ON THE BOUND STATES OF HEAVY QUARKS

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A complete solution of the Schrödinger equation for a quark confining fractional power potential is presented. The S-state energy levels of the Ψ and Υ families are calculated. Good results are obtained in comparison with the experimental data.

Introduction

The discovered resonances Ψ and Υ are interpreted as bound states of quark-antiquark ($c\bar{c}$) and ($b\bar{b}$), respectively. Several attempts have been made to discuss the nature of the interquark interaction with the linear potential receiving the greatest attention. As a result of the heavy mass of the constituent quarks and because of the asymptotic freedom [1] which is required by QCD, it seems plausible to apply the nonrelativistic potential description. From gauge theories, one guesses that $V(r) \approx 1/r$ at short distances or its modified version $1/(r \ln r)$ whereas the linear part of the potential represents the long-range part of the forces. The simplest choice is the Cornell model [2] for which the first extensive study of quarkonia was carried out: $V(r) = -(4/3)(\alpha_s/r)+kr$. Besides the linear potential [3], other types of empirical potentials providing confinement of the quarks have been envisaged, e.g. the logarithmic [4], harmonic oscillator [5], and the fractional power potentials: $Ar^{0.1} + B$ [6]. In order to obtain spin-dependent spectra, one may introduce spin-dependent terms into the potential analogous to those contained in the Fermi—Breit interaction [7] or alternatively one may use the covariant Bethe—Salpeter equation with some approximations [8]. The potential proposed in the present work has an $r^{2/3}$ term which represents the confining part of the potential. This part of the potential gives a good description for the spectrum of the Υ family when potentials of the form: $V(r) = Ar^{0.33\pm0.23}$ are used [9]. The second part $r^{-2/3}$ is softer at the origin than the Coulomb potential as required by QCD calculations [7]. The third part r^{-2} term is needed to obtain analytical expressions for the eigenvalues and eigenfunctions. This of course is useful in some of the calculations which are sensitive to the form of a wave function, e.g. the leptonic decay rates which depend on the wave function at the origin. Exact bound state solutions for fraction power potentials of the form: $V(r) = \alpha r^{2/3} + \beta r^{-2/3} + \gamma r^{-4/3}$ were given by using a suitable ansatz [10]. For each solution (energy level), there is an interrelation between the parameters α, β, and γ of the potential and the orbital angular momentum λ. Such relations become more tedious for higher levels. No unique set of the parameters α, β, and γ can be extracted to fit the whole spectrum but there will be a set of parameters for each level and they are different from each other. Our potential is simple, and contains only two adjustable parameters. In the next section, we show how to obtain the eigenvalues and eigenfunctions for this potential and, in Section 3, we present the results obtained for the Ψ and Υ spectra and compare them with the experimental data.

1. Solution of the Schrödinger Equation

Hereby we follow the notations and the methods used in [11]. Let us consider the one-dimensional Schrödinger equation

$$\phi''(x) + B(x)\phi(x) = 0 \qquad (1)$$

with suitable boundary conditions, here

$$B(x) = 2E - 2V(x), \quad \mu = h/2\pi = 1,$$

where E is the total energy, μ is the reduced mass of the two quarks, and $V(x)$ is the potential energy function. For the radial wave function in 3-dimensions,

$$R(r) = \phi(r)/r$$

where ϕ is the solution to Eq. (1) with

$$B(x) = 2E - 2V - \lambda(\lambda+1)/x^2. \qquad (2)$$

Assume that $\phi(x)$ takes the form

$$\phi(x) = f(x)\phi(g(x)), \qquad (3)$$

where $f(x)$ and $g(x)$ are functions of x to be chosen later on.

Hence, substituting in Eq. (1), we obtain the differential equation satisfied by ϕ,

$$\phi'' + \Big[\frac{2f'}{fg'} + \frac{g''}{g'^2}\Big]\phi' + \Big[\frac{f''}{fg'^2} + \frac{B}{g'^2}\Big]\phi = 0. \qquad (4)$$

To eliminate the ϕ' term in the above equation we require that f be determined in terms of g in the following manner,

$$f = C_o g'^{-1/2} \qquad (5)$$

where C_o is a non-zero constant.

Hence, the differential equation for ϕ becomes

$$\phi'' + \Big[\frac{B}{g'^2} - \frac{g'''}{2g'^3} + \frac{3g''^2}{4g'^4}\Big]\phi(g) = 0. \qquad (6)$$

Furthermore, if we demand that the coefficient of ϕ in the above equation be quadratic in g, i.e. the function ϕ satisfy the general differential equation for a parabolic cylinder function, then we have

$$B = \frac{g'''}{2g'} - \frac{3g''^2}{4g'^2} + g'^2(ag^2 + bg + c) = 0 \qquad (7)$$

where a, b, and c are arbitrary constants, and the differential equation for ϕ now becomes

$$\phi''(g) + (ag^2 + bg + c)\phi(g) = 0. \qquad (8)$$

Choosing $g(x) = x^{2/3}$, equation (7) yields

$$B(x) = \frac{5}{36x^2} + \frac{4}{9}(ax^{2/3} + b + cx^{-2/3}) = 2(E - V). \quad (9)$$

The term x^{-2} does not prevent the occurrence of eigenstates [11], hence, we proceed to determine the eigenvalues and eigenfunctions for this case where

$$V(x) = -\frac{5}{72x^2} - \frac{2}{9}ax^{2/3} - \frac{2}{9}cx^{-2/3} \qquad (10)$$

and $E = 2b/9$.

Following [11], the general solution in our case is

$$\phi(x) = C_o\sqrt{3/2}x^{1/6}U(A, Z), \qquad (11)$$

where U is the parabolic cylinder function [12],

$$A = -\frac{b^2}{8a^{3/2}} - \frac{c}{2a^{1/2}}, \qquad (12)$$

and

$$Z = \sqrt{2}a^{1/4}x^{2/3} - \frac{b}{\sqrt{2}a^{3/4}}. \qquad (13)$$

Here, $x = 0$ is a zero of $U(A, Z)$ [12], i.e.

$$U\Big(-\frac{b^2}{8a^{3/2}} - \frac{c}{2a^{1/2}}, -\frac{b}{\sqrt{2}a^{3/4}}\Big) = 0. \qquad (14)$$

Equation (14) will be satisfied only for a discrete set of b values which, through Eq. (10), determines the energy spectrum. To deduce the eigenvalues, we use a graphical analysis in the A, Z plane [11]. For $x = 0$ and using Eqs. (12) and (13), we obtain,

$$A = -\frac{Z^2}{4} - \frac{cZ^{2/3}}{2^{2/3}b^{2/3}}. \qquad (15)$$

We note the following property for the zeros of $U(A, Z)$ that each curve in the $A-Z$ plane intersects the negative A-axis once and only once at a point of the form $-(2m+3/2)$, with $m = 0, 1, 2, ...$ In the present problem, we notice that, for $a < 0$ and $c < 0$, Eq. (10) describes a confining plus short-range potential. Using the above-mentioned property, we obtain the energy eigenvalues

$$E_m = \frac{2}{9}\sqrt{16a^{3/2}(m + \frac{3}{4}) - 4ca}. \qquad (16)$$

Equation (16) is the desired equation for the eigenvalues where $E_m \propto m^{1/2}$. Hence, by fitting the parameters a and c with the experimental data for the Ψ and Υ particle spectra, we obtain the theoretical estimate for the eigenvalues using our phenomenological potential. Equation (11) gives the exact wave function for the system under consideration. This wave function can by used to calculate the leptonic decay width.

2. Results and Discussion

Since the quark mass is twice the reduced mass μ, the mass M of the bound quark-antiquark is given by

$$M = 4\mu + E. \qquad (17)$$

Now, writing Eq. (16) by restoring the original dimensions, hence

$$E_m = 2/9\sqrt{16a^{3/2}(m + 3/4)(\hbar^2/\mu)^{1/2} - 4ca}. \qquad (18)$$

Hence, by fixing the charmed quark mass to be equal to 1.2 GeV and taking the value $m_b = 4.6$ GeV for the bottom quark mass, the values of a and c can be calculated using the lowest S-states as input values. The results are shown in Table which shows a good agreement with the experimental values. The values obtained for a and c are as follows:

$a^\Psi = 2.2856, \quad c^\Psi = 0.0797,$

Masses in the Ψ and Υ families

n $^{2s+1}L_J$	masses (GeV)(present work)	Experiment*
	J/Ψ	
1 $^3S_1$	3.0969	3.0969
2 $^3S_1$	3.4868	3.6859
3 $^3S_1$	3.7699	3.7699
4 $^3S_1$	4.0038	4.0400
5 $^3S_1$	4.2077	4.1590
6 $^3S_1$	4.3908	4.4150
	Υ	
1 $^3S_1$	9.4604	9.4603
2 $^3S_1$	10.0146	10.0233
3 $^3S_1$	10.3222	10.3552
4 $^3S_1$	10.5621	10.5800
5 $^3S_1$	10.7656	10.8650
6 $^3S_1$	10.9455	11.0190

*Experimental data are taken from a review of particle physics: Europ. Phys. J. — 2000. – **15**. – P. 1.

$$a^{\Upsilon} = 3.2264, \quad c^{\Upsilon} = 0.5947$$

where $[a] = \text{GeV} \cdot \text{fm}^{-2/3}$ and $[c] = \text{GeV} \cdot \text{fm}^{2/3}$

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ПРО ЗВ'ЯЗАНІ СТАНИ ВАЖКИХ КВАРКІВ

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Р е з ю м е

Представлено повний розв'язок рівняння Шредингера для кварка, який утримується потенціалом дробового степеня. Розраховано енергетичні рівні S-стану для сімейств $J\Psi$ та Υ. Одержані результати добре узгоджуються з експериментальними даними.

О СВЯЗАННЫХ СОСТОЯНИЯХ ТЯЖЕЛЫХ КВАРКОВ

Х.М.М. Мансур

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Представлено полное решение уравнения Шредингера для кварка, который удерживается потенциалом дробной степени. Вычислены энергетические уровни S-состояния для семейств $J\Psi$ и Υ. Полученные результаты хорошо согласуются с экспериментальными данными.