p_6 + E)^2\} + 4p_6^2(p^2 + p_5^2)]u_i = 0, \quad i = 1, 2, \tag{47}$$

showing that u_1, u_2 can be chosen arbitrary, on choosing for the eigenvalues of E,

$$E^2 = p^2 + p_5^2 + (p_6 \pm m)^2, \tag{48}$$

taken twice. This clearly shows that the present model describes two lepton states of different masses M_1, M_2 given by

$$M_1^2 = p_5^2 + (p_6 + m)^2, \quad M_2^2 = p_5^2 + (p_6 - m)^2. \tag{49}$$

3. Extended Helicity Eigen Functions

In order to obtain the spinors corresponding to the different eigenvalues $\pm E_1$, $\pm E_2$, we start first to arrange them according to their helicities. We extend the definition of the helicity operator to be

$$S = \frac{1}{|\vec{p}|}\begin{pmatrix} \vec{\Sigma}\cdot\vec{p} & \cdot \\ \cdot & \vec{\Sigma}\cdot\vec{p} \end{pmatrix}, \tag{50}$$

where $\vec{\Sigma}$ is defined by Eq. (19). From expression (42) for the Hamiltonian H, it is clear that the helicity operator S commutes with the Hamiltonian. Hence, H and S have the same eigenfunctions.

For positive helicity, simple eigenfunctions are given by

$$w_i = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \psi_1 \\ \psi_2 \end{pmatrix}, \tag{51}$$

$i = 1, 2, 3, 4$; and ψ_1 ,ψ_2 are given by Eq. (11).

Also any combination of w_1, w_2, w_3, w_4 is a positive helicity eigenfunction of the helicity operator given by Eq. (50).

On the other hand, the negative helicity eigenfunctions are any linear combination of the following simple eigenfunctions:

$$w_i = \begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \psi'_1 \\ \psi'_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \psi'_1 \\ \psi'_2 \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \psi'_1 \\ \psi'_2 \end{pmatrix}, \tag{52}$$

$i=5,6,7,8$; and ψ_1', ψ_2' are given by Eq. (12).

We now choose eigenfunctions for positive and negative helicity w_+, w_-, respectively, as linear combinations:

$$w_+ = c_1w_1 + c_2w_2 + c_3w_3 + c_4w_4,$$

$$w_- = c_5w_5 + c_6w_6 + c_7w_7 + c_8w_8, \qquad (53)$$

where w_i $(i=1,2,3,4)$ are given by Eq. (51), and w_i $(i=5,6,7,8)$ are given by Eq. (52), and such that they are, at the same time, eigenfunctions of H:

$$Hw_+ = Ew_+, \quad Hw_- = Ew_-. \qquad (54)$$

On using Eq. (42) for H, one finds that c_i $(i=1\to 8)$ are given by

$$-p_6c_1 + (p+ip_5)c_2 - imc_3 = Ec_1,$$

$$(p-ip_5)c_1 + p_6c_2 + imc_4 = Ec_2,$$

$$imc_1 - p_6c_3 + (p+ip_5)c_4 = Ec_3,$$

$$-imc_2 + (p-ip_5)c_3 + p_6c_4 = Ec_4,$$

$$-p_6c_5 - (p-ip_5)c_6 - imc_7 = Ec_5,$$

$$-(p+ip_5)c_5 + p_6c_6 + imc_8 = Ec_6,$$

$$imc_5 - p_6c_7 - (p-ip_5)c_8 = Ec_7,$$

$$-imc_6 - (p+ip_5)c_7 + p_6c_8 = Ec_8. \qquad (55)$$

To simplify the obtained expressions, we use the following definitions of $\theta_1, \theta_2, \varphi$:

$$p_6 + m = E_1\cos 2\theta_1, \quad p_6 - m = E_2\cos 2\theta_2, \qquad p \pm ip_5 = E_1 \sin 2\theta_1 e^{\pm i\varphi} = E_2 \sin 2\theta_2 e^{\pm i\varphi}. \qquad (56)$$

Thus, we get the values of c_i $(i=1,2,3,4)$ for different particle energies $(E=E_1, E_2, -E_1, -E_2)$. For positive helicity,

Value	E_1	E_2	$-E_1$	$-E_2$
$c_1\sqrt{2}$	$sin\theta_1 e^{i\varphi}$	$\sin\theta_2 e^{i\varphi}$	$\cos\theta_1 e^{i\varphi}$	$\cos\theta_2 e^{i\varphi}$
$c_2\sqrt{2}$	$\cos\theta_1$	$\cos\theta_2$	$-\sin\theta_1$	$-\sin\theta_2$
$c_3\sqrt{2}$	$-i\sin\theta_1 e^{i\varphi}$	$i\sin\theta_2 e^{i\varphi}$	$-i\cos\theta_1 e^{i\varphi}$	$i\cos\theta_2 e^{i\varphi}$
$c_4\sqrt{2}$	$-i\cos\theta_1$	$i\cos\theta_2$	$i\sin\theta_1$	$-i\sin\theta_2$.

(57)

For negative helicity eigenfunctions, we get similarly

Value	E_1	E_2	$-E_1$	$-E_2$
$c_5\sqrt{2}$	$-\sin\theta_1 e^{-i\varphi}$	$-\sin\theta_2 e^{-i\varphi}$	$-\cos\theta_1 e^{-i\varphi}$	$-\cos\theta_2 e^{-i\varphi}$
$c_6\sqrt{2}$	$\cos\theta_1$	$\cos\theta_2$	$-\sin\theta_1$	$-\sin\theta_2$
$c_7\sqrt{2}$	$i\sin\theta_1 e^{-i\varphi}$	$-i\sin\theta_2 e^{-i\varphi}$	$i\cos\theta_1 e^{-i\varphi}$	$-i\cos\theta_2 e^{-i\varphi}$
$c_8\sqrt{2}$	$-i\cos\theta_1$	$i\cos\theta_2$	$i\sin\theta_1$	$-i\sin\theta_2$.

(58)

Substituting these values of c_i $(i=1\to 8)$ in (53) for w_+, w_-, we get the following expressions of $u^{(1)}, u^{(2)}; u'^{(1)}, u'^{(2)}$, the eigenstates (positive and negative helicities) corresponding to the eigenvalues E_1, E_2, respectively:

$$u^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sin\theta_1 e^{i\varphi}\psi_1 \\ \sin\theta_1 e^{i\varphi}\psi_2 \\ \cos\theta_1\psi_1 \\ \cos\theta_1\psi_2 \\ -i\sin\theta_1 e^{i\varphi}\psi_1 \\ -i\sin\theta_1 e^{i\varphi}\psi_2 \\ -i\cos\theta_1\psi_1 \\ -i\cos\theta_1\psi_2 \end{pmatrix}, \quad u^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin\theta_1 e^{-i\varphi}\psi_1' \\ -\sin\theta_1 e^{-i\varphi}\psi_2' \\ \cos\theta_1\psi_1' \\ \cos\theta_1\psi_2' \\ i\sin\theta_1 e^{-i\varphi}\psi_1' \\ i\sin\theta_1 e^{-i\varphi}\psi_2' \\ -i\cos\theta_1\psi_1' \\ -i\cos\theta_1\psi_2' \end{pmatrix}, \qquad (59)$$

$$u'^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin\theta_2 e^{i\varphi}\psi_1 \\ \sin\theta_2 e^{i\varphi}\psi_2 \\ \cos\theta_2\psi_1 \\ \cos\theta_2\psi_2 \\ i\sin\theta_2 e^{i\varphi}\psi_1 \\ i\sin\theta_2 e^{i\varphi}\psi_2 \\ i\cos\theta_2\psi_1 \\ i\cos\theta_2\psi_2 \end{pmatrix}, \quad u'^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin\theta_2 e^{-i\varphi}\psi'_1 \\ -\sin\theta_2 e^{-i\varphi}\psi'_2 \\ \cos\theta_2\psi'_1 \\ \cos\theta_2\psi'_2 \\ -i\sin\theta_2 e^{-i\varphi}\psi'_1 \\ -i\sin\theta_2 e^{-i\varphi}\psi'_2 \\ i\cos\theta_2\psi'_1 \\ i\cos\theta_2\psi'_2 \end{pmatrix}. \qquad (60)$$

We similarly get the following expressions of $\nu^{(1)}$, $\nu^{(2)}$, $\nu'^{(1)}$, $\nu'^{(2)}$ [the eigenstates (negative and positive helicities) corresponding to the eigenvalues of the anti-particles of energies $-E_1$, $-E_2$, respectively]:

$$\nu^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos\theta_1 e^{-i\varphi}\psi'_1 \\ -\cos\theta_1 e^{-i\varphi}\psi'_2 \\ -\sin\theta_1\psi'_1 \\ -\sin\theta_1\psi'_2 \\ i\cos\theta_1 e^{-i\varphi}\psi'_1 \\ i\cos\theta_1 e^{-i\varphi}\psi'_2 \\ i\sin\theta_1\psi'_1 \\ i\sin\theta_1\psi'_2 \end{pmatrix}, \quad v^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta_1 e^{i\varphi}\psi_1 \\ \cos\theta_1 e^{i\varphi}\psi_2 \\ -\sin\theta_1\psi_1 \\ -\sin\theta_1\psi_2 \\ -i\cos\theta_1 e^{i\varphi}\psi_1 \\ -i\cos\theta_1 e^{i\varphi}\psi_2 \\ i\sin\theta_1\psi_1 \\ i\sin\theta_1\psi_2 \end{pmatrix}, \qquad (61)$$

$$\nu'^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos\theta_2 e^{-i\varphi}\psi'_1 \\ -\cos\theta_2 e^{-i\varphi}\psi'_2 \\ -\sin\theta_2\psi'_1 \\ -\sin\theta_2\psi'_2 \\ -i\cos\theta_2 e^{-i\varphi}\psi'_1 \\ -i\cos\theta_2 e^{-i\varphi}\psi'_2 \\ -i\sin\theta_2\psi'_1 \\ -i\sin\theta_2\psi'_2 \end{pmatrix}, \quad \nu'^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta_2 e^{i\varphi}\psi_1 \\ \cos\theta_2 e^{i\varphi}\psi_2 \\ -\sin\theta_2\psi_1 \\ -\sin\theta_2\psi_2 \\ i\cos\theta_2 e^{i\varphi}\psi_1 \\ i\cos\theta_2 e^{i\varphi}\psi_2 \\ -i\sin\theta_2\psi_1 \\ -i\sin\theta_2\psi_2 \end{pmatrix}. \qquad (62)$$

The eigenstates given by Eqs. (59)—(62) are mutually orthogonal. The eight columns representing the eight states can then represent a unitary 8×8 matrix.

1. *Källén G.*// Dan.Mat.Fys.Medd. **27**, N12 (1953).
2. *Sabry A.A.*// Proc. Intern. Conf., Uzhgorod, Ukraine, August 18–20 1997, P. 83–87.
3. *Sabry A.A.*// Crimean Summer School-Seminar "New Trends in High Energy Physics" Crimea, Ukraine, May 27–June 4, 2000, P. 287–290.
4. *Krivsky Yu., Simulik V.M.*//Adv. Appl. Clifford Algebra, **6**, 249 (1996).
5. *Krivsky Yu., Simulik V.M.* // Proc. of the Rence, Uzhgorod, August 13–16, 1997, P. 200 – 209.
6. *Sabry A.A.*// Proc. 2nd Intern. Conf. on Basic Sciences and Advanced Technology, Assiut, Nov. 5–23.
7. *Sabry A.A.*// Proc. Intern. Conf. on Trends in High Energy Physics, Yalta, Crimea, Sept.22–29, 2001, P. 328–335.
8. *Onsager L.*// Phys. Rev., **65**, N3,4 (1944) P.117.
9. *Kaufman B.*// Ibid., **76**, N8 (1949) P.1232.
10. *Newton R.H.*// Intern. J. Theor. Phys., **26** (1987), P. 1227–1246.
11. *Sabry A.A.*// Proc. Ninth Annual Seminar NPCS'2000,(Nonlinear Phenomena in Complex Systems), May 16–19, 2000, Minsk, Belarus, P. 223–227.

ІЗОМОРФІЗМ, ЩО ПОВ'ЯЗУЄ РІВНЯННЯ ДІРАКА ДЛЯ ВІЛЬНОГО ПОЛЯ ТА РІВНЯННЯ МАКСВЕЛЛА

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Р е з ю м е

Відомо, якщо розглядати спінорний стан $\psi(x)$ як оператор поля, що задовольняє рівняння Дірака при взаємодії з полем Максвелла, то він задовольняє рівняння (1)

$$(i\gamma\frac{\partial}{\partial x} - m)\langle 0|\psi(x)|p\rangle = -\bar{\Sigma}_1^{(0)}(m^2)\Big\langle 0|\psi^{(0)}(x)|p\Big\rangle,$$

де права частина утворюється із взаємодії вищих порядків по оператору струму $j^\mu(x) = \big\langle\bar{\psi}(x)|\gamma^\mu\psi(x)\big\rangle$. В наведеному вище рівнянні теоретична маса електрона має бути нескінченною,

щоб компенсувати масу ренормалізованої частини $\bar{\Sigma}_1^{(0)}(m^2)$. В даній роботі при обговоренні взаємодії між рівняннями Максвелла для вільного поля і рівнянням Дірака для вільного поля ми вважаємо голу масу лептона скінченною. Також обговорюється ізоморфізм, що пов'язує розширене рівняння Дірака (2), (3) з рівнянням Максвелла для вільних полів.

ИЗОМОРФИЗМ, СВЯЗЫВАЮЩИЙ УРАВНЕНИЕ ДИРАКА ДЛЯ СВОБОДНОГО ПОЛЯ И УРАВНЕНИЯ МАКСВЕЛЛА

А. А. Сабри

Р е з ю м е

Известно, что если рассматривать спинорное состояние $\psi(x)$ как оператор поля, удовлетворяющий уравнению Дирака, при взаимодействии с полем Максвелла, то оно удовлетворяет уравнению (1)

$$(i\gamma\frac{\partial}{\partial x} - m)\langle 0|\psi(x)|p\rangle = -\bar{\Sigma}_1^{(0)}(m^2)\Big\langle 0|\psi^{(0)}(x)|p\Big\rangle,$$

где правая часть образуется из взаимодействий высших порядков по оператору тока $j^\mu(x) = \langle\bar{\psi}(x)|\gamma^\mu\psi(x)\rangle$. В приведенном выше уравнении теоретическая масса электрона должна быть бесконечной, чтобы компенсировать массу ренормализационной части $\bar{\Sigma}_1^{(0)}(m^2)$. В настоящей работе при обсуждении взаимосвязи между уравнениями Максвелла для свободного поля и уравнением Дирака для свободного поля мы считаем голую массу лептона конечной. Также обсуждается изоморфизм, связывающий расширенное уравнение Дирака (2), (3) с уравнением Максвелла для свободных полей.