ON SYMMETRIES IN (2+1)-DIMENSIONAL QUANTUM GRAVITY

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It is shown by Nelson, Regge and Zertuche that the algebra of observables of quantum gravity in the (2+1)-dimensional de Sitter space with space being a torus is related to the Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$. The symmetry group of the algebra of observables turns out to be the modular group PSL $(2,\mathbb{Z})$ of a torus. We construct representations of this group, corresponding to finite-dimensional representations of the algebra of observables.

Let us briefly describe the approach to (2+1)-dimensional gravity based on the first-order formalism by Witten [1] and Nelson, Regge, Zertuche [2], see also review [3]. The key idea of these papers is the fact that (2+1)-dimensional gravity is a gauge theory with Chern–Simons action. In [1], Witten gives some arguments about impossibility to present (3+1)-dimensional gravity as a gauge theory. Thus, the case of (2+1)-dimensional space-time is specific. Despite this, physicists working in this area believe that investigation of (2+1)-dimensional case will shed light to problems of quantization of (3+1)-dimensional gravity.

The standard Einstein–Hilbert action in the $3\,D$ space-time M (without matter) topologically equivalent to $\mathbb{R}\times\Sigma$ ($\mathbb{R}$ corresponds to time and a closed $2\,D$ surface Σ corresponds to space) is

$$I=\frac{1}{16\pi G}\int_M d^3x\sqrt{-g}(R-2\Lambda),$$

where R is the scalar curvature and Λ is the cosmological constant. Classical solutions of the corresponding Einstein equation are constant curvature spaces M.

In the first-order formalism, the fundamental variables are "dreibein" e^a_μ such that $\eta_{ab}{e_\mu}^a{e_\nu}^b=g_{\mu\nu}$, $\eta_{ab}=\mathrm{diag}(-\,+\,+)$, and the spin connection ${\omega_\mu}^{ab}$. We set e^a_μ and ${\omega_\mu}^{ab}$ to be independent. They can be treated as components of 1-forms $e^a=e^a_\mu dx^\mu$ and $\omega^a=\frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu$. The action I (up to a constant multiplier) becomes

$$I=2\int_M e^a\wedge\left(d\omega_a+\frac{1}{2}\epsilon_{abc}\omega^b\wedge\omega^c-\frac{\Lambda}{6}\epsilon_{abc}e^b\wedge e^c\right).$$

In the anti-de Sitter case (that is, for $\Lambda=-1/\ell^2<0$), introducing the variables $A^{(\pm)a}=\omega^a\pm(1/\ell)e^a$, we rewrite the Einstein–Hilbert action I in the completely equivalent form of Chern–Simons action with the $\mathrm{SO}(2,1)\times\mathrm{SO}(2,1)$ gauge potential

$$I[A^{(+)},A^{(-)}]=I_{\mathrm{CS}}[A^{(+)}]-I_{\mathrm{CS}}[A^{(-)}],$$

where

$$I_{\mathrm{CS}}[A]=\frac{k}{4\pi}\int_M \mathrm{tr}\Big(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\Big)$$

and $k=\ell\sqrt{2}/8G$. Variation of the action I in e^a_μ and ω^a_μ gives the equations of motion (which contain the derivatives with respect to time of dynamic variables) and constraint equations (which do not contain derivatives with respect to time). It is easy to see that dynamic variables are space-like components e^a_i and ω^a_i, $i=1,2$. The Poisson brackets of dynamic variables on a slice of constant time can be read off directly from the action I:

$$\Big\{\omega^a_i(x),e^b_j(y)\Big\}=\frac{1}{2}\epsilon_{ij}\eta^{ab}\delta^2(x-y),$$

$$\Big\{\omega^a_i(x),\omega^b_j(y)\Big\}=\Big\{e^a_i(x),e^b_j(y)\Big\}=0.$$

Here, the variables $\omega_1^a(x)$, $\omega_2^a(x)$ can be interpreted as coordinates and $-e_2^a(x)/2$, $e_1^a(x)/2$ as the corresponding momentum variables. The variables e_i^a and ω_i^a do not give yet the *physical* phase space, since it is necessary to impose the constraint equations. As argued in [1], the only gauge-invariant observables that do not vanish when the constraints are imposed are global variables, such as the traces of holonomies around possible non-contractible loops γ in Σ. In terms of variables $A_i^{(\pm)a}(x)$, $i=1,2$, Poisson brackets have the form

$$\left\{A_i^{(\pm)a}(x), A_j^{(\pm)b}(y)\right\} = \pm\frac{1}{\ell}\epsilon_{ij}\eta^{ab}\delta^2(x-y),$$

while variables $A^{(+)}$ commute with variables $A^{(-)}$. These brackets, in turn, induce Poisson brackets [2] among the traces of holonomies

$$G_\gamma^{(\pm)} = \mathrm{Pexp}\left\{\int_\gamma A_i^{(\pm)a}T_a dx^i\right\},$$

where T_a, $a=1,2,3$, are the generators of the Lie algebra corresponding to the Lie group $\mathrm{SO}(2,1)$. Since the connections $A^{(+)}$ and $A^{(-)}$ are flat (it follows from the equations of motion of the Chern–Simons action), the holonomies along two homotopic paths are equal. The traces of these holonomies are of special interest because after quantization they constitute the algebra of observables in quantum (2+1)-dimensional gravity [1]. This article is devoted to the construction of representations of the symmetry group of the algebra of observables in the case where Σ is a torus with one boundary component (hole). Let γ_1 and γ_2 be two basic cycles on the torus and

$$R_1^\pm = \frac{1}{2}\mathrm{Tr}\, G_{\gamma_1}^{(\pm)}, \quad R_2^\pm = \frac{1}{2}\mathrm{Tr}\, G_{\gamma_2}^{(\pm)}, \quad R_3^\pm = \frac{1}{2}\mathrm{Tr}\, G_{\gamma_1\gamma_2}^{(\pm)}.$$

Then, as shown in [2],

$$\{R_1^\pm, R_2^\pm\} = \pm\frac{1}{4\ell}\left(R_3^\pm - R_1^\pm R_2^\pm\right), \ \textit{cyclic permutations.}$$

In order to quantize this quadratic Poisson algebra, we have to replace $R_1^\pm$, $R_2^\pm$ and $R_3^\pm$ by operators $\hat{R}_1^\pm$, $\hat{R}_2^\pm$ and $\hat{R}_3^\pm$, respectively. Nelson, Regge and Zertuche proposed [2] to replace the quadratic combination from the right-hand sides of these relations by the corresponding symmetrization, that is, $R_1^\pm R_2^\pm \to (\hat{R}_1^\pm\hat{R}_2^\pm + \hat{R}_2^\pm\hat{R}_1^\pm)/2$ and so on. Thus, the resulting algebra of observables is defined by the relations

$$\hat{R}_1^\pm\hat{R}_2^\pm e^{\pm i\theta} - \hat{R}_2^\pm\hat{R}_1^\pm e^{\mp i\theta} = \pm 2i\sin\theta\hat{R}_3^\pm,$$

and cyclic permutations of $\hat{R}_1^\pm, \hat{R}_2^\pm, \hat{R}_3^\pm$.

It turns out that this algebra is isomorphic to the direct sum $U_q'(\mathrm{so}_3)\oplus U_{q^{-1}}'(\mathrm{so}_3)$, where $q=e^{2i\theta}$ and $\tan(\theta) = -\hbar/8\ell$. The first part of the direct sum corresponds to fields $A^{(+)}$ and the second part corresponds to $A^{(-)}$. In the de Sitter case (to which the article is devoted), the situation is similar, but the quantized algebra of observables is connected to $U_q'(\mathrm{so}_3)$ with real $q=e^{2\tilde{\theta}}$, $\tanh\tilde{\theta} = i\hbar/8\ell$. The cases of surfaces Σ with other topology and the corresponding algebras of observables are discussed in [4–6]. These algebras are also related to deformed algebras. In [7], they are obtained in a completely geometric way as Kauffman skein algebras and interpreted as quantization of the Poisson algebra of $\mathrm{SL}(2,\mathbb{C})$-characters of the fundamental group of a surface. The case of $\Lambda=0$ was considered in [1, 3]. In this case, gravity is equivalent to the Chern–Simons theory with the gauge group $\mathrm{ISO}(2,1)$ (Poincaré group). The quantization procedure was described there.

The Fairlie–Odesskii algebra $U_q'(\mathrm{so}_3)$ [8, 9] is an associative algebra with the generating elements I_1, I_2, I_3 and defining relations

$$q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \tag{1}$$

$$q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \tag{2}$$

$$q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2, \tag{3}$$

where q is a non-zero complex number called deformation parameter. In the limit $q\to 1$, the algebra $U_q'(\mathrm{so}_3)$ reduces to the Lie algebra so_3. Substituting I_3 from Eq. (1) into Eqs. (2) and (3), we obtain other equivalent formulation of $U_q'(\mathrm{so}_3)$ in terms of two generating elements I_1 and I_2:

$$I_1I_2^2 + I_2^2I_1 - (q+q^{-1})I_2I_1I_2 = -I_1, \tag{4}$$

$$I_2I_1^2 + I_1^2I_2 - (q+q^{-1})I_1I_2I_1 = -I_2. \tag{5}$$

The generators I_1 and I_2 correspond to holonomies along two basis cycles γ_1 and γ_2 on a torus. In what follows, we suppose that $0<q<1$, thus restricting ourselves to the de Sitter case.

Diffeomorphisms of a torus $\mathrm{Diff}(\Sigma)$ induce automorphisms of the algebra of observables, that is *symmetries of quantum (2+1)-dimensional gravity.* These automorphisms are generated by two automorphisms $\mathcal{S}$ and $\mathcal{T}$. Their action on $U_q'(\mathrm{so}_3)$ is uniquely defined by

$$\mathcal{S}(I_1) = I_2, \qquad \mathcal{S}(I_2) = I_1,$$

$$\mathcal{T}(I_1) = I_1, \qquad \mathcal{T}(I_2) = q^{1/2}I_2I_1 - q^{-1/2}I_1I_2.$$

These automorphisms satisfy the relations

$$\mathcal{S}^2 = 1, \qquad (\mathcal{S}\mathcal{T})^3 = 1, \tag{6}$$

which are the defining relations for the modular group $\mathrm{PSL}(2,\mathbb{Z}) \simeq \mathrm{SL}(2,\mathbb{Z})/\{1,-1\}$ of a torus (isomorphic to the quotient group $\mathrm{Diff}(\Sigma)/\mathrm{Diff}_0(\Sigma)$, where $\mathrm{Diff}_0(\Sigma)$ is the connected component of the identical diffeomorphism in $\mathrm{Diff}(\Sigma)$). This group is also known as the mapping class group. It is very important for quantum gravity to study the representations of this symmetry group, which are induced from the representations of the algebra of observables $U_q'(\mathrm{so}_3)$.

The algebra $U_q'(\mathrm{so}_3)$ *has irreducible finite-dimensional representations of classical and non-classical type* [10]. The classical type representations are a deformation of spin l representations of the Lie algebra so_3, the non-classical type representations have no such analogs. In what follows, we consider only the irreducible representations of the algebra $U_q'(\mathrm{so}_3)$ which are of classical type with integral spin l, that is $l = 0, 1, 2, \ldots$, on the $(2l+1)$-dimensional space $\mathcal{V}_l$. These representations will be denoted by T_l. The space $\mathcal{V}_l$ of the representation T_l is spanned by the basis vectors $|m\rangle$, $m = -l, -l+1, \ldots, l$, which are supposed to constitute the orthonormal basis: $\langle m'|m\rangle = \delta_{mm'}$. The action formulas for the operators $T_l(I_1)$ and $T_l(I_2)$ are:

$$T_l(I_1)|m\rangle = \mathrm{i}[m]|m\rangle, \tag{7}$$

$$T_l(I_2)|m\rangle = \mathrm{i}A_{l,m}|m+1\rangle + \mathrm{i}A_{l,m-1}|m-1\rangle, \tag{8}$$

where

$$A_{l,m} = \Big(\frac{[m][m+1]}{[2m][2m+2]}[l-m][l+m+1]\Big)^{1/2}$$

and the notation of q-number

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \tag{9}$$

is used.

If ψ is an automorphism of $U_q'(\mathrm{so}_3)$, then a mapping $T_l \circ \psi$ from $U_q'(\mathrm{so}_3)$ into the space of linear operators on $\mathcal{V}_l$ defines a representation of $U_q'(\mathrm{so}_3)$. *If* ψ *is an element of PSL*$(2,\mathbb{Z})$, *that is a sequence of automorphisms* $\mathcal{S}$ *and* $\mathcal{T}$, *then the representation* $T_l \circ \psi$ *is equivalent to* T_l. It is sufficient to prove this statement in the cases $\psi = \mathcal{T}$ and $\psi = \mathcal{S}$. The proofs in the both cases are based on the classification of representations of $U_q'(\mathrm{so}_3)$ [10]. Proof in the case $\psi = \mathcal{T}$ (resp. $\psi = \mathcal{S}$) is based on the observation that T_l is a unique, up to equivalence, irreducible $(2l+1)$-dimensional representation T of $U_q'(\mathrm{so}_3)$ such that the spectrum of operator $T(I_1)$ is $\{-\mathrm{i}[l], -\mathrm{i}[l-1], \ldots, \mathrm{i}[l]\}$ (resp., such that $\mathrm{Tr}(T(I_1)) = \mathrm{Tr}(T(I_2)) = 0$). Thus, for $\psi \in \mathrm{PSL}(2,\mathbb{Z})$, there exists an intertwining operator C_ψ such that $C_\psi^{-1} T_l(a) C_\psi = T_l(\psi(a))$, $\forall a \in U_q'(\mathrm{so}_3)$. It is defined up to a scalar multiplier (it follows from the Schur lemma). Since automorphisms $\mathcal{S}$ and $\mathcal{T}$ generate the modular group $\mathrm{PSL}(2,\mathbb{Z})$, the corresponding intertwining operators $T_\mathcal{S}$ and $T_\mathcal{T}$ give a projective representation of this group. The mentioned scalar multipliers can be chosen to make this representation to be an exact representation. Our main task is to find this representation.

Let us find $C_\mathcal{T}$. Since $T_l(\mathcal{T}(I_1)) = T_l(I_1)$ is diagonal (in the basis $\{|m\rangle\}$) with pairwise non-equal diagonal elements, it follows from the Schur lemma that $C_\mathcal{T}$ is also diagonal (in the same basis). Let $\langle m'|C_\mathcal{T}|m\rangle = \delta_{m,m'} t_m$. Further, we have

$$\langle m+1|T_l(\mathcal{T}(I_2))|m\rangle = (\mathrm{i}[m]q^{1/2} - \mathrm{i}[m+1]q^{-1/2})\times$$

$$\times\langle m+1|T_l(I_2)|m\rangle = -\mathrm{i}q^{-m-1/2}\langle m+1|T_l(I_2)|m\rangle.$$

From other hand,

$$\langle m+1|T_l(\mathcal{T}(I_2))|m\rangle = \langle m+1|C_\mathcal{T}^{-1} T_l(I_2) C_\mathcal{T}|m\rangle =$$

$$= t_{m+1}^{-1} t_m \langle m+1|T_l(I_2)|m\rangle.$$

Thus, $t_{m+1}^{-1} t_m = -\mathrm{i}q^{-m-1/2}$, and therefore $t_m = \mathrm{i}^m q^{m^2/2} t_0$.

Let us find $C_\mathcal{S}$. To this end, it is useful to diagonalize operator $T_l(I_2)$. The eigenvectors of this operator have the form

$$\widetilde{|x\rangle} = \sum_{m=-l}^{l} a_m(x)|m\rangle, \tag{10}$$

with eigenvalues $\mathrm{i}[x]$, that is,

$$T_l(I_2)\widetilde{|x\rangle} = \mathrm{i}[x]\widetilde{|x\rangle}. \tag{11}$$

Our next task is to find all the values of x and matrix elements $a_m(x)$ of the transformation matrix. Substituting Eq. (10) into Eq. (11) and using the action formula, Eq. (8), we obtain the recurrent relation

$$A_{l,n-l}P_{n+1}(x) + A_{l,n-l-1}P_{n-1}(x) = [x]P_n(x), \tag{12}$$

where $n = l + m$, $P_n(x) = a_{n-l}(x)$, $n = 0, 1, 2, \ldots, 2l$. Since $A_{l,-l-1} = A_{l,l} = 0$, the relation has a solution only at some fixed x. Now we rewrite this relation as

$$[2l-n]P'_{n+1}(x)+[n]P'_{n-1}(x) = (q^{n-l}+q^{l-n})[x]P'_n(x), \quad (13)$$

where functions $P'_n(x)$ (in fact, they are polynomials of $[x]$ if $P_0(x)$ is taken to be independent of x) are defined as

$$P_n(x) = \left(\frac{[l][2l-1]![2l-2n]}{[n]![2l-n]![l-n]}\right)^{1/2} P'_n(x), \quad (14)$$

and $[r]! = [r][r-1]\cdots[2][1]$ is the definition of q-factorial. Using the definition of q-numbers (9), we obtain from Eq. (13)

$$\frac{1-q^{2n-4l}}{1+q^{2n-2l}}P'_{n+1}(x) - \frac{(1-q^{2n})q^{-2l}}{1+q^{2n-2l}}P'_{n-1}(x) =$$

$$= (q^x - q^{-x})q^{-l}P'_n(x). \quad (15)$$

The obtained recurrent relation has the form of a recurrent relation for q-Racah polynomials of discrete variables (see [11]) if one imposes $P'_0(x) = 1$. Thus

$$P'_n(x) = {}_4\phi_3\left(\begin{matrix} q^{-y},\ \gamma\delta q^{y+1},\ q^{-n},\ \alpha\beta q^{n+1} \\ \alpha q,\ \beta\delta q,\ \gamma q \end{matrix};\ q,q\right),$$

where $\alpha = \beta = -\gamma = -\delta = \mathrm{i}q^{-l-1/2}$ and $y = l - x$. We have used the definition of *basic hypergeometric function*:

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix};\ q, z\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n,$$

where $(c_1, c_2, \ldots, c_k; q)_n = (c_1, q)_n (c_2, q)_n \cdots (c_k; q)_n$, $(c; q)_n = (1-c)(1-qc)\cdots(1-q^{n-1}c)$.

The orthogonality relation for functions $P'_n(x)$ has the form (see [11])

$$\sum_{y=0}^{2l} w_y P'_n(l-y)P'_k(l-y) = \delta_{nk} v_n^{-1} \sum_{y=0}^{2l} w_y, \quad (16)$$

where

$$w_y = \frac{(\gamma\delta q; q)_y (1-\gamma\delta q^{2y+1})(\alpha q, \gamma q, \beta\delta q; q)_y}{(q;q)_y (1-\gamma\delta q)(\delta q, \gamma\delta q/\alpha, \gamma q/\beta; q)_y}(\alpha\beta q)^{-y} =$$

$$= \frac{[l][2l-1]!}{[y]![2l-y]!}\frac{[2l-2y]}{[l-y]},$$

$$v_n = \frac{(\alpha\beta q; q)_n (1-\alpha\beta q^{2n+1})(\alpha q, \gamma q, \beta\delta q; q)_n}{(q;q)_n (1-\alpha\beta q)(\beta q, \alpha\beta q/\gamma, \alpha q/\delta; q)_n}(\gamma\delta q)^{-n} =$$

$$= \frac{[l][2l-1]![2l-2n]}{[n]![2l-n]![l-n]}.$$

Note that Eq. (14) has the form $P_n(x) = v_n^{1/2}P'_n(x)$. Taking into account (see [11]) that

$$\sum_{n=0}^{2l} v_n = \sum_{y=0}^{2l} w_y = \left(2\frac{[2l-2]!!}{[l-1]!}\right)^2,$$

where $[k]!! = [k][k-2]\cdots[2](\mathrm{or}[1])$, $[0]!! = 1$, we rewrite Eq. (16) as

$$\sum_{x=-l}^{l} \hat{w}(x)P_n(x)P_k(x) = \delta_{nk},$$

where

$$\hat{w}(x) = \frac{w_{l-x}}{\sum_{y=0}^{2l} w_y} = \frac{[l][2l-1]!}{[l-x]![l+x]!}\frac{[2x]}{[x]}\left(\frac{[l-1]!}{2[2l-2]!!}\right)^2.$$

Therefore, $s_m(x)$, $m = -l, -l+1, \ldots, l$, defined as

$$s_m(x) = (-1)^{x-l}\hat{w}^{1/2}(x)P_{m+l}(x) = \frac{[l]![2l-1]!}{2[2l-2]!!}\times$$

$$\times(-1)^{x-l}\left(\frac{[2x]}{[x][l-x]![l+x]!}\frac{[2m]}{[m][l-m]![l+m]!}\right)^{1/2}\times$$

$$\times {}_4\phi_3\left(\begin{matrix} q^{-l+x},\ -q^{-l-x},\ q^{-m-l},\ -q^{-l+m} \\ \mathrm{i}q^{-l+1/2},\ q^{-2l},\ -\mathrm{i}q^{-l+1/2} \end{matrix};\ q,q\right), \quad (17)$$

satisfy the orthonormality condition

$$\sum_{x=-l}^{l} s_m(x)s_{m'}(x) = \delta_{mm'},$$

that is, $s_m(x)$ are the matrix elements of an orthogonal matrix. Hence,

$$\sum_{m=-l}^{l} s_m(x)s_m(x') = \delta_{xx'}.$$

These relations lead to orthonormality of the set of basis vectors

$$|x\rangle = \sum_{m=-l}^{l} s_m(x)|m\rangle, \qquad x = -l, -l+1, \ldots, l. \quad (18)$$

Note that these vectors are proportional to $\widetilde{|x\rangle}$, and therefore they are also eigenvectors for $T_l(I_2)$ with the same eigenvalues $\mathrm{i}[x]$. For the matrix elements $s_m(x)$, we have $s_m(x) = s_x(m)$. To make this symmetry explicit, we use the Sears transformation (see formula (III.15) in [11]) for the series ${}_4\phi_3$ from Eq. (17). The result is

$$s_m(x) = \frac{[l]![2l-1]!}{2[2l-2]!!}\times$$

$$\times\left(\frac{[2x]}{[x][l-x]![l+x]!}\frac{[2m]}{[m][l-m]![l+m]!}\right)^{1/2}(-1)^{x+m}\times$$

$$\times{}_4\phi_3\left(\begin{matrix} -q^{-l+x},\ q^{-l-x},\ -q^{-l+m},\ q^{-l-m} \\ q^{-2l},\ \mathrm{i}q^{-l+1/2},\ -\mathrm{i}q^{-l+1/2}\end{matrix};\ q,q\right). \quad (19)$$

Let us show that $C_\mathcal{S}$ may be defined as

$$C_\mathcal{S}|m\rangle = \sum_{m'=-l}^{l} s_{m'}(m)|m'\rangle. \quad (20)$$

Indeed, we have $T_l(I_2)C_\mathcal{S}|m\rangle = \mathrm{i}[m]C_\mathcal{S}|m\rangle$, and therefore $\langle m'|C_\mathcal{S}^{-1}T_l(I_2)C_\mathcal{S}|m\rangle = \langle m'|T_l(I_1)|m\rangle$. Thus, $C_\mathcal{S}^{-1}T_l(I_2)C_\mathcal{S} = T_l(I_1)$. Since $C_\mathcal{S}$ has an orthogonal and symmetric matrix, we have $C_\mathcal{S}^{-1} = C_\mathcal{S}^{\mathrm{T}} = C_\mathcal{S}$. Using this fact, the former relation can be rewritten as $C_\mathcal{S}^{-1}T_l(I_1)C_\mathcal{S} = T_l(I_2)$. Thus, $C_\mathcal{S}$ is, indeed, an intertwining operator corresponding to automorphism $\mathcal{S}$. Moreover, we automatically obtain that $C_\mathcal{S}$ satisfies the first of the defining relations (6) of $\mathrm{PSL}(2,\mathbb{Z})$.

In order to obtain the representation of the second of relations (6), we have to fix the matrix element t_0 of operator $C_\mathcal{T}$ in an appropriate way. First, we rewrite the second of the defining relations (6)

$$C_\mathcal{T}C_\mathcal{S}C_\mathcal{T}C_\mathcal{S}C_\mathcal{T}C_\mathcal{S} = \mathbf{1}$$

in the form

$$C_\mathcal{S}C_\mathcal{T}C_\mathcal{S} = C_\mathcal{T}^{-1}C_\mathcal{S}C_\mathcal{T}^{-1},$$

where we have used $C_\mathcal{S}^2 = \mathbf{1}$. Now we take the matrix elements $\langle m'|\cdots|m''\rangle$ of the both sides of this relation. We obtain

$$\sum_{m=-l}^{l} t_0 s_m(m'')q^{m^2/2}\mathrm{i}^m s_m(m') =$$

$$= t_0^{-2}q^{-(m'^2+m''^2)/2}\mathrm{i}^{-m'-m''}s_{m'}(m''). \quad (21)$$

It follows from the explicit formula (19) for $s_m(x)$ that

$$s_m(-l) = \frac{[l]!}{2[2l-2]!!}(-1)^{m-l}\left(\frac{[2l-1]!}{[l]}\right)^{1/2}\times$$

$$\times\left(\frac{[2m]}{[m][l-m]![l+m]!}\right)^{1/2}.$$

Fixing in Eq. (21) $m' = m'' = -l$ and using

$$s_{-l}(-l) = \frac{[l-1]!}{2[2l-2]!!},$$

we rewrite Eq. (21) as

$$t_0\frac{[l]![2l-1]!!}{2}\sum_{m=-l}^{l}\frac{\mathrm{i}^m q^{\frac{m^2}{2}}(q^m+q^{-m})}{[l-m]![l+m]!} = t_0^{-2}q^{-l^2}(-1)^l.$$

The left-hand side of this relation is equal to $t_0q^{-l(l-1)/2}$. Hence, $t_0 = (-1)^l q^{-l(l+1)/6}$ and the operator $C_\mathcal{T}$ is

$$C_\mathcal{T}|m\rangle = (-1)^l q^{-l(l+1)/6}\mathrm{i}^m q^{m^2/2}|m\rangle. \quad (22)$$

Then the intertwining operators $C_\mathcal{S}$ and $C_\mathcal{T}$ given by Eqs. (20) and (22) satisfy the defining relations for $\mathrm{PSL}(2,\mathbb{Z})$, giving a representation of the modular group of torus, which is the symmetry group in (2 + 1)-dimensional quantum gravity with space being a torus.

The investigation described in this article is close to the investigation made by R. Kashaev in [12], where he considered the spaces of general genus and infinite-dimensional representations of their mapping class groups.

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ПРО СИМЕТРІЇ В (2+1)-ВИМІРНІЙ КВАНТОВІЙ ГРАВІТАЦІЇ

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Р е з ю м е

В роботі Нельсон, Редже та Зертуче показано, що алгебра спостережуваних в квантовій гравітації в (2+1)-вимірному деситтерівському просторі-часі з тором в ролі простору пов'язана з алгеброю Фарлі–Одеського $U_q'(\mathrm{so}_3)$. Виявляється, що групою симетрій алгебри спостережуваних є модулярна група тора $\mathrm{PSL}(2,\mathbb{Z})$. Для цієї групи ми будуємо представлення, що відповідають скінченновимірним представленням алгебри спостережуваних.

О СИММЕТРИЯХ В (2+1)-МЕРНОЙ КВАНТОВОЙ ГРАВИТАЦИИ

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В работе Нельсон, Редже и Зертуче показано, что алгебра наблюдаемых в квантовой гравитации в (2+1)-мерном деситтеровском пространстве-времени с тором в качестве пространства связана с алгеброй Фарли–Одесского $U_q'(\mathrm{so}_3)$. Оказывается, что группой симметрий алгебры наблюдаемых является модулярная группа тора $\mathrm{PSL}(2,\mathbb{Z})$. Для этой группы мы строим представления, соответствующие конечномерным представлениям алгебры наблюдаемых.