SUBQUANTUM LEVEL OF MATTER

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Not long ago spinors were considered as the simplest and most fundamental objects of particle physics. However it turns out that there exist more fundamental objects, from which spinors may be built (and, in this sense, they play the role of Kummer ideal numbers). We call they fundors. They are used in the description of a new more deep level of physical reality – pre-matter, from which usual elementary particles arise. Here, a mathematical carcass of the theory is built.

Pythagor had considered that numbers govern the world. Physicians well know the usual integer numbers **Z** (called Euclidean). They form the ring (the algebraic structure with two operations: summation and multiplication). It turns out there are another *more fundamental* numbers (so-called algebraic, Kummer has named them ideal ones), of which Euclidean numbers consist. In the framework of ideal numbers, the prime numbers of **Z** (namely, 2, 3, 5, 7, 11, 13,...) become decomposable, i.e., having multipliers (now the latter are called divisors). Therefore, we have to consider that **Z** is not a closed ring under the *division operation* (which is a new operation in comparison with initial ring operations; hereby, to multiply by zero is possible but to divide by it is impossible[1]) and may be completed (by means of extension) by ideal numbers.

1. As is well known, the ring extension operation is connected with some trouble. For instance, in the ring $\mathbf{Z}[\sqrt{-5}]$, there is no one-valued decomposition of numbers $a+b\sqrt{-5}$ ($a,b\in\mathbf{Z}$) into prime ones (in fact, 3, 7, $1+2\sqrt{-5}$, $1-2\sqrt{-5}$ are primes in $\mathbf{Z}[\sqrt{-5}]$, but we have $21 = 3\cdot 7 = (1+2\sqrt{-5})\times(1-2\sqrt{-5})$; namely such a sutuation takes place when one considers that every elementary particle consists of another ones). In order to restore the one-valued decomposition property, it is needed now to extend $\mathbf{Z}[\sqrt{-5}]$ in such a way that $3 = AB$, $7 = CD$, $1+2\sqrt{-5} = AC$, $1-2\sqrt{-5} = BD$, where A, B, C, D are new prime numbers (ideal symbols) of a new commutative ring. So, the prime numbers of initial ring **Z** are not prime (simple) indeed. Hereby, it is important to achieve such a level when a further ring extension will not possible (in such a case, one speaks about *algebraic closedness* of the final ring).

2. In particle physics at calculus of total particle number and boson occupation number, the ring **Z** is used (ring $\mathbf{Z}_2 = \mathbf{Z}/\bmod 2$ is used for description of fermion occupation number). In addition, representations of the rotation group SO(3) and its covering SU(2) are used for particle *spin* description. They are numbers too.

The general definition of numbers as some representations of (discrete) groups was given by Galois. In the case of continuous groups, the situation is the same (see F. Klein's Erlagen program and the group representation theory connected with it). Irreducible representations of a group forming the ring (see further) are the numbers connected with the group.

The especially important role in physics is played by the rotation group SO(3). Its representations are used for description of spin properties of particles. We are interested namely in this group and its finite-dimensional representations $D(l)$, $l\in\mathbf{Z}_+$, used for description of bosons. Let us consider their pure number properties.

The whole collection of these representations is denoted by $D = \{D(l)\}$. Its elements are finite sums $\sum_l a_l D(l)$, where $a_l\in\mathbf{Z}$. Due to the well-known Clebsch – Gordan theorem

$$D(l_1)\times D(l_2) = \sum_{l=|l_1-l_2|}^{l_1+l_2} D(l), \qquad (1)$$

the set D is closed under operation of usual sum + and Kronecker multiplication × and therefore it is a ring – the ring of finite-dimensional representations of the rotation group SO(3). One can say that the D

[1] Ring (algebra) as an algebraic structure is irreproachable: all its elements are enjoying equal rights in respect of ring operations. On the contrary, not all elements of bodies (in particular, fields) are enjoying equal rights because, for example, zero is damageable: to divide by zero is impossible (ideal element ∞ arises). With this circumstance, uncertainties of the types ∞ – ∞, ∞/∞, 0/0 arise. Therefore, we prefer to deal with rings only.

is a transcendental extension of the ring **Z**, got by means of the addition of the element $D(1)$ to **Z** — the fundamental representation of the rotation group SO(3). Therefore, we can write $D = \mathbf{Z}\,[D(1)]$.

As known, every linear representation $D(l)$ is connected with the carrier vector space $\mathbf{R}_{2l+1}$ (see [1]), and another ring $T\,[\mathbf{R}_3]$ (a tensor algebra over $\mathbf{R}_3$, where $\mathbf{R}_3$ is the module of the fundamental representation $D(1)$) is connected with the ring D. Therefore, we can pass in our reasoning from one ring to another one and to speak about properties of one ring as of another one.

So it turns out that the ring D (and, hence, the ring $T\,[\mathbf{R}_3]$) permits the simple algebraic extension as it is algebraically non-closed (non-closed relatively to the new operation — *square root extraction*, see [2]). Indeed, let us consider the representation $D(1)$ and its carrier space $\mathbf{R}_3 \supset \vec{p} = (p_1, p_2, p_3)$ (it means that there exists a basis $\vec{e_1}, \vec{e_2}, \vec{e_3}$ in $\mathbf{R}_3$, and we can write $\vec{p} = \vec{e_i} p_i$ where p_1, p_2, p_3 are the components of the vector $\vec{p}$ in this basis which are real numbers from **R**). On $\mathbf{R}_3$, there exists a quadratic symmetric form $(\vec{p}, \vec{p}) = p_1^2 + p_2^2 + p_3^2$, which is invariant under the trans-formations $O \in \mathrm{SO}(3) : (O\,\vec{p}, O\,\vec{p}) = (\vec{p}, \vec{p})$. This form as a polynomial of p_1, p_2, p_3 may be decomposed into linear forms $\hat{p} = \sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3$, i.e. $(\vec{p}, \vec{p}) = = \hat{p}^{\,2}$ [2], where the coefficients $\sigma_i\,(i = 1, 2, 3)$ must satisfy the relations

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\,, \tag{2}$$

which determine the Clifford algebra C_3 with the basis $\vec{\sigma}, 1$ (this algebra is isomorphic to the Hamiltonian algebra of quaternions with the basis $i\,\vec{\sigma}, 1$). The algebra C_3, as well known, has the realization by means of 2×2 Pauli matrices (that is very important for physical applications), acting in a two-dimensional complex vector space $S_2 \supset \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$, named spinor space. We see that the square root extraction applied to a vector leads, first, to the mapping $T\,[\mathbf{R}]_3 \to C_3$ which is a homomorphism of the ring $T\,[\mathbf{R}_3]$ (with tensor multiplication) into the Clifford algebra C_3 (the latter is obtained from $T\,[\mathbf{R}]_3$ under factorization upon the ideal determined by relations (2), see [1]) and, secondly, to the spinor ring $U\,[S_2]$ (here U is taking the envelope algebra with usual multiplication) extending the ring $U\,[T_3]$ (see further)[2].

The transformation $p \to p' = Op$ leads to the transformation $\hat{p} \to \hat{p}' = \sigma p' = \sigma' p$, where $\sigma' = \sigma O$, hereby matrices σ' obey the same relations (2) as σ due to the orthogonal condition $O^T O = 1$. Therefore, σ' and σ are equivalent, and hence there exists such a matrix $S\,(O)$ that

$$S\,(O)\,\sigma S^{-1}\,(O) = \sigma' = \sigma O. \tag{3}$$

It is not difficult to show that, in the neighbourhood of 1, the matrix $S\,(O)$ has the same structure as O, i.e., if $O = 1 + I_\alpha \theta_\alpha$ (θ_α are parameters of the group SO(3) and I_α are its generators), $S(O) = 1 + + S(I)\,\theta_\alpha = 1 + S_\alpha \theta_\alpha$, hereby

$$[S_\alpha, \sigma_i] = \sigma_k\,(I_\alpha)_{ki}\,. \tag{4}$$

It follows from here that

$$S_\alpha = \frac{1}{4}\,\sigma I_\alpha \sigma. \tag{5}$$

If $[I_\alpha, I_\beta] = c^{\gamma}_{\alpha\beta} I_\gamma$, where $c^{\gamma}_{\alpha\beta}$ are the structure constants of the Lie algebra of the group SO(3), we have $[S_\alpha, S_\beta] = c^{\gamma}_{\alpha\beta} S_\gamma$.

As well known, this isomorphism of Lie algebras does not continue to a homomorphism between groups: it does not follows from (3) that $S(O')\,S(O) = = S(O'\,O)$. The matter is that $S(O)$ is a covering mapping depending in general case on the path $\tilde{O}$ in SO(3) that outgoes from 1 and finishes in O. However, in the given case, $S(O)$ depend on the class of homotopic paths in SO(3) (SO(3) is a linearly two-connected group) and are the elements of a new group Spin(3) twice covering the SO(3). Hereby SO(3) =

[2] Ring $U[\mathbf{R}_3] = U[p_1, p_2, p_3]$ of polynomials of p_1, p_2, p_3 is obtained from the ring $T\,[\mathbf{R}_3]$ by means of factorization over some ideal, see [1]. There is a diagram, into which the Clifford enclosure (not homomorphism): $p_l \xrightarrow{\rho} \sigma_l$ enters:

$$\begin{tikzcd} T\,[\mathbf{R}_3] \arrow[r, "h_U"] \arrow[d, "h_C"'] & U\,[p_1, p_2, p_3] \arrow[dl, "\rho"] \\ C_3 & \end{tikzcd}$$

Here h_U and h_C are homomorphisms (and I_U, I_C are the corresponding ideals, so that we have $U\,[p_1, p_2, p_3] = T\,[\mathbf{R}_3]/I_U$, $C_3 = T\,[\mathbf{R}_3]/I_C$). By considering the tensor algebra $T\,[C_3]$ and its module $T\,[S_2]$, $T\,[S_2]$ extends the ring $T\,[\mathbf{R}_3]$. It permits us to speak about algebraically non-closedness of the latter.

Obviously, the Clifford algebra may be considered as a non-commutative extension of the real **R** (or complex **C**) number ring.

$\mathrm{Spin}(3)/\mathbf{Z}_2$, where $\mathbf{Z}_2 = \{-1,1\}$ is the central normal devisor in Spin(3), hereat the reverse mapping $S(O) \to O$ is a homomorphism. Using the isomorphism $\mathrm{Spin}(3) \approx \mathrm{SU}(2)$, we will further consider the group SU(2).

As $(I_\alpha)_{ik} = \varepsilon_{\alpha k i}$ (it is completely skew-symmetric tensor), so $S_\alpha = \frac{i}{2}\sigma_\alpha$ are anti-Hermitian matrices. Therefore, the global transformation $S(O) = = \exp(\frac{i}{2}\vec{\sigma}\vec{\theta})$ is a unitary matrix. It follows from the transformation law for spinors $\varphi \to S(O)\,\varphi$, $\overline{\varphi} \to \overline{\varphi}\, S^{-1}(O)$ that the magnitudes $p_\alpha = \overline{\varphi}\,\sigma_\alpha\,\varphi$ composed from spinors are transformed like components of a vector. So that, in this construction, spinors play the role of ideal numbers (elements, of which vectors consist). And we may write that $T[\mathbf{R}_3] \subset T[S_2]$.

As $\vec{S}^2 = 3/4 = s(s+1)$, $s = 1/2$ and hence matrices $S(O)$ give the representation $D(1/2)$ with spin 1/2 in the space S_2. Admitting liberty of speech, we speak about the representation $D(1/2)$ of the group SU(2) as about a spinor representation of the group SO(3). As usual, the existence of spinor representations is connected with two-connectedness of the group SO(3). We connect their existence with the algebraic non-closedness of the ring D [3].

It follows from Clebsch — Gordan formula $D(1/2) \times D(1/2) = D(1) + D(0)$ generalized to spinor representations that, with respect to the irreducible representations $D(1)$ and $D(0)$ of the group SO(3), the spinor representation $D(1/2)$ plays the role of ideal number (representation). By adding $D(1/2)$ to the ring D, we get a wider ring $\overline{D} = D[D(1/2)]$ — the ring of finite-dimensional representations of the group SU(2). In particle theory, spinor representations with half integer spin are used for description of fermions. And we can identify the ring $\overline{D}$ with the elementary particle world.

3. Now the following question is arisen again: is the ring $\overline{D}$ algebraically closed or not? (In other words, is the particle world closed or not?) It turns out that, in spite of one-connectedness of the group SU(2), the ring of its representations $\overline{D}$ is algebraically non-closed. This theorem is exceptionally important for understanding the particle nature.

We may apply the same operation of square root extraction but to the space S_2. For this purpose, we will use isomorphism $\mathrm{SU}(2) \approx \mathrm{Sp}(1)$. Then we can consider the space S_2 as a simplectic space over the Grassmannian algebra G (see [3]), i.e., S_2 is a G-module G_2 [4]

$$\varphi_\alpha \varphi_\beta = -\varphi_\beta \varphi_\alpha, \quad \varphi_\alpha^2 = 0. \tag{6}$$

Note that the algebra G_2 may be obtained from the tensor algebra $T[S_2]$ by means of factorization of the latter upon the ideal determined by relations (6), see [1]. On S_2, there exists a skew-symmetric invariant quadratic form $[\varphi, \varphi] = \varphi_\alpha \varepsilon^{\alpha\beta} \varphi_\beta$ $\left(\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$: $[u\varphi, u\varphi] = [\varphi, \varphi]$, where $u \in \mathrm{Sp}(1)$ (we recall that, in the case of the space $\mathbf{R}_3$, the symmetric form was considered). As S_2 is a G-module, we have $[\varphi, \varphi] = 2\varphi_1\varphi_2 \neq 0$ (it is important to note that, in a pure geometric approach to the space S_2 used in twistor theory, we would have $[\varphi, \varphi] = 0$ and a further consideration would be impossible).

Again $[\varphi, \varphi]$ as a polynomial of the second degree, may be decomposed into linear multipliers $\hat{\varphi} = \sqrt{2}\, a^\alpha \varphi_\alpha$: $[\varphi, \varphi] = \hat{\varphi}^2$, where the coefficients a^α must obey the relations

$$a^\alpha a^\beta - a^\beta a^\alpha = \varepsilon^{\alpha\beta}, \tag{7}$$

which determine the Heisenberg algebra h_2. The Heisenberg ring $U[h_2]$ like two previous (the Clifford and Grassmannian ones) may be obtained from the tensor algebra $T[S_2]$ upon the factorization determined by relations (7), see [1]. It is very important to notice that the Grassmannian numbers (6) have neither matrix nor operator realizations (φ_α are only symbols). But their extension — algebra (7) — has, see [4].

[3] Namely this property of initial rings (their algebraic non-closedness) leads to the existence of ideal elements that we use systematically. So, the ideal numbers of the group SO(3) are spinors. It is only the first echelon of ideal numbers in physics (see further).

[4] This means that there exists the basis e^1, e^2 in S_2 and every element $\varphi \in S_2$ may be written in the form $e^\alpha \varphi_\alpha$, where $\varphi_\alpha \in G$, i.e. φ_α satisfy relations (6).

Again there exists the enclosure σ of the module $S_2(G)$ into the Heisenberg algebra $h_2 : \varphi_\alpha \to a_\alpha$, and the diagram

$$\begin{tikzcd} T[S_2] \arrow[r, "h_G"] \arrow[d, "h_H"] & \wedge[\varphi_1, \varphi_2] = G_2 \arrow[dl, "\sigma"] \\ U[a_1, a_2] = W_2 & \end{tikzcd}$$

where h_G and h_H homomorphisms (and I_G, I_H are the corresponding ideals so that we have $G_2 = T[S_2]/I_G$, $W_2 = T[S_2]/I_H$), but σ is not a homomorphism.

By considering the tensor algebra $T[U[h_2]]$ ($U[h_2]$ is called also the Weyl algebra and labeled as W_2) and its module $T[F]$, $T[F]$ extends the ring $T[S_2]$.

From the point of viev of hypercomplex number theory, the ring W_2 may be considered as a non-commutative (infinite-dimensional) extension of the complex number ring $\mathbf{C}$.

So algebra (7) has an irreducible representation realized by operators a^α, acting in the infinite-dimensional vector space F playing an important role in quantum theory of particles, see [3].

The transformations $\varphi \to \varphi' = u\varphi$ lead to the transformations $\hat{\varphi} \to \hat{\varphi}' = a\varphi' = a'\varphi = (au)\varphi$. Hereby, as u is a simplectic, i.e. $u^T \varepsilon u = \varepsilon$, so the operators a' obey the same commutation relations as a. Therefore, there exists such a transformation $T(u)$ that

$$T(u)\, aT^{-1}(u) = a' = au\,. \tag{8}$$

As above, it follows from here that u and $T(u)$ are locally isomorphic, i.e., if $u = 1 + \frac{i}{2}\vec{\sigma}\vec{\theta}$, we may write $T(u) = 1 + iT(\frac{1}{2}\vec{\sigma})\vec{\theta} = 1 + i\vec{L}\vec{\theta}$. We have

$$[\vec{L}, a^\alpha] = a^\beta (\tfrac{1}{2}\vec{\sigma})^\alpha_\beta. \tag{9}$$

Therefore,

$$\vec{L} = \frac{1}{4} a^\beta \vec{\sigma}^\alpha_\beta a_\alpha, \tag{10}$$

where $a_\alpha = -\varepsilon_{\alpha\beta} a^\beta$, and $\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, hereby

$$[L_i, L_j] = i\,\varepsilon_{ijk} L_k. \tag{11}$$

The mapping T: $\frac{1}{2}\vec{\sigma} \to \vec{L}$ gives an isomorphism between the Lie algebras su(2) (generators $\frac{1}{2}\vec{\sigma}$) and $s\tilde{u}(2)$ (generators $\vec{L}$) enclosed in the ring $U[h_2]$.

However, the groups $SU(2) \supset u$ and $S\tilde{U}(2) \supset T(u)$ are quite different in whole. It will be seen if one builds the linear representation given by the operators $T(u)$ acting in a topological vector space F, see [3]. As above, the mapping $u \to T(u)$ is not a homomorphism, it is a kind of covering mapping (we recall that SU(2) is a one-connected group and, according to the well-known Weyl theorem [5], it has no covering manifold). Operators $T(u)$ do not depend on point $u \in$ SU(2), but depend on the path in SU(2) that outgoes from $1 \in SU(2)$ and finishes in $u \in SU(2)$ [3]. They realize the representation of a new Lie group $-$ 1-chain group $S\tilde{U}(2)$, building over SU(2). The reverse mapping $T(\tilde{u}) \to u$ is a homomorphism, whose kernel is a subgroup of circles $\tilde{\Omega} \subset S\tilde{U}(2)$ (see [3]). The topological group $S\tilde{U}(2)$ (it is not a manifold in whole) is a fibration $(SU(2), \tilde{\Omega})$,see [3] (hereby SU(2) may be considered to be a fibration too: SU(2) = (SO(3), $\mathbf{Z}_2$), where $\mathbf{Z}_2 = \{-1, 1\}$).

Elements f of the space F are called semispinors. They are transformed by the formula $f \to T(\tilde{u})f$. For F, there exists a dual space $\dot{F} \supset \dot{f}$ connected with F by a sesquilinear form $\langle \dot{f}, f\rangle$ invariant under the transformations $T(u)$ as $\bar{\dot{f}}$ are transformed by the formula $\bar{\dot{f}} \to \bar{\dot{f}}\, T^{-1}(\tilde{u})$. Hereby the entities $\varphi^\alpha = \langle \dot{f}, a^\alpha f\rangle$ are transformed like spinor components. In this construction, semispinors f enter into spinors, spinors consist of, as if, semispinconductors and, in this sense, semispinors play the role of ideal elements. *Semispinors are more fundamental entities than spinors.* It turns out that a new physical reality deeper than particles stands behind them, see further. Recently spinors are considered to be the most fundamental objects of the physical world (may be due to the Weyl theorem). But it is not so.

If follows from the definition of $\vec{L}$ that the Casimir operator $\vec{L}^2 = -\frac{3}{16} = \lambda(\lambda + 1)$. This yields $\lambda = -1/4, -3/4$. Representations corresponding to these spin values are labeled as $D^+(-1/4)$ and $D^+(-3/4)$. Thus, only the pair of semispinor representations, realized in the space F, are connected with the algebra h_2 [3][5].

In general case, semispinor representations of the Lie algebra su(2) are infinite-dimensional irreducible representations $D^+(\lambda)$ with junior Cartan vector corresponding to an arbitrary complex weight (spin) λ. The ring of semispinor representations labeled as $\tilde{D}$ is the extension of the ring $\bar{D}$ obtained by addition of the element $D^+(\lambda)$ to it, i.e. $\tilde{D} = \bar{D}[D^+(\lambda)]$. It follows from here that it consists of only the representations $D^+(\lambda + \frac{n}{2})$, where λ is fixed and $n \subset \mathbf{Z}$[6]. In [3], it was shown that the ring $\tilde{D}$ is connected with the Heisenberg algebra h_4. The following formula

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[5] In [6] , Sorokin and Volkov have suggested to consider some hypothetical particles called quartions behind these irreducible representations (strictly speaking, those of another SU(1,1)-group). See also Wilczek and Zec [7], where similar objects called anyons or semions were used for descriptions of the fractional Hall effect. In [6], field equations for quartions were proposed. Earlier in a general approach to the arbitrary spin problem, such equations have been obtained in [3].

Now we consider that, behind semispinor representations (more exactly, behind some infinite direct sum of such representations or a ring), a new form of matter (not separate particles but pre-matter in whole) stands, see further. In this situation, to use separate irreducible representations is insufficient.

[6] Strictly speaking, only the case $\lambda = 0$ corresponds to the ring. Hereby the infinite-dimensional representation $D^+(n/2)$ is a tail area of the finite-dimensional representation $D(n/2)$, which exists at the point $n/2$. $D^+(n/2)$ is equivalent to the representation $D^+(-n/2 - 1)$ at the point $-n/2 - 1$, see [3].

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was established there:

$$\tilde{D} = \overline{D} \otimes D_h(0), \tag{12}$$

where $D_h(0)$ is a non-trivial unit representation of the Lie algebra su(2), corresponding to the supplementary variable $\varphi \doteq \varphi_2$ which is an SU(2)-scalar called a hidden parameter. Its existence is connected with the $\tilde{\Omega}$ group.

So the square root extraction from Grassmannian 2-spinors leads, first, to the mapping $S_2(G) \to h_2$ of a Grassmannian ring into any dynamical system (the Heisenberg algebra is always some canonical system) and, secondly, to the semispinor ring $T[F]$. Elements of the latter are called *fundors*. In the fundor ring, we achieve the level of algebraically closedness: no any extension exists, see [3].

In the conclusion of this part, we have to note that the spinor structure there exists only for the *real* Euclidean or pseudo-Euclidean space $\mathbf{R}_n$ (its symmetry group SO(n) is a two-connected manifold having the spinor covering Spin(n)). We would like to emphasize once more that the Grassmannian algebra has no operator realization but it has a semispinor or fundor structure. All this means the non-common reticence of subquantum entities standing behind this structure. The latter is dynamical in its nature described by one of the Heisenberg algebras. We consider that this structure underlies our physical world.

In connection with this, we would like to recall another pure philosophical idea concerning Leibnizian monadas as a foundation of the world. What are monadas? They are infinite small entities with various degrees of smallness. The idea about their existence grew from the notion of infinitely small numbers. However, such numbers do not exist indeed (only one such a number exists — zero). Introduction of this kind of numbers is connected with a not permitted contradiction, see [4]. Therefore, we have to consider monadas to be nonreal objects not existing from the mathematical point of view.

4. Now we show how the fundor structure appears in particle physics and enters in the mathematical carcass of the fundamental physics.

We begin from the base — space-time continuum. In the whole (on the scale of the Universe), it is a Riemannian space. But even in its small part, considered in elementary particle physics where it is flat, this space is not a vector space that has been considered in part 2. It is the affine or Poincare space $\mathbf{A}_{3,1}$ with coordinates X_μ. Its symmetry group is the Poincare group $P = L \times) T_{3,1}$, where $L = \mathrm{SO}(3,1) \otimes P \otimes T$ is the general Lorentzian group (P, T are reflections of space and time), and $T_{3,1}$ are translations.

For description of particle motion in this space, Newton postulated the so-called differential structure on it. As a result, with every point X of this space, the pair of vector spaces, namely a co-tangent space T^*_X with base dX_μ and a tangent one T_X with base $\partial/\partial X_\mu$ were connected. Hence, the vector structure has evidently a local character. But it quite sufficient for us.

Since the local ring $T[T^*_X]$ is non-closed and permits a spinor extension (see formally) we can write $(dX_\mu)_X = \overline{\psi}(X)\,\gamma_\mu\,\psi(X)\,ds$, where γ_μ are elements of the Clifford algebra (Dirac matrices), $ds = \sqrt{(dX_\mu)^2_X}$, and $\psi(X)$ are cross sections of the spinor fibration $(\mathbf{A}_{3,1}, S^{(*)}_8)$. Here $X \in \mathbf{A}_{3,1}$, $\psi, \overline{\psi} \in S^{(*)}_8$ ($S^{(*)}_8$ is the Dirac spinor space).

So, having the co-tangent space T^*_X at every point $X \in \mathbf{A}_{3,1}$, one can, first, build a co-tangent fibration $(\mathbf{A}_{3,1}, T^*\mathbf{A}_{3,1})$, where $T^*\mathbf{A}_{3,1} = \bigcup_{X \in \mathbf{A}_{3,1}} T^*_X$ (and one can do the same in the case of the tangent space T_X, i.e., to build a tangent fibration $(\mathbf{A}_{3,1}, T^*\mathbf{A}_{3,1})$, where $T\mathbf{A}_{3,1} = \bigcup_{X \in \mathbf{A}_{3,1}} T_X$). Secondly, by expanding (at every point X) into a wider spinor space $(S^{(*)}_8)_X$ (or the local ring $T[(S^{(*)}_8)_X]$ connected with it), one can build a spinor fibration $(\mathbf{A}_{3,1}, S^{(*)}_8)$ by means of a joint of all local spinor spaces $(S^{(*)}_8)_X$. Hereby it is clear that the space $(S^{(*)}_8)_X$, extending the space T^*_X, must be used in the same sence as T^*_X, i.e. as a fiber grown from a point $X \in \mathbf{A}_{3,1}$. It is important to understand that this statement is not an axiom but the consequence of our construction.

It is important to emphasize also that, according to this construction, spinors are not used as any substructure of the space-time $\mathbf{A}_{3,1}$ substituting it like in the twistor theory, but in the sence of any supplementary structure to the space-time, namely, as the fiber of a new space — the spinor fibration $(\mathbf{A}_{3,1}, S^{(*)}_8)$.

We have already mentioned that $\psi(X)$ is any cross section of this spinor fibration, i.e., the fields corresponding to a Dirac's particle. We will not stop at the equations for these fields, which follow from the structure of these fields as objects of the Lorentz and Poincare groups (concerning this, see [8]). Underline only that Dirac unerringly guessed the meaning of spinors as matter variables (but not space). Now we can say that the base $\mathbf{A}_{3,1}$ is the empty space-time (without matter) and the fibration $(\mathbf{A}_{3,1}, S)$ (here S is not only a spinor but other vector fibers too) is the space-time filling by matter (or physical space-time).

Hereby, in the classical description of matter, it is sufficient to consider only the co-tangent fibration

$(\mathbf{A}_{3,1}, T^*\mathbf{A}_{3,1})$ which is a phase space of classical mechanics. In the quantum description, the canonical variables of matter (without spin) are connected with the tangent fibration $(\mathbf{A}_{3,1}, T\mathbf{A}_{3,1})$ which is an associative algebra. Transition from a co-tangent fibration to a tangent one is called spatial (or the first) quantization [4]. Taking into account spin of particles, the field variables of matter are connected with the wider (namely matter) vector fibration $(\mathbf{A}_{3,1}, S)$ mentioned above. We emphasize that the both spinor structure and wave nature of matter have *local* character. Therefore, it is a nonsence to speak about spinor or wave properties of macroobjects (a star, a galaxy, or the Universe).

So, the first echelon of ideal numbers – spinors – is used in the mathematical carcass of fundamental physics as a fiber of the matter fibration $(\mathbf{A}_{3,1}, S_8^{(*)})$ of the space-time $\mathbf{A}_{3,1}$ and as the cross sections of this fibration – Dirac's fields $\psi(X)$.

A further consequent description of these objects is connected usually with the second quantized field theory, in which fields $\psi(X)$ are considered to be operators $\hat{\psi}(X)$, acting in a Fock – Hilbert space and obeying the permutation relations

$$\{\hat{\psi}(\vec{X}, t), \hat{\bar{\psi}}(\vec{X}', t)\} = \gamma_4 \delta^3(\vec{X} - \vec{X}'). \tag{13}$$

Due to the singularity of the right hand side in (13), quantized field theory meets ultraviolet divergences. They are connected obviously with ultra small distances when $\vec{X} = \vec{X}'$

It is very important to understand that it follows from nowhere that the Newtonian conception of space as a differential manifold with measure and the Faraday – Maxwell field conception may be used at very small distances (very high energies). Analysis of the situation which takes place in this region shows [9] that, in the neighbourhood of a singular point of the Universe (the so-called cosmological singularity) and also in high-energy particle collisions, the particle wave functions are compressed (due to the Fitz Gerald – Lorentzian space contraction) so much that they become non-differentiable. As a result, the configuration space loses the differential structure and measure. In such circumstances, the simple joint of axioms of the usual Heisenberg – Schrödinger quantum theory based on a separable Hilbert space [10] with the demands of special relativity given by Feynman and others [11] is insufficient to construct a consequent quantum field theory. Axiomatic approach to the problem [12] showed that further investigations in the framework of this scheme is without looking (negative results of Haag and Wightman).

It was shown in [4] that, in this region, we have to reject the usual field conception (a field is a function on the continuum) and Newtonian model of space-time (a differential manifold with measure) and to accept another field concept – a field is a function on the discontinuum. This new quantum field theory breaks the well-known symmetry between the configuration and momentum spaces inherent in the usual quantum theory (due to the usual Fourier transformation, the latter is characterized by some equilibrium between these spaces). In the new theory, this symmetry is completely broken because the configuration space is completely non-closed here (discontinuum) and a new measure – H.Bohr's one – is considered on the momentum space [13]. A class of almost periodic functions and a *non-separable* Hilbert space are connected with this measure. Such functions describe particle constituents – granules. Granule fields (quantized Dirac's fields on the discontinuum) obey the permutation relations [4]

$$\{\hat{\psi}(\vec{X}, t), \hat{\bar{\psi}}(\vec{X}', t)\} = 0, \tag{14}$$

which are essentially distinguished from (13). These relations have no solutions in the form of oparators densely defined in the Fock – Hilbert space, see [4].

The question is raisen: what do these relations mean? They look like (6), which determine a Grassmannian algebra. As $\vec{X}$ take an infinite lot of values, (14) determine an infinite-dimensional Grassmann algebra. However, at $\vec{X} = \vec{X}'$ ($\vec{X}$ are fixed), we have a finite-dimensional algebra labeled as $S_8^{(*)}(G) = U[\psi, \bar{\psi}]$. In part 3, it has been shown that such rings are algebraically non-closed and permit the extension to the fundor ring connected with the Heisenber algebra. In this case, it will be the algebra $h_8^{(*)}$. Mapping (enclosure) $S_8^{(*)}(G) \to h_8^{(*)}$ (i.e., $\psi \to \Phi$, $\bar{\psi} \to \bar{\Phi}$, where Φ, $\bar{\Phi}$ are generators of $h_8^{(*)}$ was called in [3] as the quantization of a Dirac – Grassmann fiber.

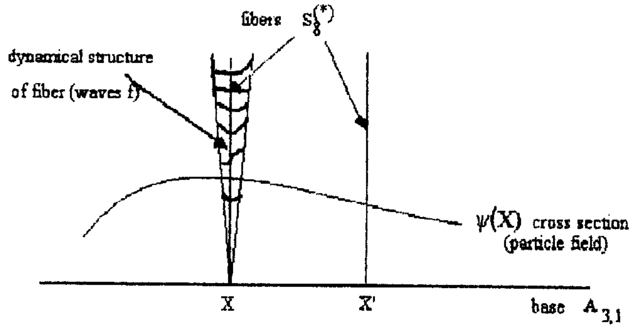
A new kind of dynamical systems – relativistic bi-Hamiltonian – is connected with the algebra $h_8^{(*)}$). For its description, a non-Fock (non-self-adjoin) representation of $h_8^{(*)}$ is well adopted. Such a representation is realized in the pair of topological vector spaces $(\dot{\mathbf{F}}, \mathbf{F})$ dual with respect to some non-Hermitian sesquilinear form $\langle .,. \rangle$. This representation describes a non-standard complex oscillator, having no ground state. Such an oscillator has an infinite lot of states with negative occupation numbers. With the latter, the existence of a supplementary variable φ is connected, see part 3.

The carrier space $\mathbf{F}$ of an irreducible representation of the algebra $h_8^{(*)}$ is named a fundor one. In it, the ring of semispinor representations of the Lorentz group (more exactly, those of the group $\widetilde{SL}(2, \mathbf{C})$ which is a 1-chain group over SL(2, $\mathbf{C}$) – any kind of covering) is realized. Elements of the space $\mathbf{F}$ are called fundors.

In **F**, the representation of SL(2, **C**) is completely reducible to the infitite system of irreducible semispinor representations. Fundors are the second echelon of ideal numbers. They describe a hidden dynamical structure of granules. They are the most fundamental numbers of the physical world[7].

5. We summarize. At present time, the most fundamental from finite (hypercomplex) numbers are Caley numbers **Ca**. In [14], it was shown that there is the reason to put them in the ground of creation of our Universe. They determine the dynamics of the most early stage of formation of the Universe and also the most fundamental properties of it. Due to the non-associativity of the Caley algebra, it has no any matrix realization and its dynamics goes out from the frame of the Lie group theory. In order to improve it, we should consider a tensor algebra $T\,[\mathbf{Ca}]$ over **Ca** (recall that tensor multiplication is associative).

It is very important that the Caley algebra contains the associative subalgebra of the Hamiltonian quaternions **H**, which has already a matrix realization. Non-associativity of **Ca** is connected, strictly speaking, with one Caley's number E, so that $\mathbf{Ca} = \mathbf{H} + \mathbf{H}E$. If the algebra **Ca** is enclosed in the Heisenberg algebra $h_8^{(*)}$ [8], which has the infinite-dimensional *non-self-adjoint* representation (we have underlined the very important property of this representation), so the number E finds a realization in the form of $E = T^+ T = -TT^+$ ($T^2 = \pm 1$, so that $E^2 = -1$), where T is time reflection: $\Phi \to \overline{\Phi}$, see [14]. Here $T^+ \neq T, T^{-1}$, that is fulfilled only at the regime of a non-self-adjoint representation. Hereby, the algebra $U\,[h_4^{(*)}]$ is realized in the subspace $\mathbf{F}_0 \subset \mathbf{F}$, and $T\,[U\,[h_4^{(*)}]]$ (connected with $T\,[\mathbf{Ca}]$) does in the tensor algebra $T\,[\mathbf{F}_0]$. It is essential that the multiplication in $T\,[U\,[h_4^{(*)}]]$ is external (Cartan) multiplication.



Spinor fidration of spase-time

The ring $T\,[\mathbf{F}]$ describes the ensemble of pre-matter quanta (or ether waves) f, of which our Universe consists, in the zero circle of its evolution (before the Bog Bang). After the Big Bang (the total quantum transition $f \to \dot{f}$), fundamental particles arose and the ring $\overline{D}$ (and its sub-ring D) included in $T\,[\mathbf{F}]$ came into force. The latter is considered over the ring **Z**, and when the spin properties of particles become disregarded (it is possible at macroscopic level), only the ring **Z** stays from the whole tower of number rings.

All extensions considered here of the initial ring **Z** are the achievements of the algebraic science of the 19th and 20th centuries. Search for the most fundamental numbers was the main principle of these investigations. Modern algebra grew from this principle. We consider that this principle underlies the fundamental particle physics too. In the figure, the main elements of the mathematical carcass of fundamental physics are shown.

Finishing our exposition, we present the tower of number rings underlain the fundamental physics. For representations and their modules, it looks so:

$$D \subset \overline{D} \subset \widetilde{D},$$

$$T\,[\mathbf{R}_{3,1}] \subset T\,[S_8^{(*)}(G)] \subset T\,[\mathbf{F}].$$

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[7] Fundor ring $\widetilde{D}$ of the group SL(2,**C**) consists of the representations $\{(\lambda + p/2, \lambda' + q/2)^+\}$, $p, q \in \mathbf{Z}$ (where it must be $\lambda = \lambda' = 0$). The ring $\widetilde{D}$ is equivalent to the ring $\overline{D} \otimes (0, 0)_h$ (compare with (12)), where $\overline{D}$ is the ring of finite-dimensional representations $(n/2, m/2)$, $n, m \in \mathbf{Z}_+$ of the group SL(2, **C**), and $(0,0)_h$ is a non-trivial unit representation connected with additional variables, see part 3, and also [3].

[8] We have already written about three Heisenberg algebras $h_4^{(*)}$, $h_8^{(*)}$, $h_{16}^{(*)}$. Here, the each next is a doubling of the previous. Algebra $h_4^{(*)}$ underlies the Universe and is connected with the Caley algebra **Ca**. The latter is the Caley – Dickson doubling of the Hamilton algebra H. In our Universe, the Caley – Dickson doubling reduces to the doubling of the algebra $h_4^{(*)}$, in result of which the algebra $h_8^{(*)}$ arises. The latter contains isotopic symmetry in the form of $U_i(2)$. Representations of both algebras $h_4^{(*)}$ and $h_8^{(*)}$ are given in the subspace $\mathbf{F}_0 \subset \mathbf{F}$ of functions depending on the additional variables $\varphi_k \doteq \varphi_{2k}$ ($k = 1,2$ are isotopic indices; in the case of $h_4^{(*)}$, it is one variable $\varphi \doteq \varphi_2$, see part 3). On the extension of the $h_8^{(*)}$-representation from the subspace $\mathbf{F}_0$ onto the total carrier space $\mathbf{F} = \mathbf{F}_F \otimes \mathbf{F}_0$, the doubling of the algebra $h_8^{(*)}$ is happened and the algebra $h_{16}^{(*)}$ arises. This algebra contains isotopic and Dirac's indices: $\mathbf{F}_F$ is the space of functions depending on the variables $\varphi_{\alpha k}$, where $\alpha = 1, 2$ are Dirac's indices. Vice versa, the contraction of the $h_{16}^{(*)}$- representation from the space **F** into the subspace $\mathbf{F}_0$ is accompanied by algebra contraction $h_{16}^{(*)} \supset h_8^{(*)}$, on one hand, and the space-time contraction $\mathbf{A}_{3,1} \supset \mathbf{A}_{1,1}$, on the other hand. The process $\mathbf{F}_0 \subset \mathbf{F}$ plays a very important role in the subquantum theory.

Here, $\mathbf{R}_{3,1}$ is the vector space associated with the affine space-time $\mathbf{A}_{3,1}$ (i.e., this is the co-tangent space $T^*_X\mathbf{A}_{3,1}$ in fact). Here, the appearance of the Grassmannian algebra G is connected with the spinor properties of matter.

We give also the tower for hypercomplex numbers, which has been used here:

$$\mathbf{Z}\rightarrow\mathbf{R}\rightarrow\mathbf{C}\rightarrow\mathbf{H}\rightarrow\mathbf{Ca}\rightarrow U[h_4^{(*)}].$$

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